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SOME NOTES ON THE EULER OBSTRUCTION OF A FUNCTION

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Abstract. In this paper, we present an alternative proof of the Brasselet, Massey, Parameswaran and Seade formula for the Euler obstruction of a function [5] using Ebeling and Gusein-Zade's results on the radial index and the Euler obstruction of 1-forms [11].

1. Introduction

Let $(X,0) \subset (\mathbb{C}^N,0)$ be an equidimensional reduced complex analytic germ. The Euler obstruction $\mathrm{Eu}_X(0)$ was defined by MacPherson [20] as a tool to prove the conjecture about existence and unicity of Chern classes in the singular case. Since that the Euler obstruction has been deeply investigated by many authors as Brasselet, Schwartz, Seade, Sebastiani, Gonzalez-Sprinberg, Lê, Teissier, Sabbah, Dubson, Kato and others. For an overview about the Euler obstruction see [2, 3].

In [4] a Lefschetz type formula for the Euler obstruction was given by Brasselet, Lê and Seade. This formula relates the Euler obstruction $\mathrm{Eu}_X(0)$ to the topology of the Milnor fibre of a generic linear form $l : (X,0) \to (\mathbb{C},0)$. It shows that the Euler obstruction, as a constructible function, satisfies the Euler condition relatively to generic linear forms (Theorem 2.3).

In [5], the authors studied how far the equality given in the above theorem is from being true if we replace the generic linear form l with some other analytic function on X with at most an isolated stratified critical point at 0. For this, they defined the Euler obstruction $\mathrm{Eu}_{f,X}(0)$ of a function f on a complex analytic variety X, which can be seen as a generalization of the Milnor number, and they established a Lefschetz type formula for this new invariant (Theorem 2.5).

The definition of the Euler obstruction of a function was extended by Ebeling and Gusein-Zade in [11] to the case of complex 1-forms. When the 1-form is the differential of a holomorphic function f, they recovered the Euler obstruction of the function (up to sign). They also define the radial index of a 1-form, which is a generalization to the singular case of the classical Poincaré-Hopf index. Then they established relations between the local Euler obstruction of a 1-form, the radial index and Euler characteristics of complex links.

In this paper, we use the results of Ebeling and Gusein-Zade to give an alternative proof of the Brasselet, Massey, Parameswaran and Seade formula for the Euler obstruction of a function (Theorem 2.5).

The main idea of the original proof of Theorem 2.5 was to construct a vector field that combines all the properties needed to prove the result, essentially using Poincaré-Hopf type theorems. Let us say some words about our proof, which uses combinatorial techniques and is a less extensive and less constructive proof than the original one in [5]. We first give an expression of the Euler obstruction of a 1-form in terms of the radial indices of this form on the closures of the strata of X and Euler characteristics of complex links (this relation appears first in [11],

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Corollary 1, with a different proof). As a corollary, we obtain a formula for $\mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0)$ in terms of Euler characteristics of complex links and the Euler characteristics of the Milnor fibre of f on the closures of the stata of X. Then we use the addivity of the Euler characteristic to get a relation between $\mathrm{Eu}_{f,X}(0)$ and the Euler characteristics of the Milnor fibres of f on the strata of X.

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2. The Euler obstruction

Let us now introduce some objects in order to define the Euler obstruction.

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an equidimensional reduced complex analytic germ of dimension d in an open set $U \subset \mathbb{C}^N$. We consider a complex analytic Whitney stratification $\{V_i\}$ of U adapted to X and we assume that $\{0\}$ is a stratum. We choose a small representative of $(X, 0)$ such that 0 belongs to the closure of all the strata. We denote it by X and we write $X = \cup_{i=0}^{q} V_i$ where $V_0 = \{0\}$ and $V_q = X_{\mathrm{reg}}$, the set of smooth points of X. We assume that the strata $V_0, \ldots, V_{q-1}$ are connected and that the analytic sets $\overline{V_0}, \ldots, \overline{V_{q-1}}$ are reduced. We set $d_i = \dim V_i$ for $i \in \{1, \ldots, q\}$ (note that $d_q = d$).

Let $G(d, N)$ denote the Grassmanian of complex d-planes in $\mathbb{C}^N$. On the regular part X_{reg} of X the Gauss map $\phi : X_{\mathrm{reg}} \to U \times G(d, N)$ is well defined by $\phi(x) = (x, T_x(X_{\mathrm{reg}}))$.

Definition 2.1. *The Nash transformation (or Nash blow-up) $\widetilde{X}$ of X is the closure of the image $\mathrm{Im}(\phi)$ in $U \times G(d, N)$. It is a (usually singular) complex analytic space endowed with an analytic projection map $\nu : \widetilde{X} \to X$ which is a biholomorphism away from $\nu^{-1}(\mathrm{Sing}(X))$.*

The fiber of the tautological bundle $\mathcal{T}$ over $G(d, N)$, at the point $P \in G(d, N)$, is the set of vectors v in the d-plane P. We still denote by $\mathcal{T}$ the corresponding trivial extension bundle over $U \times G(d, N)$. Let $\widetilde{T}$ be the restriction of $\mathcal{T}$ to $\widetilde{X}$, with projection map π. The bundle $\widetilde{T}$ on $\widetilde{X}$ is called *the Nash bundle* of X.

