Journal *of* Singularities
Volume 12 (2015), 92-104

Proc. of Singularities in Geometry and Appl. III, Edinburgh, Scotland, 2013
DOI: 10.5427/jsing.2015.12f

SOME CONJECTURES ON STRATIFIED-ALGEBRAIC VECTOR BUNDLES

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Abstract. We investigate relationships between topological and stratified-algebraic vector bundles on real algebraic varieties. We propose some conjectures and prove them in special cases.

1. Introduction

In the recent joint paper with K. Kurdyka [23], we introduced and investigated stratified-algebraic vector bundles on real algebraic varieties. They occupy an intermediate position between algebraic and topological vector bundles. A challenging problem is to find a characterization of topological vector bundles admitting a stratified-algebraic structure. In the present paper, we propose Conjecture A, whose proof would provide a complete solution of this problem. Conjecture B and Conjecture C, which also are concerned with relationships between stratified-algebraic and topological vector bundles, easily follow from Conjecture A. We prove these three conjectures in some special cases. Furthermore, we show that they are connected with certain problems involving transformation of compact smooth (of class $\mathcal{C}^\infty$) submanifolds of nonsingular real algebraic varieties onto subvarieties. All results announced in this section are proved in Section 2.

In the present paper, we develop the same new direction of research in real algebraic geometry as the authors of [5, 16, 20, 21, 22, 23].

Throughout this paper the term *real algebraic variety* designates a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^N$, for some N, endowed with the Zariski topology and the sheaf of real-valued regular functions (such an object is called an affine real algebraic variety in [7]). The class of real algebraic varieties is identical with the class of quasi-projective real varieties, cf. [7, Proposition 3.2.10, Theorem 3.4.4]. Morphisms of real algebraic varieties are called *regular maps*. Each real algebraic variety carries also the Euclidean topology, which is induced by the usual metric on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties refer to the Euclidean topology.

Let X be a real algebraic variety. By a *stratification* of X we mean a finite collection $\mathcal{S}$ of pairwise disjoint Zariski locally closed subvarieties whose union is X. Each subvariety in $\mathcal{S}$ is called a *stratum* of $\mathcal{S}$. A map $f\colon X \to Y$, where Y is a real algebraic variety, is said to be *stratified-regular* if it is continuous and for some stratification $\mathcal{S}$ of X, the restriction $f|_S\colon S \to Y$ of f to each stratum S in $\mathcal{S}$ is a regular map, cf. [23]. The notion of stratified-regular map is closely related to those of hereditarily rational function [20] and fonction régulue [16].

Let $\mathbb{F}$ stand for $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions). All $\mathbb{F}$-vector spaces will be left $\mathbb{F}$-vector spaces. When convenient, $\mathbb{F}$ will be identified with $\mathbb{R}^{d(\mathbb{F})}$, where

$$d(\mathbb{F}) = \dim_{\mathbb{R}} \mathbb{F}.$$

2010 *Mathematics Subject Classification.* 14P25, 14P99, 14F25, 19A49.

Key words and phrases. Real algebraic variety, stratification, stratified-algebraic vector bundle, stratified-regular map.

The author was partially supported by NCN grant 2011/01/B/ST1/01289.

For any nonnegative integer n, let $\varepsilon_X^n(\mathbb{F})$ denote the standard trivial $\mathbb{F}$-vector bundle on X with total space $X \times \mathbb{F}^n$, where $X \times \mathbb{F}^n$ is regarded as a real algebraic variety.

An *algebraic* $\mathbb{F}$*-vector bundle on* X is an algebraic $\mathbb{F}$-vector subbundle of $\varepsilon_X^n(\mathbb{F})$ for some n (cf. [7, Chapters 12 and 13] for various characterizations of algebraic $\mathbb{F}$-vector bundles).

We now recall the fundamental notion introduced in [23]. A *stratified-algebraic* $\mathbb{F}$*-vector bundle on* X is a topological $\mathbb{F}$-vector subbundle ξ of $\varepsilon_X^n(\mathbb{F})$, for some n, such that for some stratification $\mathcal{S}$ of X, the restriction $\xi|_S$ of ξ to each stratum S of $\mathcal{S}$ is an algebraic $\mathbb{F}$-vector subbundle of $\varepsilon_S^n(\mathbb{F})$.

A topological $\mathbb{F}$-vector bundle η on X is said to *admit an algebraic structure* if it is isomorphic to an algebraic $\mathbb{F}$-vector bundle on X. Similarly, η is said to *admit a stratified-algebraic structure* if it is isomorphic to a stratified-algebraic $\mathbb{F}$-vector bundle on X. These two classes of $\mathbb{F}$-vector bundles have been extensively investigated in [3, 4, 6, 7, 8, 9, 10, 13] and [23], respectively. In general, their behaviors are quite different, cf. [23, Example 1.11]. The $\mathbb{F}$-vector bundle η can be regarded as an $\mathbb{R}$-vector bundle, which is indicated by $\eta_{\mathbb{R}}$. If η admits an algebraic structure or a stratified-algebraic structure, then so does $\eta_{\mathbb{R}}$.

Some preparation is necessary to formulate a conjectural characterization of topological $\mathbb{F}$-vector bundles admitting a stratified-algebraic structure.

Let V be a compact nonsingular real algebraic variety. A cohomology class in $H^k(V;\mathbb{Z}/2)$ is said to be *algebraic* if the homology class Poincaré dual to it can be represented by a Zariski closed subvariety of V of codimension k. The set $H^k_{\mathrm{alg}}(V;\mathbb{Z}/2)$ of all algebraic cohomology classes in $H^k(V;\mathbb{Z}/2)$ forms a subgroup. The groups $H^k_{\mathrm{alg}}(-;\mathbb{Z}/2)$ have been studied by many authors. Their basic properties can be found in [4, 7, 11, 14].

The following notion was introduced and investigated in [23]. A cohomology class u in $H^k(X;\mathbb{Z}/2)$ is said to be *stratified-algebraic* if there exists a stratified-regular map $\varphi\colon X \to V$, into a compact nonsingular real algebraic variety V, such that $u = \varphi^*(v)$ for some algebraic cohomology class v in $H^k(V;\mathbb{Z}/2)$. The set $H^k_{\mathrm{str}}(X;\mathbb{Z}/2)$ of all stratified-algebraic cohomology classes in $H^k(X;\mathbb{Z}/2)$ forms a subgroup. The groups $H^k_{\mathrm{str}}(-;\mathbb{Z}/2)$ have many expected, "good" properties. In particular, if ξ is a stratified-algebraic $\mathbb{F}$-vector bundle on X, then the kth Stiefel–Whitney class $w_k(\xi_{\mathbb{R}})$ of the $\mathbb{R}$-vector bundle $\xi_{\mathbb{R}}$ is in $H^k_{\mathrm{str}}(X;\mathbb{Z}/2)$ for every nonnegative integer k. Furthermore, a topological $\mathbb{R}$-line bundle λ on X admits a stratified-algebraic structure if and only if $w_1(\lambda)$ is in $H^1_{\mathrm{str}}(X;\mathbb{Z}/2)$.