Let us recall the original definition of the Euler obstruction, due to MacPherson [20]. Let $z = (z_1, \ldots, z_N)$ be local coordinates in $\mathbb{C}^N$ around $\{0\}$, such that $z_i(0) = 0$. We denote by B_ε and S_ε the ball and the sphere centered at $\{0\}$ and of radius ε in $\mathbb{C}^N$. Let us consider the norm $\|z\| = \sqrt{z_1\overline{z}_1 + \cdots + z_N\overline{z}_N}$. Then the differential form $\omega = d\|z\|^2$ defines a section of the real vector bundle $T(\mathbb{C}^N)^*$, cotangent bundle on $\mathbb{C}^N$. Its pull-back restricted to $\widetilde{X}$ becomes a section of the dual bundle $\widetilde{T}^*$ which we denote by $\widetilde{\omega}$. For ε small enough, the section $\widetilde{\omega}$ is nonzero over $\nu^{-1}(z)$ for $0 < \|z\| \leq \varepsilon$. The obstruction to extend $\widetilde{\omega}$ as a nonzero section of $\widetilde{T}^*$ from $\nu^{-1}(S_\varepsilon)$ to $\nu^{-1}(B_\varepsilon)$, denoted by $Obs(\widetilde{T}^*, \widetilde{\omega})$ lies in $H^{2d}(\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon); \mathbb{Z})$. Let us denote by $\mathcal{O}_{\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon)}$ the orientation class in $H_{2d}(\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon); \mathbb{Z})$.

Definition 2.2. *The local Euler obstruction of X at 0 is the evaluation of $Obs(\widetilde{T}^*, \widetilde{\omega})$ on $\mathcal{O}_{\nu^{-1}(\mathbb{B}_\varepsilon), \nu^{-1}(\mathbb{S}_\varepsilon)}$, i.e.:*

$$\mathrm{Eu}_X(0) = \langle Obs(\widetilde{T}^*, \widetilde{\omega}), \mathcal{O}_{\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon)} \rangle.$$

An equivalent definition of the Euler obstruction was given by Brasselet and Schwartz in the context of vector fields [6].

The idea of studying the Euler obstruction using hyperplane sections appears in the works of Dubson [8] and Kato [13], but the approach we follow here comes from [4, 5].

Theorem 2.3 ([4]). *Let $(X, 0)$ and $\{V_i\}$ be given as before, then for each generic linear form l, there is ε_0 such that for any ε with $0 < \varepsilon < \varepsilon_0$ and $\delta \neq 0$ sufficiently small, the Euler obstruction of $(X, 0)$ is equal to:*

$$\mathrm{Eu}_X(0) = \sum_{i=1}^{q} \chi\big(V_i \cap B_\varepsilon \cap l^{-1}(\delta)\big) \cdot \mathrm{Eu}_X(V_i),$$

where χ denotes the Euler-Poincaré characteristic, $\mathrm{Eu}_X(V_i)$ is the value of the Euler obstruction of X at any point of V_i, $i = 1, \ldots, q$, and $0 < |\delta| \ll \varepsilon \ll 1$.

We define now an invariant introduced by Brasselet, Massey, Parameswaran and Seade in [5], which measures in a way how far the equality given in Theorem 2.3 is from being true if we replace the generic linear form l with some other function on X with at most an isolated stratified critical point at 0. Let $f : X \to \mathbb{C}$ be a holomorphic function which is the restriction of a holomorphic function $F : U \to \mathbb{C}$. A point x in X is a critical point of f if it is a critical point of $F_{|V(x)}$, where $V(x)$ is the stratum containing x. We assume that f has an isolated singularity (or an isolated critical point) at 0, i.e. that f has no critical point in a punctured neighborhood of 0 in X. In order to define this new invariant, the authors constructed in [5] a stratified vector field on X, denoted by $\overline{\nabla}_X f$. This vector field is homotopic to $\overline{\nabla} F|_X$ and one has $\overline{\nabla}_X f(x) \neq 0$ unless $x = 0$.

Let $\tilde{\zeta}$ be the lifting of $\overline{\nabla}_X f$ as a section of the Nash bundle $\widetilde{T}$ over $\widetilde{X}$ without singularity over $\nu^{-1}(X \cap S_\varepsilon)$. Let $\mathcal{O}(\tilde{\zeta}) \in H^{2n}\big(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon)\big)$ be the obstruction cocycle to the extension of $\tilde{\zeta}$ as a nowhere zero section of $\widetilde{T}$ inside $\nu^{-1}(X \cap B_\varepsilon)$.

Definition 2.4. *The local Euler obstruction $\mathrm{Eu}_{f,X}(0)$ is the evaluation of $\mathcal{O}(\tilde{\zeta})$ on the fundamental class of the pair $(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon))$.*

The following result is the Brasselet, Massey, Parameswaran and Seade formula [5] that compares the Euler obstruction of the space X with that of a function on X.

Theorem 2.5. *Let $(X, 0)$ and $\{V_i\}$ be given as before and let $f : (X, 0) \to (\mathbb{C}, 0)$ be a function with an isolated singularity at 0. For $0 < |\delta| \ll \varepsilon \ll 1$ we have:*

$$\mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0) = \left(\sum_{i=1}^{q} \chi\big(V_i \cap B_\varepsilon \cap f^{-1}(\delta)\big) \cdot \mathrm{Eu}_X(V_i) \right).$$

In this paper, we present an alternative proof for this result using Ebeling and Gusein-Zade's work [11] . In order to do this, let us consider the Nash bundle $\widetilde{T}$ on $\widetilde{X}$. The corresponding dual bundles of complex and real 1-forms are denoted, respectively, by $\widetilde{T}^* \to \widetilde{X}$ and $\widetilde{T}^*_{\mathbb{R}} \to \widetilde{X}$.

Definition 2.6. *Let $(X, 0)$ and $\{V_\alpha\}$ be given as before. Let ω be a (real or complex) 1-form on X, i.e. a continuous section of either $T^*_{\mathbb{R}}\mathbb{C}^N|_X$ or $T^*\mathbb{C}^N|_X$. A singularity of ω in the stratified sense means a point x where the kernel of ω contains the tangent space of the corresponding stratum.*

This means that the pull-back of the form to V_α vanishes at x. Given a section η of $T^*_{\mathbb{R}}\mathbb{C}^N|_A$, $A \subset V$, there is a canonical way of constructing a section $\tilde{\eta}$ of $\widetilde{T}^*_{\mathbb{R}}|_{\tilde{A}}$, $\tilde{A} = \nu^{-1}A$, such that if η has an isolated singularity at the point $0 \in X$ (in the stratified sense), then we have a never-zero section $\widetilde{\eta}$ of the dual Nash bundle $\widetilde{T}^*_{\mathbb{R}}$ over $\nu^{-1}(S_\varepsilon \cap X) \subset \widetilde{X}$. Let

$$o(\eta) \in H^{2d}(\nu^{-1}(B_\varepsilon \cap X), \nu^{-1}(S_\varepsilon \cap X); \mathbb{Z})$$

be the cohomology class of the obstruction cycle to extend this to a section of $\widetilde{T}^*_{\mathbb{R}}$ over $\nu^{-1}(B_\varepsilon \cap X)$. Then we can define (c.f. [7]):

Definition 2.7. *The local Euler obstruction of the real differential form η at an isolated singularity is the integer $\mathrm{Eu}_{X,0}\ \eta$ obtained by evaluating the obstruction cohomology class $o(\eta)$ on the orientation fundamental cycle $[\nu^{-1}(B_\varepsilon \cap X), \nu^{-1}(S_\varepsilon \cap X)]$.*

In the complex case, one can perform the same construction, using the corresponding complex bundles. If ω is a complex differential form, section of $T^*\mathbb{C}^N|_A$ with an isolated singularity, one can define the local Euler obstruction $\mathrm{Eu}_{X,0}\ \omega$. Notice that, as explained in [7] p.151, it is equal to the local Euler obstruction of its real part up to sign:

$$\mathrm{Eu}_{X,0}\ \omega\ =\ (-1)^d \mathrm{Eu}_{X,0} \mathrm{Re}\ \omega.$$

This is an immediate consequence of the relation between the Chern classes of a complex vector bundle and those of its dual. Remark also that when we consider the differential of a function f, we have the following equality (see [11]):

$$\mathrm{Eu}_{X,0}\ df = (-1)^d \mathrm{Eu}_{f,X}(0).$$

3. The complex link, radial index and Euler obstruction

In this section, we recall the definition of the complex link and of the radial index. We also present a formula of Ebeling and Gusein-Zade which expresses the radial index of a 1-form in terms of Euler characteristics of complex links and Euler obstructions.

The complex link is an important object in the study of the topology of complex analytic sets. It is analogous to the Milnor fibre and was studied first in [15]. It plays a crucial role in complex stratified Morse theory (see [12]) and appears in general bouquet theorems for the Milnor fibre of a function with isolated singularity (see [16, 17, 22, 23]). It is related to the multiplicity of polar varieties and also the local Euler obstruction (see [8, 9, 18, 19]). Let us recall briefly its definition. Let M be a complex analytic manifold equipped with a Riemannian metric and let $Y \subset M$ be a complex analytic variety equipped with a Whitney stratification. Let V be a stratum of Y and let p be a point in V. Let N be a complex analytic submanifold of M which meets V transversally at the single point p. By choosing local coordinates on N, in some neighborhood of p we can assume that N is an Euclidian space $\mathbb{C}^k$.

Definition 3.1. *The complex link of V in Y is the set denoted by $\mathrm{lk}^{\mathbb{C}}(V,Y)$ and defined as follows:*

$$\mathrm{lk}^{\mathbb{C}}(V,Y) = Y \cap N \cap B_\varepsilon \cap l^{-1}(\delta),$$

where $l : N \to \mathbb{C}$ is a generic linear form and $0 < |\delta| \ll \varepsilon \ll 1$.

The fact that the complex link of a stratum is well-defined, i.e. independent of all the choices made to define it, is explained in [19, 9, 12]. It is also independent of the embedding of the analytic variety Y (see [19]).

In [11], Ebeling and Gusein-Zade established relations between the local Euler obstruction of a 1-form, its radial index and Euler characteristics of complex links. The radial index is a generalization to the singular case of the Poincaré-Hopf index.

This index for 1-forms is a natural extension of the equivalent notion for vector fields, a notion first introduced by King and Trotman in a 1995 preprint only recently published [14] and then studied by Ebeling and Gusein-Zade in [10] and by Aguilar, Seade and Verjovsky in [1].