Convention. Henceforth, we assume for simplicity that all vector bundles are of constant rank.

If X is a compact real algebraic variety with $\dim X \leq d(\mathbb{F})$, then each topological $\mathbb{F}$-vector bundle on X admits a stratified-algebraic structure, cf. [23, Corollary 3.6]. Without any restrictions on the dimension of X, we propose the following.

Conjecture A. *Let X be a compact real algebraic variety and let ξ be a topological $\mathbb{F}$-vector bundle on X. If the Stiefel–Whitney class $w_k(\xi_{\mathbb{R}})$ is in $H^k_{\mathrm{str}}(X;\mathbb{Z}/2)$ for every positive integer $k < \dim X$, then ξ admits a stratified-algebraic structure.*

Any compact real algebraic variety X is traingulable [7, Theorem 9.2.1]. In particular, $H^k(X;\mathbb{Z}/2) = 0$ for every $k > \dim X$. Furthermore, by Hopf's theorem and [23, Theorem 2.5],

$$H^k_{\mathrm{str}}(X;\mathbb{Z}/2) = H^k(X;\mathbb{Z}/2) \text{ for } k = \dim X.$$

This explains why the condition "$w_k(\xi_{\mathbb{R}})$ is in $H^k_{\mathrm{str}}(X;\mathbb{Z}/2)$" in Conjecture A is imposed only for $k < \dim X$.

One may argue that Conjecture A is unlikely to be true since the Stiefel–Whitney classes $w_k(\xi_{\mathbb{R}})$ carry only limited information on the $\mathbb{F}$-vector bundle ξ, especially when $\mathbb{F} = \mathbb{C}$ or

$\mathbb{F} = \mathbb{H}$. However, according to [23, Theorem 1.7], the $\mathbb{F}$-vector bundle ξ admits a stratified-algebraic structure if and only if the $\mathbb{R}$-vector bundle $\xi_{\mathbb{R}}$ admits a stratified-algebraic structure.

For $\mathbb{R}$-vector bundles, we have the following.

Theorem 1.1. *Let X be a compact real algebraic variety and let ξ be a topological $\mathbb{R}$-vector bundle on X. If $\dim X \leq 3$ and the Stiefel–Whitney class $w_k(\xi)$ is in $H^k_{\mathrm{str}}(X;\mathbb{Z}/2)$ for $k = 1, 2$, then ξ admits a stratified-algebraic structure.*

This result cannot be regarded as a strong evidence for Conjecture A since the assumption $\dim X \leq 3$ is very restrictive.

We also prove Conjecture A in another special case.

Theorem 1.2. *Let X be a compact real algebraic variety and let ξ be a topological $\mathbb{F}$-vector bundle on X. If $\dim X \leq d(\mathbb{F}) + 1$ and the Stiefel–Whitney class $w_{d(\mathbb{F})}(\xi_{\mathbb{R}})$ is in $H^{d(\mathbb{F})}_{\mathrm{str}}(X;\mathbb{Z}/2)$, then ξ admits a stratified-algebraic structure.*

Even if Conjecture A does not hold in general, it may be true, with no restrictions on $\dim X$, for $\mathbb{F}$-vector bundles of low rank. In particular, it remains open for $\mathbb{R}$-vector bundles of rank 2 and $\mathbb{F}$-line bundles with $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{H}$.

We now concentrate our attention on $\mathbb{C}$-line bundles. For any $\mathbb{C}$-line bundle λ and any positive integer r, let $\lambda^{\otimes r}$ denote the rth tensor power of λ. Note that

$$w_k((\lambda^{\otimes 2})_{\mathbb{R}}) = 0$$

for every positive integer k, cf. [24, p. 171]. Consequently, Conjecture A implies

Conjecture B. *For any compact real algebraic variety X and any topological $\mathbb{C}$-line bundle λ on X, the $\mathbb{C}$-line bundle $\lambda^{\otimes 2}$ admits a stratified-algebraic structure.*

We can reformulate Conjecture B as follows. Let $\mathrm{VB}^1_{\mathbb{C}}(X)$ be the group of isomorphism classes of topological $\mathbb{C}$-line bundles on X (with operation induced by tensor product). Denote by $\mathrm{VB}^1_{\mathbb{C}\text{-str}}(X)$ the subgroup of $\mathrm{VB}^1_{\mathbb{C}}(X)$ consisting of the isomorphism classes of $\mathbb{C}$-line bundles admitting a stratified-algebraic structure. Conjecture B is equivalent to the assertion that every element of the quotient group $\mathrm{VB}^1_{\mathbb{C}}(X)/\mathrm{VB}^1_{\mathbb{C}\text{-str}}(X)$ is of order at most 2.

According to Theorem 1.2, Conjecture B holds if $\dim X \leq 3$. We can however prove a little more.

Theorem 1.3. *Let X be a compact real algebraic variety of dimension at most 4. For any topological $\mathbb{C}$-line bundle λ on X, the $\mathbb{C}$-line bundle $\lambda^{\otimes 2}$ admits a stratified-algebraic structure.*

Furthermore, we have the following result.

Theorem 1.4. *Let X be a compact real algebraic variety of dimension 5. For any topological $\mathbb{C}$-line bundle λ on X, the $\mathbb{C}$-line bundle $\lambda^{\otimes 4}$ admits a stratified-algebraic structure.*

Conjecture B is related to seemingly quite different problems. Let V be a nonsingular real algebraic variety. A bordism class in the nth unoriented bordism group $\mathfrak{N}_n(V)$ of V is said to be *algebraic* if it can be represented by a regular map from an n-dimensional compact nonsingular real algebraic variety into V, cf. [1, 2]. The set $\mathfrak{N}^{\mathrm{alg}}_n(V)$ of all algebraic bordism classes in $\mathfrak{N}_n(V)$ forms a subgroup.

Approximation Conjecture. *For any nonsingular real algebraic variety V, the following condition is satisfied: If M is a compact smooth submanifold of V and the unoriented bordism class of the inclusion map $M \hookrightarrow V$ is algebraic, then M is ε-isotopic to a nonsingular Zariski locally closed subvariety of V.*

Here "ε-isotopic" means isotopic via a smooth isotopy that can be chosen arbitrarily close, in the $\mathcal{C}^\infty$ topology, to the inclusion map $M \hookrightarrow V$. A sightly weaker assertion than the one in the Approximation Conjecture is known to be true: If the unoriented bordism class of the inclusion map $M \hookrightarrow V$ is algebraic, then the smooth submanifold $M \times \{0\}$ of $V \times \mathbb{R}$ is ε-isotopic to a nonsingular Zariski locally closed subvariety of $V \times \mathbb{R}$, cf. [1, Theorem F].