In order to define this index, let us consider first the real case. Let $Z \subset \mathbb{R}^n$ be a closed subanalytic set equipped with a Whitney stratification $\{S_\alpha\}_{\alpha\in\Lambda}$. Let ω be a continuous 1-form defined on $\mathbb{R}^n$. We say that a point P in Z is a zero (or a singular point) of ω on Z if it is a zero of $\omega_{|S}$, where S is the stratum that contains P. In the sequel, we define the radial index of ω at P, when P is an isolated zero of ω on Z. We can assume that $P = 0$ and we denote by S_0 the stratum that contains 0.

Definition 3.2. *A 1-form ω is radial on Z at 0 if, for an arbitrary non-trivial subanalytic arc $\varphi : [0,\nu[\to Z$ of class C^1, the value of the form ω on the tangent vector $\dot{\varphi}(t)$ is positive for t small enough.*

Let $\varepsilon > 0$ be small enough so that in the closed ball B_ε, the 1-form has no singular points on $Z \setminus \{0\}$. Let $S_0, \dots, S_r$ be the strata that contain 0 in their closure. Following Ebeling and Gusein-Zade, there exists a 1-form $\tilde{\omega}$ on $\mathbb{R}^n$ such that:

(1) The 1-form $\tilde{\omega}$ coincides with the 1-form ω on a neighborhood of S_ε.
(2) The 1-form $\tilde{\omega}$ is radial on Z at the origin.
(3) In a neighborhood of each zero $Q \in Z \cap B_\varepsilon \setminus \{0\}$, $Q \in S_i$, $\dim S_i = k$, the 1-form $\tilde{\omega}$ looks as follows. There exists a local subanalytic diffeomorphism $h : (\mathbb{R}^n, \mathbb{R}^k, 0) \to (\mathbb{R}^n, S_i, Q)$ such that $h^*\tilde{\omega} = \pi_1^*\tilde{\omega}_1 + \pi_2^*\tilde{\omega}_2$ where π_1 and π_2 are the natural projections $\pi_1 : \mathbb{R}^n \to \mathbb{R}^k$ and $\pi_2 : \mathbb{R}^n \to \mathbb{R}^{n-k}$, $\tilde{\omega}_1$ is a 1-form on a neighborhood of 0 in $\mathbb{R}^k$ with an isolated zero at the origin and $\tilde{\omega}_2$ is a radial 1-form on $\mathbb{R}^{n-k}$ at 0.

Definition 3.3. *The radial index $\mathrm{ind}^{\mathbb{R}}_{Z,0}\ \omega$ of the 1-form ω on Z at 0 is the sum:*

$$1 + \sum_{i=0}^{r} \sum_{Q | \tilde{\omega}_{|S_i}(Q)=0} \mathrm{ind}_{PH}(\tilde{\omega}, Q, S_i),$$

where $\mathrm{ind}_{PH}(\tilde{\omega}, Q, S_i)$ is the Poincaré-Hopf index of the form $\tilde{\omega}_{|S_i}$ at Q and where the sum is taken over all zeros of the 1-form $\tilde{\omega}$ on $(Z \setminus \{0\}) \cap B_\varepsilon$. If 0 is not a zero of ω on Z, we put $\mathrm{ind}^{\mathbb{R}}_{Z,0}\ \omega = 0$.

A straightforward corollary of this definition is that the radial index satisfies the law of conservation of number (see Remark 9.4.6 in [7] or the remark before Proposition 1 in [11]).

Let us go back to the complex case. As in Section 2, $(X,0) \subset (\mathbb{C}^N, 0)$ is an equidimensional reduced complex analytic germ of dimension d in an open set $U \subset \mathbb{C}^N$. Let ω be a complex 1-form on U with an isolated singular point on X at the origin.

Definition 3.4. *The complex radial index $\mathrm{ind}^{\mathbb{C}}_{X,0}\ \omega$ of the complex 1-form ω on X at the origin is $(-1)^d$ times the index of the real 1-form given by the real part of ω.*

Let us write $n_i = (-1)^{d-d_i-1}\left(\chi\big(\mathrm{lk}^{\mathbb{C}}(V_i, X)\big) - 1\right)$, where $\{V_i\}$ is the Whitney stratification of $(X,0)$ considered before. In particular for an open stratum V_i of X, $\mathrm{lk}^{\mathbb{C}}(V_i, X)$ is empty and so $n_i = 1$. Let us define the Euler obstruction $\mathrm{Eu}_{Y,0}\ \omega$ to be equal to 1 for a zero-dimensional connected variety Y. Under this conditions Ebeling and Gusein-Zade proved in [11] the following result which relates the radial index of a 1-form to Euler obstructions.

Theorem 3.5. *Let $(X,0) \subset (\mathbb{C}^N, 0)$ be the germ of a reduced complex analytic space at the origin, with a Whitney stratification $\{V_i\}$, $i = 0, \dots, q$, where $V_0 = \{0\}$ and V_q is the regular part of X. Then:*

$$\mathrm{ind}^{\mathbb{C}}_{X,0}\ \omega = \sum_{i=0}^{q} n_i \cdot \mathrm{Eu}_{\overline{V_i},0}\ \omega.$$

4. Corollaries of Theorem 3.5 and alternative proof of Theorem 2.5

In this section, we give some corollaries of Theorem 3.5, among them an alternative proof of Theorem 2.5.