The following is a special case of the Approximation Conjecture.

Conjecture B(k). *For any compact nonsingular real algebraic variety V, the following condition is satisfied: If M is a compact smooth codimension k submanifold of V and the unoriented bordism class of the inclusion map $M \hookrightarrow V$ is zero, then M is isotopic to a nonsingular Zariski locally closed subvariety of V.*

In the context of this paper, Conjecture B(2) is of particular interest.

Proposition 1.5. *Conjecture $B(2)$ implies Conjecture B.*

Denote by $e(\mathbb{F})$ the integer satisfying $d(\mathbb{F}) = 2^{e(\mathbb{F})}$, that is,

$$e(\mathbb{F}) = \begin{cases} 0 & \text{if } \mathbb{F} = \mathbb{R} \\ 1 & \text{if } \mathbb{F} = \mathbb{C} \\ 2 & \text{if } \mathbb{F} = \mathbb{H}. \end{cases}$$

Given a nonnegative integer n, set

$$a(n) = \min\{l \in \mathbb{Z} \mid l \geq 0, 2^l \geq n\},$$
$$a(n, \mathbb{F}) = \max\{0, a(n) - e(\mathbb{F})\}.$$

For any topological $\mathbb{F}$-vector bundle ξ and any positive integer r, let

$$\xi(r) = \xi \oplus \cdots \oplus \xi$$

be the r-fold direct sum.

Conjecture C. *For any compact real algebraic variety X and any topological $\mathbb{F}$-vector bundle ξ on X, the $\mathbb{F}$-vector bundle $\xi(2^{a(\dim X, \mathbb{F})})$ admits a stratified-algebraic structure.*

Equivalently, Conjecture C can be stated as follows. Let $K_{\mathbb{F}}(X)$ be the Grothendieck group of topological $\mathbb{F}$-vector bundles (of constant rank) on X. Denote by $K_{\mathbb{F}\text{-str}}(X)$ the subgroup of $K_{\mathbb{F}}(X)$ generated by the classes of $\mathbb{F}$-vector bundles admitting a stratified-algebraic structure. Conjecture C implies the inclusion

$$2^{a(\dim X, \mathbb{F})} K_{\mathbb{F}}(X) \subseteq K_{\mathbb{F}\text{-str}}(X).$$

Conversely, according to [23, Corollary 3.14], this inclusion implies Conjecture C.

With notation as in Conjecture C, we have

$$w_k(\xi(2^{a(\dim X, \mathbb{F})})_{\mathbb{R}}) = 0$$

for every positive integer $k < \dim X$. Indeed, this assertion follows from the Whitney formula for Stiefel–Whitney classes. Consequently, Conjecture A implies Conjecture C. In particular, by Theorems 1.1 and 1.2, Conjecture C holds if $\dim X \leq 3$ or if $\dim X \leq 5$ and $\mathbb{F} = \mathbb{H}$. This can be generalized as follows.

Theorem 1.6. *Let X be a compact real algebraic variety of dimension $n \leq 5$. For any topological $\mathbb{F}$-vector bundle ξ on X, the $\mathbb{F}$-vector bundle $\xi(2^{a(n,\mathbb{F})})$ admits a stratified-algebraic structure.*

Theorem 1.6 is in some sense optimal. This is made precise in Theorem 1.7 below.

Any topological $\mathbb{R}$-vector bundle ξ gives rise to an $\mathbb{F}$-vector bundle $\mathbb{F} \otimes \xi$. Here $\mathbb{R} \otimes \xi = \xi$, $\mathbb{C} \otimes \xi$ is the complexification of ξ, and $\mathbb{H} \otimes \xi$ is the quaternionization of ξ.

Theorem 1.7. *Let n be an integer satisfying $0 \leq n \leq 5$. Then there exist an n-dimensional compact irreducible nonsingular real algebraic variety X and a topological $\mathbb{R}$-line bundle λ on X with the following property: For a positive integer r, the $\mathbb{F}$-vector bundle $(\mathbb{F} \otimes \lambda)(r)$ admits a stratified-algebraic structure if and only if r is divisible by $2^{a(n,\mathbb{F})}$.*

Other results related to the conjectures above are contained in Section 2, cf. Theorems 2.3 and 2.8.

2. Proofs and further results

To begin with, we recall some properties of stratified-algebraic cohomology classes. For any real algebraic variety X, the direct sum

$$H^*_{\mathrm{str}}(X;\mathbb{Z}/2) = \bigoplus_{k \geq 0} H^k_{\mathrm{str}}(X;\mathbb{Z}/2)$$

is a subring of the cohomology ring $H^*(X;\mathbb{Z}/2)$. The rings $H^*_{\mathrm{str}}(-;\mathbb{Z}/2)$ have the following functorial property: If $f\colon X \to Y$ is a stratified-regular map between real algebraic varieties, then

$$f^*(H^*_{\mathrm{str}}(Y;\mathbb{Z}/2)) \subseteq H^*_{\mathrm{str}}(X;\mathbb{Z}/2).$$

For the proofs, the reader can refer to [23].

By a *multiblowup* of a real algebraic variety X we mean a regular map $\pi\colon X' \to X$ which is the composition of a finite collection of blowups with nonsingular centers. If C is a Zariski closed subvariety of X and the restriction $\pi_C\colon X' \setminus \pi^{-1}(C) \to X \setminus C$ of π is a biregular isomorphism, then we say that the multiblowup π is *over* C.

A *filtration* of X is a finite sequence $\mathcal{F} = (X_{-1}, X_0, \dots, X_m)$ of Zariski closed subvarieties satisfying

$$\varnothing = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_m = X.$$

The following result will be frequently referred to.

Theorem 2.1 ([23, Theorem 5.4])**.** *Let X be a compact real algebraic variety. For a topological $\mathbb{F}$-vector bundle ξ on X, the following conditions are equivalent:*

(a) *The $\mathbb{F}$-vector bundle ξ admits a stratified-algebraic structure.*
(b) *There exists a filtration $\mathcal{F} = (X_{-1}, X_0, \dots, X_m)$ of X, and for each $i = 0, \dots, m$, there exists a multiblowup $\pi_i\colon X'_i \to X_i$ over X_{i-1} such that the pullback $\mathbb{F}$-vector bundle $\pi_i^*(\xi|_{X_i})$ on X'_i admits a stratified-algebraic structure.*

In applications, Theorem 2.1 will often be combined with the following observation.

Remark 2.2. Any real algebraic variety X has a filtration $\mathcal{F} = (X_{-1}, X_0, \dots, X_m)$ such that $\dim X_{i-1} < \dim X_i$ and $X_i \setminus X_{i-1}$ is a nonsingular variety of pure dimension for $0 \leq i \leq m$. Furthermore, according to Hironaka's theorem on resolution of singularities [17] (cf. also [19] for a very readable exposition), for each $i = 0, \dots, m$, there exists a multiblowup $\pi_i\colon X'_i \to X_i$ over X_{i-1} with X'_i nonsingular.

We are ready to prove the first result announced in Section 1.

Proof of Theorem 1.1. In view of the functoriality of $H^*_{\mathrm{str}}(-;\mathbb{Z}/2)$, Theorem 2.1 and Remark 2.2, we may assume without loss of generality that the variety X is nonsingular. Then

$$H^k_{\mathrm{str}}(X;\mathbb{Z}/2) = H^k_{\mathrm{alg}}(X;\mathbb{Z}/2)$$

for every nonnegative integer k, cf. [23, Proposition 7.7]. Now it suffices to make use of [8, Theorem 1.6]. Indeed, since $\dim X \leq 3$ and $w_k(\xi)$ is in $H^k_{\mathrm{alg}}(X;\mathbb{Z}/2)$ for $k = 1, 2$, it follows that the $\mathbb{R}$-vector bundle ξ admits an algebraic structure. We obtained this stronger conclusion, but only for X nonsingular. □

As usual, the kth Chern class of a $\mathbb{C}$-vector bundle η will be denoted by $c_k(\eta)$. In [23], we defined for any real algebraic variety X a subgroup $H^{2k}_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$ of the cohomology group $H^{2k}(X;\mathbb{Z})$. Here we only need the group $H^2_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$, which consists of all cohomology classes in $H^2(X;\mathbb{Z})$ of the form $c_1(\lambda)$ for some stratified-algebraic $\mathbb{C}$-line bundle λ on X. Thus a topological $\mathbb{C}$-line bundle μ on X admits a stratified-algebraic structure if and only if $c_1(\mu)$ is in $H^2_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. Furthermore, if ξ is a stratified-algebraic $\mathbb{C}$-vector bundle on X, then $c_1(\xi)$ is in $H^2_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. The groups $H^2_{\mathbb{C}\text{-str}}(-;\mathbb{Z})$ have the following functorial property: If $f\colon X \to Y$ is a stratified-regular map between real algebraic varieties, then

$$f^*(H^2_{\mathbb{C}\text{-str}}(Y;\mathbb{Z})) \subseteq H^2_{\mathbb{C}\text{-str}}(X;\mathbb{Z}).$$

The proofs of these facts are contained in [23].

Theorem 2.3. *Let X be a compact real algebraic variety and let ξ be a topological $\mathbb{C}$-vector bundle on X. Then ξ admits a stratified-algebraic structure, provided that one of the following two conditions is satisfied:*

(i) $\dim X \leq 4$ *and* $c_1(\xi)$ *is in* $H^2_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$*;*
(ii) $\dim X = 5$, $c_1(\xi)$ *is in* $H^2_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$ *and* $w_4(\xi_{\mathbb{R}})$ *is in* $H^4_{\text{str}}(X;\mathbb{Z}/2)$*.*

Proof. In view of the functoriality of $H^2_{\mathbb{C}\text{-str}}(-;\mathbb{Z}/2)$ and $H^*_{\text{str}}(-;\mathbb{Z}/2)$, Theorem 2.1 and Remark 2.2, we may assume without loss of generality that the variety X is nonsingular.

Suppose that $\dim X \leq 5$, $\operatorname{rank}\xi \geq 1$ and $c_1(\xi)$ is in $H^2_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. Let λ be a stratified-algebraic $\mathbb{C}$-line bundle on X with

$$c_1(\lambda) = -c_1(\xi).$$

Since $\dim X \leq 5$, we have

$$\xi \oplus \lambda = \eta \oplus \varepsilon,$$

where η and ε are topological $\mathbb{C}$-vector bundles on X, $\operatorname{rank}\eta = 2$ and ε is trivial, cf. [18, p. 99]. According to [23, Corollary 3.14], it suffices to prove that the $\mathbb{C}$-vector bundle η admits a stratified-algebraic structure if either (i) or (ii) is satisfied.

Since the variety X is nonsingular, we may assume that the $\mathbb{C}$-vector bundle η is smooth.

Assertion 1. The $\mathbb{C}$-line bundle $\det\eta$, where $\det\eta$ is the second exterior power of η, is trivial.

It suffices to prove that $c_1(\det\eta) = 0$. The last equality holds since $c_1(\det\eta) = c_1(\eta)$ and

$$c_1(\eta) = c_1(\xi \oplus \lambda) = c_1(\xi) + c_1(\lambda) = 0.$$

The proof of Assertion 1 is complete.

Let $s\colon X \to \eta$ be a smooth section transverse to the zero section. The zero locus

$$Z(s) = \{x \in X \mid s(x) = 0\}$$

of s is a smooth submanifold (possibly empty) of X of codimension 4.

Assertion 2. The smooth submanifold $Z(s)$ is isotopic to a nonsingular Zariski locally closed subvariety of X.

If condition (i) is satisfied, then $Z(s)$ is a finite set, and hence Assertion 2 holds.

Now suppose that condition (ii) is satisfied. Then $Z(s)$ is a smooth curve in X. Since the $\mathbb{R}$-vector bundle $\eta_{\mathbb{R}}$ is orientable, the restriction $\eta_{\mathbb{R}}|_{Z(s)}$ is a trivial vector bundle on $Z(s)$. Consequently, the normal bundle to $Z(s)$ in X is trivial, being isomorphic to $\eta_{\mathbb{R}}|_{Z(s)}$. Suppose that the homology class $[Z(s)]_X$ in $H_1(X;\mathbb{Z}/2)$ represented by $Z(s)$ can also be represented by an algebraic (possibly singular) curve B in X. Moving $Z(s)$ by an isotopy, we may assume that $Z(s) \cap B = \varnothing$. Now Assertion 2 follows from the argument used in the proof of Theorem 1.5 in [21].