As in the previous sections, $(X,0) \subset (\mathbb{C}^N, 0)$ is an equidimensional reduced complex analytic germ of dimension d in an open set U, equipped with a Whitney stratification $\{V_i\}$ such that 0 belongs to the closure of all the strata. We write $X = \cup_{i=0}^{q} V_i$ where $V_0 = \{0\}$ and $V_q = \overline{X_{\text{reg}}}$. We assume that the strata $V_0, \ldots, V_{q-1}$ are connected and that the analytic sets $\overline{V_0}, \ldots, \overline{V_{q-1}}$ are reduced. We set $d_i = \dim V_i$ for $i \in \{1, \ldots, q\}$. Let $f : X \to \mathbb{C}$ be a holomorphic function which is the restriction of a holomorphic function $F : U \to \mathbb{C}$. We assume that f has an isolated singularity at 0.

Let us see what happens when we apply Theorem 3.5 to the form $\sum \overline{z_k} dz_k$. Let us consider $(z_1, z_2, \ldots, z_N)$ as complex coordinates of $\mathbb{C}^N$, where $z_k = u_k + \sqrt{-1} v_k$. This implies that $(u_1, v_1, \ldots, u_N, v_N)$ are real coordinates of $\mathbb{R}^{2N}$. Let ω be a 1-form defined by $\omega = \sum_k \overline{z_k} dz_k$, it means that:

$$\omega = \sum_k (u_k - \sqrt{-1} v_k)(du_k + \sqrt{-1} dv_k),$$

and so that:

$$\omega = \sum_k (u_k du_k + v_k dv_k) + \sqrt{-1} \sum (u_k dv_k - v_k du_k).$$

In this case, the real 1-form $\operatorname{Re} \omega = \sum (u_k du_k + v_k dv_k)$ is a radial 1-form, and $\operatorname{ind}^{\mathbb{R}}_{X,0} \operatorname{Re} \omega = 1$. Since $\operatorname{ind}^{\mathbb{C}}_{X,0} \omega = (-1)^d \operatorname{ind}^{\mathbb{R}}_{X,0} \operatorname{Re} \omega$, we find that:

$$\operatorname{ind}^{\mathbb{C}}_{X,0} \omega = (-1)^d \operatorname{ind}^{\mathbb{R}}_{X,0} \operatorname{Re} \omega = (-1)^d.$$

As it was remarked before,

$$\operatorname{Eu}_{X,0} \omega = (-1)^d \operatorname{Eu}_{X,0} \operatorname{Re} \omega.$$

Using this information and the definition of n_i given in Section 3, we have the next equality:

$$n_i \operatorname{Eu}_{\overline{V_i},0} \omega = (-1)^{d-d_i-1} \Big(\chi\big(\operatorname{lk}^{\mathbb{C}}(V_i, X)\big) - 1 \Big) (-1)^{d_i} \operatorname{Eu}_{\overline{V_i}}(0).$$

Therefore, by Theorem 3.5 we conclude that:

$$(-1)^d = (-1)^d \left[\sum_{i=0}^{q-1} \Big(1 - \chi\big(\operatorname{lk}^{\mathbb{C}}(V_i, X)\big) \Big) \operatorname{Eu}_{\overline{V_i}}(0) + \operatorname{Eu}_X(0) \right],$$

and so we arrive to the following lemma:

Lemma 4.1. *We have:*

$$\operatorname{Eu}_X(0) = 1 + \sum_{i=0}^{q-1} \Big(\chi\big(\operatorname{lk}^{\mathbb{C}}(V_i, X)\big) - 1 \Big) \operatorname{Eu}_{\overline{V_i}}(0). \tag{1}$$

When we apply Theorem 3.5 to the form df, we obtain a similar result for the Euler obstruction of the function f.

Lemma 4.2. *We have:*

$$1 - \chi(f^{-1}(\delta) \cap X \cap B_\varepsilon) = \sum_{i=0}^{q} \Big(1 - \chi\big(\operatorname{lk}^{\mathbb{C}}(V_i, X)\big) \Big) \operatorname{Eu}_{f,\overline{V_i}}(0).$$

Proof. On the one hand, applying Theorem 3.5 to the form df, we have:

$$\operatorname{ind}^{\mathbb{C}}_{X,0} df \;=\; \sum_{i=0}^{q} n_i \operatorname{Eu}_{\overline{V_i},0} df \;=\; (-1)^{d-d_i-1} \Big(\chi\big(\operatorname{lk}^{\mathbb{C}}(V_i, X)\big) - 1 \Big) (-1)^{d_i} \operatorname{Eu}_{f,\overline{V_i}}(0).$$

On the other hand, by Theorem 3 of [11] we have:

$$\operatorname{ind}^{\mathbb{C}}_{X,0} df = (-1)^d \left(1 - \chi(f^{-1}(\delta) \cap X \cap B_\varepsilon) \right).$$

It follows that:

$$1-\chi(f^{-1}(\delta)\cap X\cap B_\varepsilon)=\sum_{i=0}^{q}\Big(1-\chi\big(\mathrm{lk}^{\mathbb{C}}(V_i,X)\big)\Big)\,\mathrm{Eu}_{f,\overline{V_i}}(0).$$

□

Before stating the next result, let us set $\mathrm{B}_{f,X}(0)=\mathrm{Eu}_X(0)-\mathrm{Eu}_{f,X}(0)$.