The existence of B can be proved as follows. Since

$$c_2(\eta) = c_2(\xi \oplus \lambda) = c_2(\xi) + c_1(\xi) \cup c_1(\lambda) = c_2(\xi) - c_1(\lambda) \cup c_1(\lambda),$$

we get

$$w_4(\eta_{\mathbb{R}}) = w_4(\xi_{\mathbb{R}}) - w_2(\lambda_{\mathbb{R}}) \cup w_2(\lambda_{\mathbb{R}}),$$

cf. [24, p. 171]. Consequently, $w_4(\eta_{\mathbb{R}})$ is in $H^4_{\mathrm{str}}(X; \mathbb{Z}/2)$. Furthermore,

$$H^4_{\mathrm{str}}(X; \mathbb{Z}/2) = H^4_{\mathrm{alg}}(X; \mathbb{Z}/2),$$

the variety X being compact and nonsingular, cf. [23, Proposition 7.7]. In conclusion, the cohomology class $w_4(\eta_{\mathbb{R}})$ is algebraic. On the other hand, the Stiefel–Whitney class $w_4(\eta_{\mathbb{R}})$ is Poincaré dual to the homology class $[Z(s)]_X$ in $H_1(X; \mathbb{Z}/2)$. Hence there exists an algebraic curve B in X satisfying the required condition. The proof of Assertion 2 is complete.

According to [23, Theorem 1.9], Assertions 1 and 2 imply that the $\mathbb{C}$-vector bundle η admits a stratified-algebraic structure.

□

For the convenience of the reader, we recall the following result.

Theorem 2.4 ([23, Theorem 1.7])**.** *Let X be a compact real algebraic variety. A topological $\mathbb{F}$-vector bundle ξ on X admits a stratified-algebraic structure if and only if the $\mathbb{R}$-vector bundle $\xi_{\mathbb{R}}$ admits a stratified-algebraic structure.*

We can now easily derive Theorem 1.2.

Proof of Thorem 1.2. If $\dim X \leq 3$, it suffices to make use of Theorems 1.1 and 2.4.

Since $\dim X \leq d(\mathbb{F}) + 1$, it remains to consider the case $\mathbb{F} = \mathbb{H}$. Denote by $\xi_{\mathbb{C}}$ the $\mathbb{H}$-vector bundle ξ regarded as a $\mathbb{C}$-vector bundle. Then $c_1(\xi_{\mathbb{C}}) = 0$ and the Stiefel–Whitney class $w_4((\xi_{\mathbb{C}})_{\mathbb{R}}) = w_4(\xi_{\mathbb{R}})$ is in $H^4_{\mathrm{str}}(X; \mathbb{Z}/2)$. Hence, according to Theorem 2.3, the $\mathbb{C}$-vector bundle $\xi_{\mathbb{C}}$ admits a stratified-algebraic structure. Consequently, the $\mathbb{R}$-vector bundle $(\xi_{\mathbb{C}})_{\mathbb{R}} = \xi_{\mathbb{R}}$ admits a stratified-algebraic structure. The proof is complete in view of Theorem 2.4. □

For any topological $\mathbb{C}$-vector bundle ξ, let $\overline{\xi}$ denote the conjugate vector bundle, cf. [24]. Recall that $\xi_{\mathbb{R}} = \overline{\xi}_{\mathbb{R}}$ and $c_k(\overline{\xi}) = (-1)^k c_k(\xi)$ for each nonnegative integer k.

Corollary 2.5. *Let X be a compact real algebraic variety of dimension at most 4. For any topological $\mathbb{C}$-vector bundle ξ on X, the $\mathbb{C}$-vector bundle $\xi(2)$ admits a stratified-algebraic structure.*

Proof. Since

$$c_1(\xi \oplus \overline{\xi}) = c_1(\xi) + c_1(\overline{\xi}) = 0,$$

according to Theorem 2.3, the $\mathbb{C}$-vector bundle $\xi \oplus \overline{\xi}$ admits a stratified-algebraic structure. Consequently, the $\mathbb{R}$-vector bundle

$$(\xi \oplus \overline{\xi})_{\mathbb{R}} = \xi_{\mathbb{R}} \oplus \overline{\xi}_{\mathbb{R}} = \xi_{\mathbb{R}} \oplus \xi_{\mathbb{R}} = \xi(2)_{\mathbb{R}}$$

admits a stratified-algebraic structure. The proof is complete in view of Theorem 2.4. □

In a similar way, we obtain

Corollary 2.6. *Let X be a compact real algebraic variety of dimension 5. For any topological $\mathbb{C}$-vector bundle ξ on X, the $\mathbb{C}$-vector bundle $\xi(4)$ admits a stratified-algebraic structure.*

Proof. We have

$$c_1((\xi \oplus \overline{\xi}) \oplus (\xi \oplus \overline{\xi})) = 0,$$
$$c_2((\xi \oplus \overline{\xi}) \oplus (\xi \oplus \overline{\xi})) = 2c_2(\xi \oplus \overline{\xi}).$$

The last equality implies

$$w_4(((\xi \oplus \overline{\xi}) \oplus (\xi \oplus \overline{\xi}))_{\mathbb{R}}) = 0,$$

cf. [24, p. 171]. Hence, according to Theorem 2.3, the $\mathbb{C}$-vector bundle $(\xi \oplus \overline{\xi}) \oplus (\xi \oplus \overline{\xi})$ admits a stratified-algebraic structure. Consequently, the $\mathbb{R}$-vector bundle

$$((\xi \oplus \overline{\xi}) \oplus (\xi \oplus \overline{\xi}))_{\mathbb{R}} = \xi(4)_{\mathbb{R}}$$

admits a stratified-algebraic structure. The proof is complete in view of Theorem 2.4. □

Proof of Theorem 1.3. It suffices to prove that the Chern class $c_1(\lambda^{\otimes 2})$ is in $H^2_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. We have

$$c_1(\lambda^{\otimes 2}) = 2c_1(\lambda) = c_1(\lambda(2)).$$

By Corollary 2.5, the $\mathbb{C}$-vector bundle $\lambda(2)$ admits a stratified-algebraic structure, and hence the Chern class $c_1(\lambda(2))$ is in $H^2_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. □

Proof of Theorem 1.4. It suffices to prove that the Chern class $c_1(\lambda^{\otimes 4})$ is in $H^2_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. We have

$$c_1(\lambda^{\otimes 4}) = 4c_1(\lambda) = c_1(\lambda(4)).$$

By Corollary 2.6, the $\mathbb{C}$-vector bundle $\lambda(4)$ admits a stratified-algebraic structure, and hence the Chern class $c_1(\lambda(4))$ is in $H^2_{\mathbb{C}\text{-str}}(X;\mathbb{Z})$. □

For the proof of Proposition 1.5, we need the following observation.

Lemma 2.7. *Let M be a compact smooth manifold and let ξ be a rank k smooth $\mathbb{R}$-vector bundle on M with $w_k(\xi) = 0$. Let $s\colon M \to \xi$ be a smooth section transverse to the zero section and let*

$$N = \{x \in M \mid s(x) = 0\}$$

be the zero locus of s. Then N is a codimension k smooth submanifold of M (possibly empty) and the unoriented bordism class of the inclusion map $e\colon N \hookrightarrow M$ is zero.

Proof. For any smooth manifold P, we denote by τ_P its tangent bundle and set $w_j(P) = w_j(\tau_P)$ for each nonnegative integer j.

The smooth manifold N is of dimension $n = \dim M - k$. Denote by $[N]$ the fundamental class of N in the homology group $H_n(N;\mathbb{Z}/2)$. According to [15, (17.3)], it suffices to prove that for any nonnegative integer l and any cohomology class v in $H^l(M;\mathbb{Z}/2)$, the equality

$$(\dagger) \qquad \langle w_{i_1}(N) \cup \ldots \cup w_{i_r}(N) \cup e^*(v), [N]\rangle = 0$$

holds for all nonnegative integers $i_1, \ldots, i_r$ satisfying $i_1 + \cdots + i_r = n - l$. This can be done as follows. Since the normal bundle to N in M is isomorphic to the pullback $e^*\xi$ (recall that s is transverse to the zero section), we have

$$\tau_N \oplus e^*\xi \cong e^*\tau_M.$$

It follows that for each nonnegative integer i, the Stiefel–Whitney class $w_i(N)$ belongs to the image of the homomorphism

$$e^*\colon H^i(M;\mathbb{Z}/2) \to H^i(N;\mathbb{Z}/2),$$

cf. [24, p. 10]. Consequently,

$$w_{i_1}(N) \cup \ldots \cup w_{i_r}(N) \cup e^*(v) = e^*(u)$$

for some cohomology class u in $H^n(M;\mathbb{Z}/2)$. Since

$$\langle e^*(u),[N]\rangle = \langle u, e_*([N])\rangle,$$

equality (†) holds if $e_*([N])=0$ in $H_n(M;\mathbb{Z}/2)$. The cohomology class $w_k(\xi)$ is Poincaré dual to the homology class $e_*([N])$. By assumption, $w_k(\xi)=0$, and hence $e_*([N])=0$. □

Proof of Proposition 1.5. Suppose that Conjecture B(2) holds. In view of Theorem 2.1, we may assume without loss of generality that the variety X is nonsingular. Therefore we may also assume that the $\mathbb{C}$-line bundle λ is smooth. Let $s\colon X\to\lambda^{\otimes 2}$ be a smooth section transverse to the zero section and let $Z(s)$ be the zero locus of s. According to [23, Theorem 1.8], it suffices to prove that the smooth submanifold $Z(s)$ is isotopic to a nonsingular Zariski locally closed subvariety of X. Since $c_1(\lambda^{\otimes 2})=2c_1(\lambda)$, we have

$$w_2((\lambda^{\otimes 2})_{\mathbb{R}})=0,$$

cf. [24, p. 171]. By Lemma 2.7, the submanifold $Z(s)$ has the required property. □

Proof of Theorem 1.6. If $\dim X\le d(\mathbb{F})$, then the $\mathbb{F}$-vector bundle $\xi=\xi(1)$ admits a stratified-algebraic structure, cf. [23, Corollary 3.6]. Henceforth, we assume that

$$d(\mathbb{F})+1\le \dim X\le 5.$$

The rest of the proof is divided into three steps.

Case 1. Suppose that $\mathbb{F}=\mathbb{C}$.

Since $3\le\dim X\le 5$, Case 1 follows from Corollaries 2.5 and 2.6.

Case 2. Suppose that $\mathbb{F}=\mathbb{R}$.

For any nonnegative integer a, the $\mathbb{R}$-vector bundles $\xi(2^{a+1})$ and $((\mathbb{C}\oplus\xi)(2^a))_{\mathbb{R}}$ are isomorphic. Since $2\le\dim X\le 5$, it suffices to make use of Case 1.

Case 3. Suppose that $\mathbb{F}=\mathbb{H}$.

Now $\dim X=5$. Denote by $\xi_{\mathbb{C}}$ the $\mathbb{H}$-vector bundle ξ regarded as a $\mathbb{C}$-vector bundle. Since $c_1(\xi_{\mathbb{C}})=0$, we get

$$c_1(\xi_{\mathbb{C}}\oplus\xi_{\mathbb{C}})=0 \text{ and } c_2(\xi_{\mathbb{C}}\oplus\xi_{\mathbb{C}})=2c_2(\xi_{\mathbb{C}}).$$

The last equality implies

$$w_4((\xi_{\mathbb{C}}\oplus\xi_{\mathbb{C}})_{\mathbb{R}})=0,$$

cf. [24, p. 171]. According to Theorem 2.3, the $\mathbb{C}$-vector bundle $\xi_{\mathbb{C}}\oplus\xi_{\mathbb{C}}$ admits a stratified-algebraic structure. Consequently, the $\mathbb{R}$-vector bundle $(\xi_{\mathbb{C}}\oplus\xi_{\mathbb{C}})_{\mathbb{R}}=\xi(2)_{\mathbb{R}}$ admits a stratified-algebraic structure. In view of Theorem 2.4, the $\mathbb{H}$-vector bundle $\xi(2)$ admits a stratified-algebraic structure. The proof of Case 3 is complete. □

We give one more result related to Conjectures A and C.

Theorem 2.8. *Let X be a compact real algebraic variety of dimension at most* 5. *Let ξ be a topological $\mathbb{F}$-vector bundle on X such that the Stiefel–Whitney class $w_k(\xi_{\mathbb{R}})$ is in $H^k_{\mathrm{str}}(X;\mathbb{Z}/2)$ for $k=1,2$. Then the $\mathbb{F}$-vector bundle $\xi(2)$ admits a stratified-algebraic structure.*

Proof. In view of Theorem 2.4, we may assume that $\mathbb{F}=\mathbb{R}$, and hence $\xi=\xi_{\mathbb{R}}$. Since $w_1(\xi)$ is in $H^1_{\mathrm{str}}(X;\mathbb{Z}/2)$, there exists a stratified-algebraic $\mathbb{R}$-line bundle λ on X with $w_1(\lambda)=w_1(\xi)$. Consider the $\mathbb{R}$-vector bundle

$$\eta=\xi\oplus\lambda.$$

Since the $\mathbb{R}$-vector bundles $\eta(2)$ and $\xi(2)\oplus\lambda(2)$ are isomorphic, in view of [23, Corollary 3.14], it suffices to prove that $\eta(2)$ admits a stratified-algebraic structure. Actually, $\eta(2)=(\mathbb{C}\otimes\eta)_{\mathbb{R}}$ and hence it remains to show that the $\mathbb{C}$-vector bundle $\mathbb{C}\otimes\eta$ admits a stratified-algebraic structure. This can be done as follows. We have

$$w_1(\eta) = w_1(\eta) + w_1(\eta) = 0,$$

which implies the equality

$$(*) \qquad c_1(\mathbb{C}\otimes\eta) = 0,$$

cf. [24, p. 182]. Furthermore,

$$w_2(\eta) = w_2(\xi) + w_1(\xi)\cup w_1(\lambda) = w_2(\xi) + w_1(\xi)\cup w_1(\xi).$$

Since $H^*_{\mathrm{str}}(X;\mathbb{Z}/2)$ is a ring, the Stiefel–Whitney class

$$(**) \qquad w_4((\mathbb{C}\otimes\eta)_{\mathbb{R}}) = w_4(\eta\oplus\eta) = w_2(\eta)\cup w_2(\eta)$$

is in $H^4_{\mathrm{str}}(X;\mathbb{Z}/2)$. The $\mathbb{C}$-vector bundle $\mathbb{C}\otimes\eta$ admits a stratified-algebraic structure in view of $(*)$, $(**)$ and Theorem 2.3. □

For any positive integer m, let $\mathbb{S}^m$ and $\mathbb{P}^m(\mathbb{R})$ denote the unit m-sphere and real projective m-space, respectively.