Corollary 4.3. *We have:*

$$\chi(f^{-1}(\delta)\cap X\cap B_\varepsilon)=\sum_{i=0}^{q}\Big(1-\chi\big(\mathrm{lk}^{\mathbb{C}}(V_i,X)\big)\Big)\,\mathrm{B}_{f,\overline{V_i}}(0).$$

Proof. By the previous lemma, we have the following equation:

$$\mathrm{Eu}_{f,X}(0)=1-\chi(f^{-1}(\delta)\cap X\cap B_\varepsilon)+\sum_{i=0}^{q-1}\Big(\chi\big(\mathrm{lk}^{\mathbb{C}}(V_i,X)\big)-1\Big)\,\mathrm{Eu}_{f,\overline{V_i}}(0). \tag{2}$$

By the difference (1) − (2) we arrive at:

$$\mathrm{B}_{f,X}(0)=\chi(f^{-1}(\delta)\cap X\cap B_\varepsilon)+\sum_{i=0}^{q-1}\Big(\chi\big(\mathrm{lk}^{\mathbb{C}}(V_i,X)\big)-1\Big)\,\mathrm{B}_{f,\overline{V_i}}(0). \tag{3}$$

Hence we find:

$$\chi(f^{-1}(\delta)\cap X\cap B_\varepsilon)=\sum_{i=0}^{q}\Big(1-\chi\big(\mathrm{lk}^{\mathbb{C}}(V_i,X)\big)\Big)\,\mathrm{B}_{f,\overline{V_i}}(0).$$

□

In [11, Corollary 1], Ebeling and Gusein-Zade give an "inverse" of the formula of Theorem 3.5. They use combinatorial theory (Möbius inverse). In the sequel, we give an inductive proof of that result. Let us recall the notations of [11]. The strata V_i of X are partially ordered: $V_i \prec V_j$ (we shall write $i\prec j$) if $V_i\subset \overline{V_j}$ and $V_i\neq V_j$. For two strata V_i and V_j with $V_i\preceq V_j$ (we shall write $i\preceq j$), let N_{ij} be the normal slice of the variety $\overline{V_j}$ to the stratum V_i at a point of it and let $M_{l|N_{ij}}$ be the complex link of V_i in $\overline{V_j}$. We denote $\chi(Z)-1$ by $\overline{\chi}(Z)$. For $i\prec j$, let m_{ij} be defined as follows:

$$m_{ij}=(-1)^{\dim\,X-\dim\,V_i}\sum_{i=k_0\prec\cdots\prec k_r=j}\overline{\chi}(M_{l|N_{k_0k_1}})\cdots\overline{\chi}(M_{l|N_{k_{r-1}k_r}}),$$

and let us set $m_{ii}=1$.

Corollary 4.4. *Let ω be a complex 1-form with an isolated zero on X at the origin. We have:*

$$\mathrm{Eu}_{X,0}\;\omega=\sum_{i=0}^{q}m_{iq}\cdot\mathrm{ind}^{\mathbb{C}}_{\overline{V_i},0}\;\omega.$$

Proof. This is clearly true if $\dim X=0$. Let us assume that $\dim X=d\geq 1$ and prove the result by induction on the depth of the stratification. The first step is to consider the case when X has an isolated singularity at the origin. In this case, the stratification will be $\{V_0=\{0\},V_1=X_{\mathrm{reg}}\}$ and

$$n_0=(-1)^{d-1}(\chi(\mathrm{lk}^{\mathbb{C}}(V_0,X)-1)=(-1)^{d-1}\overline{\chi}(M_{l|N_{01}}),$$

$\mathrm{Eu}_{X,0}\;\omega=1$, $n_1=1$ and $\mathrm{Eu}_{\overline{V_1},0}\;\omega=\mathrm{Eu}_{X,0}\;\omega$. Applying Theorem 3.5, we get:

$$\mathrm{ind}^{\mathbb{C}}_{X,0}\;\omega=(-1)^{d-1}\overline{\chi}(M_{l|N_{01}})+\mathrm{Eu}_{X,0}\;\omega,$$

and so:

$$\mathrm{Eu}_{X,0}\ \omega = \mathrm{ind}^{\mathbb{C}}_{X,0}\ \omega + (-1)^d \overline{\chi}(M_{l|N_{01}}).$$

This is exactly the expected formula because $\mathrm{ind}^{\mathbb{C}}_{V_0,0}\ \omega = 1$ and $m_{01} = (-1)^d \overline{\chi}(M_{l|N_{01}})$.

Let us prove the general case. By the induction hypothesis, for each $k \in \{0, \dots, d-1\}$, we have:

$$\mathrm{Eu}_{\overline{V_k},0}\ \omega = \sum_{j \ | \ V_j \subset \overline{V_k}} m_{ik} \cdot \mathrm{ind}^{\mathbb{C}}_{\overline{V_j},0}\ \omega.$$

But we know by Theorem 3.5 that:

$$\mathrm{Eu}_{X,0}\ \omega = \mathrm{ind}^{\mathbb{C}}_{X,0}\ \omega - \sum_{k=0}^{d-1} n_k \cdot \mathrm{Eu}_{\overline{V_k},0}\ \omega.$$