Proof of Theorem 1.7. If $n \leq d(\mathbb{F})$, then for any n-dimensional real algebraic variety X, each topological $\mathbb{F}$-vector bundle on X admits a stratified-algebraic structure, cf. [23, Corollary 3.6]. The proof is complete in this case since $a(n,\mathbb{F}) = 0$. Henceforth, we assume that $d(\mathbb{F})+1 \leq n \leq 5$. The rest of the proof is divided into two steps.

Step 1. First we consider $\mathbb{F} = \mathbb{R}$, in which case $2 \leq n \leq 5$.

For any positive integer k, let N^k be the disjoint union of two copies of $\mathbb{P}^k(\mathbb{R})$, that is,

$$N^k = N^k_0 \cup N^k_1,$$

where $N^k_0 = \{0\}\times\mathbb{P}^k(\mathbb{R})$ and $N^k_1 = \{1\}\times\mathbb{P}^k(\mathbb{R})$. Let μ_k be the $\mathbb{R}$-line bundle on N^k whose restriction to N^k_0 corresponds to the tautological $\mathbb{R}$-line bundle on $\mathbb{P}^k(\mathbb{R})$ and whose restriction to N^k_1 is the standard trivial $\mathbb{R}$-line bundle. By construction,

$$(\mathrm{c}_1) \qquad \langle w_1(\mu_k)^k, [N^k]\rangle \neq 0,$$

where $[N^k]$ is the fundamental class of N^k in $H_k(N^k;\mathbb{Z}/2)$.

We define a smooth manifold M^n by

$$M^n = N^{k(n)}\times\mathbb{S}^{d(n)},$$

where

$$(k(n), d(n)) = \begin{cases} (n-1, 1) & \text{if } n = 2,3,5 \\ (2,2) & \text{if } n = 4. \end{cases}$$

Denote by

$$\pi_n\colon M^n \to N^{k(n)}$$

the canonical projection. Let $y_{d(n)}$ be a point in $\mathbb{S}^{d(n)}$ and let α_n be the homology class in $H_{k(n)}(M^n;\mathbb{Z}/2)$ represented by the smooth submanifold

$$K^{k(n)} = N^{k(n)}\times\{y_{d(n)}\}$$

of M^n. Set

$$A(n) = \{u\in H^{k(n)}(M;\mathbb{Z}/2) \mid \langle u, \alpha_n\rangle = 0\}.$$

Since the normal bundle to $K^{k(n)}$ in M^n is trivial and $K^{k(n)}$ is the boundary of a compact smooth manifold with boundary, it follows form [12, Proposition 2.5, Thoerem 2.6] that there exist an irreducible nonsingular real algebraic variety X and a smooth diffeomorphism $\varphi\colon X \to M$ with

$$H^{k(n)}_{\mathrm{alg}}(X;\mathbb{Z}/2) \subseteq \varphi^*(A(n)).$$

Recall that $H^*_{\mathrm{alg}}(X;\mathbb{Z}/2) = H^*_{\mathrm{str}}(X;\mathbb{Z}/2)$, the variety X being compact and nonsingular, cf. [23, Proposition 7.7]. Consequently,

$$H^{k(n)}_{\mathrm{str}}(X;\mathbb{Z}/2) \subseteq \varphi^*(A(n)). \tag{c$_2$}$$

We define a topological $\mathbb{R}$-line bundle λ on X by

$$\lambda = (\pi_n \circ \varphi)^* \mu_{k(n)}.$$

Assertion 3. If $k(n)$ is divisible by a positive integer l, then

$$w_1(\lambda)^l \notin H^l_{\mathrm{str}}(X;\mathbb{Z}/2).$$

Indeed, setting $v = w_1(\mu_{k(n)})$, we get

$$w_1(\lambda) = \varphi^*(\pi_n^*(v))$$

and hence

$$w_1(\lambda)^{k(n)} = \varphi^*(\pi_n^*(v^{k(n)})).$$

Furthermore,

$$\langle \pi_n^*(v^{k(n)}), \alpha_n\rangle = \langle v^{k(n)}, (\pi_n)_*(\alpha_n)\rangle.$$

Since $(\pi_n)_*(\alpha_n) = [N^{k(n)}]$, condition (c$_1$) implies that

$$\langle \pi_n^*(v^{k(n)}), \alpha_n\rangle \neq 0.$$

In other words,

$$\pi_n^*(v^{k(n)}) \notin A(n). \tag{c$_3$}$$

In view of (c$_2$) and (c$_3$), we get

$$w_1(\lambda)^{k(n)} \notin H^{k(n)}_{\mathrm{str}}(X;\mathbb{Z}/2). \tag{c$_4$}$$

If $k(n) = lp$, then

$$w_1(\lambda)^{k(n)} = (w_1(\lambda)^l)^p. \tag{c$_5$}$$

Assertion 3 follows from (c$_4$) and (c$_5$) since $H^*_{\mathrm{str}}(X;\mathbb{Z}/2)$ is a ring.

Assertion 4. If r is a positive integer satisfying $r < 2^{a(n)}$, then the $\mathbb{R}$-vector bundle $\lambda(r)$ does not admit a stratified-algebraic structure.

First note that for any odd positive integer q, we have

$$w_1(\lambda(q)) = q w_1(\lambda) = w_1(\lambda), \tag{c$_6$}$$

$$w_2(\lambda(2q)) = w_2(\lambda(q) \oplus \lambda(q)) = w_1(\lambda(q))^2 = w_1(\lambda)^2, \tag{c$_7$}$$

$$w_4(\lambda(4q)) = w_4(\lambda(2q) \oplus \lambda(2q)) = w_2(\lambda(2q))^2 = w_1(\lambda)^4. \tag{c$_8$}$$

Observe that $k(n) = 2^{i(n)}$, where $i(n)$ is an integer satisfying $0 \le i(n) \le 2$. Furthermore, if $1 \le r \le 2^{a(n)}$, then $r = 2^j q$, where q is an odd positive integer and j is an integer satisfying $0 \le j \le i(n)$. In particular, $k(n)$ is divisible by 2^j. According to (c$_6$), (c$_7$), (c$_8$) and Assertion 3,

$$w_l(\lambda(r)) \notin H^l_{\mathrm{str}}(X;\mathbb{Z}/2) \text{ for } l = 2^j,$$

and hence Assertion 4 follows.

Now let r be an arbitrary integer. In view of Theorem 1.6 and Assertion 4, the $\mathbb{R}$-vector bundle $\lambda(r)$ admits a stratified-algebraic structure if and only if r is divisible by $2^{a(n)} = 2^{a(n,\mathbb{R})}$. This completes the proof of Step 1.

Step 2. General case.

Now $d(\mathbb{F}) + 1 \leq n \leq 5$. In particular,

$$a(n, \mathbb{R}) = a(n, \mathbb{F}) + e(\mathbb{F}). \tag{c$_9$}$$

Furthermore,

$$(\mathbb{F} \otimes \lambda)_{\mathbb{R}} \cong \lambda(d(\mathbb{F})) = \lambda(2^{e(\mathbb{F})}),$$

and hence

$$((\mathbb{F} \otimes \lambda)(r))_{\mathbb{R}} \cong (\mathbb{F} \otimes \lambda)_{\mathbb{R}}(r) \cong \lambda(2^{e(\mathbb{F})})(r) \cong \lambda(2^{e(\mathbb{F})} r).$$

By Step 1, the $\mathbb{R}$-vector bundle $((\mathbb{F} \otimes \lambda)(r))_{\mathbb{R}}$ admits a stratified-algebraic structure if and only if the integer $2^{e(\mathbb{F})} r$ is divisible by $2^{a(n,\mathbb{R})}$. In view of (c_9), the latter condition is equivalent to the divisibility of r by $2^{a(n,\mathbb{F})}$. Finally, according to Theorem 2.4, the $\mathbb{F}$-vector bundle $(\mathbb{F} \otimes \lambda)(r)$ admits a stratified-algebraic structure if and only if r is divisible by $2^{a(n,\mathbb{F})}$. The proof is complete. □

References

[1] Selman Akbulut and Henry King. On approximating submanifolds by algebraic sets and a solution to the nash conjecture. *Invent. Math.*, 107:87–98, 1992. DOI: http://dx.doi.org/10.1007/BF01231882.

[2] Selman Akbulut and Henry King. *Topology of Real Algebraic Sets*, volume 25 of *Math. Sci. Research Institute Publ.,*. Springer, 1992.

[3] Riccardo Benedetti and Alberto Tognoli. On real algebraic vector bundles. *Bull. Sci. Math.*, 104:89–112, 1980.

[4] Riccardo Benedetti and Alberto Tognoli. *Remarks and counterexamples in the theory of real algebraic vector bundles and cycles*, pages 198–211. Number 959 in Lecture Notes in Math. Springer, 1982.

[5] Marcin Bilski, Wojciech Kucharz, Anna Valette, and Guillaume Valette. Vector bundles and regulous maps. *Math Z.*, 275:403–418, 2013. DOI: http://dx.doi.org/10.1007/s00209-012-1141-6.

[6] Jacek Bochnak, Michael Buchner, and Wojciech Kucharz. Vector bundles over real algebraic varieties. *K-Theory*, 3:271–298., 1989. DOI: http://dx.doi.org/10.1007/BF00533373.

[7] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real Algebraic Geometry*, volume 36 of *Ergeb. der Math. und ihrer Grenzgeb. Folge 3*. Springer, 1998.

[8] Jacek Bochnak and Wojciech Kucharz. K-theory of real algebraic surfaces and threefolds. *Math. Proc. Cambridge Philos. Soc.*, 106:471–480, 1989.

[9] Jacek Bochnak and Wojciech Kucharz. Vector bundles on a product of real cubic curves. *K-Theory*, 6:487–497, 1992. DOI: http://dx.doi.org/10.1007/BF00997378.

[10] Jacek Bochnak and Wojciech Kucharz. Elliptic curves and real algebraic morphisms. *J. Algebraic Geometry*, 2:635–666, 1993.

[11] Jacek Bochnak and Wojciech Kucharz. On homology classes represented by real algebraic varieties. In *Singularities Symposium – Łojasiewicz 70*, number vol. 44 in Banach Center Publ., pages 21–35. Polish Acad. Sci., 1998.

[12] Jacek Bochnak and Wojciech Kucharz. Real algebraic morphisms represent few homotopy classes. *Math. Ann.*, 337:909–921, 2007. DOI: 10.1007/BF02698900.

[13] Jacek Bochnak, Wojciech Kucharz, and Robert Silhol. Morphisms, line bundles and moduli spaces in real algebraic geometry. *Inst. Hautes Études Sci. Publ. Math.*, 86:5–65, 1997. DOI: 10.1007/s00208-006-0061-3.

[14] Armand Borel and André Haefliger. La classe d'homologie fondamentale d'un espace analityque. *Bull. Soc. Math. France*, 89:461–513, 1961.

[15] Pierre E. Conner. *Differentiable Periodic Maps.* Number 738 in Lecture Notes in Math. Springer, 2 edition, 1979.

[16] Goulwen Fichou, Johannes Huisman, Frédéric Mangolte, and Jean-Philippe Monnier. Fonctions régulues. J. Reine Angew. Math., to appear.

[17] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. *Ann. of Math.*, 79:109–326, 1964. DOI: 10.2307/1970486.

[18] Dale Husemoller. *Fibre Bundles*. Springer, 1975.

[19] János Kollár. *Lectures on Resolution of Singularities*, volume 166 of *Ann. of Math. Studies*. Princeton University Press, 2007.

[20] János Kollár and Krzysztof Nowak. Continuous rational functions on real and p-adic varieties. *Math. Z.*, 279:85–97, 2015.

[21] Wojciech Kucharz. Rational maps in real algebraic geometry. *Adv. Geom.*, 9:517–539, 2009. DOI: 10.1515/ADVGEOM.2009.024.

[22] Wojciech Kucharz. Approximation by continuous rational maps into spheres. *J. Eur. Math. Soc. (JEMS)*, 16:1555–1569, 2014.

[23] Wojciech Kucharz and Krzysztof Kurdyka. Stratified-algebraic vector bundles, 2013. arXiv: 1308.4376.

[24] John. W. Milnor and James D. Stasheff. *Characteristic Classes*. Number 76 in Ann. of Math. Studies. Princeton University Press, 1974.

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