Replacing $\mathrm{Eu}_{\overline{V_k},0}\ \omega$ by its above value, we obtain:

$$\mathrm{Eu}_{X,0}\ \omega = \mathrm{ind}^{\mathbb{C}}_{X,0}\ \omega - \sum_{k=0}^{d-1} n_k \left(\mathrm{ind}^{\mathbb{C}}_{\overline{V_k},0}\ \omega + \sum_{j \ | \ V_j \subset \partial\overline{V_k}} m_{jk} \cdot \mathrm{ind}^{\mathbb{C}}_{\overline{V_j},0}\ \omega \right).$$

We see that each $\mathrm{ind}_{\overline{V_j},0}\mathbb{C}\ \omega$ appears in each term

$$n_k \left(\mathrm{ind}^{\mathbb{C}}_{\overline{V_k},0}\ \omega + \sum_{j \ | \ V_j \subset \partial\overline{V_k}} m_{jk} \cdot \mathrm{ind}^{\mathbb{C}}_{\overline{V_j},0}\ \omega \right),$$

for which $V_j \subset \overline{V_k}$. Therefore we can write:

$$\mathrm{Eu}_{X,0}\ \omega = \mathrm{ind}^{\mathbb{C}}_{X,0}\ \omega - \sum_{j=0}^{d-1} \mathrm{ind}^{\mathbb{C}}_{\overline{V_j},0}\ \omega \left(n_j + \sum_{k \ | \ V_j \subset \partial\overline{V_k}} m_{jk} \cdot n_k \right).$$

Let us examine $A_j = n_j + \sum_{k \ | \ V_j \subset \partial\overline{V_k}} m_{jk} \cdot n_k$. We have:

$$A_j = (-1)^{d-d_j-1} \overline{\chi}(M_{l|N_{jq}}) +$$

$$\sum_{k \ | \ V_j \subset \partial\overline{V_k}} \Big((-1)^{d_k-d_j-1} \sum_{j=k_0 \prec \cdots \prec k_r = k} \overline{\chi}(M_{l|N_{k_0k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1}k_r}}) \times (-1)^{d-d_k-1} \overline{\chi}(M_{l|N_{kq}}) \Big).$$

Therefore, we see that:

$$A_j = (-1)^{d-d_j-1} \overline{\chi}(M_{l|N_{jq}}) +$$

$$\sum_{k \ | \ V_j \subset \partial\overline{V_k}} \left((-1)^{d-d_j-1} \sum_{j=k_0 \prec \cdots \prec k_{r+1} = q} \overline{\chi}(M_{l|N_{k_0k_1}}) \cdots \overline{\chi}(M_{l|N_{k_rk_{r+1}}}) \right),$$

i.e. $A_j = -m_{jq}$. We get the desired result. □

When we apply this to the form $\omega = \sum \overline{z_k} dz_k$, we get:

$$\mathrm{Eu}_X(0) = \sum_{i=0}^{q} \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_{l|N_{k_0k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1}k_r}}). \qquad (*)$$

This formula is still valid if $V_0 \neq \{0\}$. In this case, we can introduce the stratum $V_{-1} = \{0\}$. The above formula becomes:

$$\mathrm{Eu}_X(0) = \sum_{i=-1}^{q} \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_{l|N_{k_0 k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1} k_r}}).$$

But since generically the linear form l has no singularity at 0 on V_0, the Milnor fibre $M_{l|\overline{V_k}}$ of $l : \overline{V_k} \to \mathbb{C}$ is contractible for $k \geq 0$, which implies that $\overline{\chi}(M_{l|N_{-1k}}) = 0$ for $k \geq 0$.

Applied to the form df, Corollary 4.4 gives:

$$\mathrm{Eu}_{f,X}(0) = -\sum_{i=0}^{q} \overline{\chi}(M_{f|\overline{V_i}}) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_{l|N_{k_0 k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1} k_r}}), \qquad (**)$$

where $M_{f|\overline{V_i}}$ denotes the Milnor fibre of $f : \overline{V_i} \to \mathbb{C}$, because $\mathrm{Eu}_{f,X}(0) = (-1)^d \mathrm{Eu}_{X,0}\ df$ and $\mathrm{ind}^{\mathbb{C}}_{\overline{V_i},0}\ df = (-1)^{d_i - 1} \overline{\chi}(M_{f|\overline{V_i}})$.

We are now in position to give the alternative proof of Theorem 2.5.

Proof. Using the two equalities $(*)$ and $(**)$ above, we find:

$$\mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0) = \sum_{i=0}^{q} \chi(M_{f|\overline{V_i}}) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_{l|N_{k_0 k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1} k_r}}).$$

By the additivity of the Euler characteristic, for each $i \in \{0, \ldots, q\}$ we have:

$$\chi(M_{f|\overline{V_i}}) = \sum_{j\ |V_j \subset \overline{V_i}} \chi(M_{f|V_j}).$$

Therefore, we have:

$$\mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0) = \sum_{i=0}^{q} \left(\sum_{j\ |V_j \subset \overline{V_i}} \chi(M_{f|V_j}) \right) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_{l|N_{k_0 k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1} k_r}}).$$

As in the proof of the previous corollary, we see that each $\chi(M_{f|V_j})$ appears in an expression

$$\left(\sum_{j\ |V_j \subset \overline{V_i}} \chi(M_{f|V_j}) \right) \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_{l|N_{k_0 k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1} k_r}}),$$

when $V_j \subset \overline{V_i}$. We can factorize $\chi(M_{f|V_j})$ in the above equality and get:

$$\mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0) = \sum_{j=0}^{q} \chi(M_{f|V_j}) \left(\sum_{i\ |V_j \subset \overline{V_i}} \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_{l|N_{k_0 k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1} k_r}}) \right).$$

But by the equality $(*)$ and the remark that follows it, we see that:

$$\sum_{i\ |V_j \subset \overline{V_i}} \sum_{i=k_0 \prec \cdots \prec k_r = q} \overline{\chi}(M_{l|N_{k_0 k_1}}) \cdots \overline{\chi}(M_{l|N_{k_{r-1} k_r}}),$$

is exactly $\mathrm{Eu}_X(V_j)$. $\square$

References

[1] AGUILAR, M., SEADE, J. and VERJOVSKY, A.: Indices of vector fields and topological invariants or real analytic singularities, *J. Reine Angew. Math.* **504** (1998), 159-176.

[2] BRASSELET, J.-P.: Local Euler obstruction, old and new, *XI Brazilian Topology Meeting (Rio Claro, 1998)*, 140-147, World Sci. Publishing, River Edge, NJ, 2000.

[3] BRASSELET, J.-P. and GRULHA JR., N. G.: Local Euler obstruction, old and new II, *London Mathematical Society - Lectures Notes Series 380 - Real and Complex Singularities*, Cambridge University Press, 2010, v., p. 23-45.

[4] BRASSELET, J.-P., LÊ, D. T. and SEADE, J.: Euler obstruction and indices of vector fields, *Topology*, **6** (2000) 1193-1208. DOI: 10.1016/S0040-9383(99)00009-9

[5] BRASSELET, J.-P., MASSEY, D. and PARAMESWARAN, A. and SEADE, J.: Euler obstruction and defects of functions on singular varieties, *Journal London Math. Soc (2)* **70** (2004), no.1, 59-76.

[6] BRASSELET, J.-P. and SCHWARTZ, M.-H.: Sur les classes de Chern d'un ensemble analytique complexe, *Astérisque* **82-83** (1981) 93-147.

[7] BRASSELET, J.-P., SEADE, J. and SUWA, T.: Vector fields on singular varieties, *Lecture Notes in Mathematics*, **1987**, Springer-Verlag, Berlin (2009).

[8] DUBSON, A.: Classes caractéristiques des variétés singulières, *C. R. Acad. Sci. Paris Sr. A-B* **287** (1978), no. 4, 237-A240.

[9] DUBSON, A.: Calcul des invariants numériques des singularités et applications, *Sonderforschungsbereich 40 Theoretische Mathematik* Universitaet Bonn (1981).

[10] EBELING, W. and GUSEIN-ZADE, S.M .: On the index of a vector field at an isolated singularity, *The Arnoldfest (Toronto, ON, 1997)*, 141-152, Fields Inst. Commun., 24, Amer. Math. Soc., Providence, RI, 1999.

[11] EBELING, W. and GUSEIN-ZADE, S.M .: Radial index and Euler obstruction of a 1-form on a singular variety, *Geom. Dedicata* **113** (2005), 231-241. DOI: 10.1007/s10711-005-2184-1

[12] GORESKY, M. and MACPHERSON, R.: Stratified Morse theory, *Springer-Verlag*, Berlin, 1988. DOI: 10.1007/978-3-642-71714-7

[13] KATO, M.: Singularities and some global topological properties, *Proc. R.I.M.S. Singularities Symposium*, Kyoto University (1978).

[14] KING, H. and TROTMAN, D.: Poincaré-Hopf theorems on singular spaces, *Proc. London Math. Soc.*, 108 (3), (2014), 682-703. DOI: 10.1112/plms/pdt039

[15] LÊ D.T.: Vanishing cycles on complex analytic sets, *Proc. Sympos., Res. Inst. Math. Sci., Kyoto, Univ. Kyoto, 1975* Sûrikaisekikenkyûsho Kókyûroku, no. 266 (1976), 299-318.

[16] LÊ D. T.: Le concept de singularité isolée de fonction analytique, *Adv. Stud. Pure Math.* **8** (1986) 215–227.

[17] LÊ D.T.: Complex analytic functions with isolated singularities, *J. Algebraic Geom.* **1** (1992), no. 1, 83-99.

[18] LÊ D. T. and TEISSIER, B.: Variétés polaires Locales et classes de Chern des variétés singulières, *Ann. of Math.* **114**, (1981), 457-491. DOI: 10.2307/1971299

[19] LÊ D. T. and TEISSIER, B.: Cycles évanescents, sections planes et conditions de Whitney II, Singularities, Part 2 (Arcata, Calif., 1981), 65-103, *Proc. Sympos. Pure Math.* **40**, Amer. Math. Soc., Providence, RI, 1983.

[20] MACPHERSON, R. D.: Chern classes for singular algebraic varieties, *Ann. of Math.* **100** (1974), 423-432. DOI: 10.2307/1971080

[21] SEADE, J., TIBĂR, M. and VERJOVSKY A.: Milnor Numbers and Euler obstruction, *Bull. Braz. Math. Soc. (N.S)* **36** (2005), no. 2, 275-283.

[22] SIERSMA, D.: A bouquet theorem for the Milnor fibre, *J. Algebraic Geom.* **4** (1995), no. 1, 51-66.

[23] TIBĂR, M.: Bouquet decomposition of the Milnor fibre, *Topology* **35** (1996), no. 1, 227-241.

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