

SECTIONS, MULTISECTIONS, AND $U(1)$ FIELDS IN F-THEORY

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ABSTRACT. We show that genus-one fibrations lacking a global section fit naturally into the geometric moduli space of Weierstrass models. Elliptic fibrations with multiple sections (nonzero Mordell-Weil rank), which give rise in F-theory to abelian $U(1)$ fields, arise as a subspace of the set of genus-one fibrations with multisections. Higgsing of certain matter multiplets charged under abelian gauge fields in the corresponding supergravity theories break the $U(1)$ gauge symmetry to a discrete gauge symmetry group. We demonstrate these results explicitly in the case of bisections, and describe the general framework for multisections of higher degree. We further show that nearly every $U(1)$ gauge symmetry arising in an F-theory model can be “unHiggsed” to an $SU(2)$ gauge symmetry with adjoint matter, though in certain situations this leads to a model in which a superconformal field theory is coupled to a conventional gauge and gravity theory. The only exceptions are cases in which the attempted unHiggsing leads to a boundary point at an infinite distance from the interior of the moduli space.

1. INTRODUCTION

F-theory [1, 2, 3] is a nonperturbative approach to string theory in which the axiodilaton $\tau = \chi + ie^{-\phi}$ of type IIB supergravity is specified by means of an auxiliary complex torus (elliptic curve), and 7-branes serve as sources for the RR scalar, providing an opportunity for $SL(2, \mathbb{Z})$ -multivaluedness of the τ field. In most work to date, F-theory is compactified on a base B_n of complex dimension n , where the tori $\mathbb{C}/\langle 1, \tau(\xi_1, \dots, \xi_n) \rangle$ parameterized by coordinates ξ_i on the base are assumed to fit together to form a Calabi-Yau $(n+1)$ -fold X_{n+1} that is elliptically fibered with section, $\pi : X_{n+1} \rightarrow B_n$, so that (after appropriate blowing down) X_{n+1} can be described by a Weierstrass model

$$(1) \quad y^2 = x^3 + fx + g,$$

where f, g are sections of line bundles $\mathcal{O}(-4K), \mathcal{O}(-6K)$ on the base B_n (locally described simply as functions of the base coordinates). The 7-branes are located at the *discriminant locus* $\{4f^3 + 27g^2 = 0\}$, in a manner specified by the Kodaira–Néron classification of singular fibers [4, 5].

Recently, Braun and Morrison [6] considered a more general class of F-theory compactification spaces, where the space X_{n+1} has a genus-one (torus) fibration, but no global section. They identified a large number of examples of such genus-one fibrations over the base $B_2 = \mathbb{P}^2$ in the comprehensive list compiled by Kreuzer and Skarke [7] of Calabi-Yau threefolds that are hypersurfaces in toric varieties. Any such X_{n+1} has a *Jacobian fibration* J_{n+1} , which is an elliptically fibered Calabi–Yau with section¹ whose τ function and discriminant locus are identical to those of X_{n+1} . The set of genus-one fibered Calabi–Yau manifolds with the same Jacobian fibration J_{n+1} is known as the Tate-Shafarevich group of J_{n+1} , denoted $\text{III}(J_{n+1})$, and is identified with

¹This statement has been mathematically proven only for $n+1 \leq 3$ [4, 8, 9], but is likely true in arbitrary dimension.

the discrete part of the gauge group of F-theory following [10, 11]². Note that $\coprod(J_{n+1})$ represents not only a disjoint set of manifolds, but also includes an abelian group structure [12, 13]. Braun and Morrison identified in the examples they studied an apparent deficit in the number of scalar hypermultiplets required for gravitational anomaly cancellation, when the massless scalars are identified only as the complex structure moduli of the smooth genus one fibrations without global section. They resolved this apparent problem by identifying additional massless hypermultiplets at nodes in the discriminant locus of the Jacobian fibrations (more specifically, in the I_1 part of that locus).

While this analysis supports the proposition that genus-one fibrations without a global section are associated with consistent F-theory backgrounds, it also raises several questions, such as whether these backgrounds are connected to other F-theory geometries or form a disjoint component of the moduli space of the theory, and how the additional massless hypermultiplets should be interpreted. In this note, we show how these genus-one fibrations and their Jacobians fit naturally into the connected moduli space of Weierstrass models, and relate them to models with $U(1)$ gauge fields arising from extra sections of the elliptic fibrations. The structure of $U(1)$ gauge fields in F-theory is rather subtle, as they are determined by global features (the Mordell-Weil group) of an elliptic fibration; F-theory models with one or more $U(1)$ fields have been the subject of significant recent research activity (see for example [14, 15, 16, 18, 19, 20, 17, 22, 21, 23]).

In rough outline, the framework developed in this note is as follows: over any complex n -dimensional base B_n , there is a space \mathcal{W} of Weierstrass models, parameterized by the sections f, g in (1). Any Calabi-Yau $(n + 1)$ -fold with a genus-one fibration has a multisection of some degree k , and its associated Jacobian fibration has a Weierstrass model which is generally singular when $k > 1$ (even in the absence of nonabelian gauge symmetry). We can map the set \mathcal{M}_k of genus-one fibrations with a k -fold multisection (a “ k -section”, or when $k = 2$, a “bisection”) to a subset $\mathcal{J}^k \subseteq \mathcal{W}$ of the set of Weierstrass models, consisting of the Jacobians of those genus-one fibrations. The set of elliptic fibrations with k independent global sections (rank $r = k - 1$ Mordell-Weil group) can also be viewed through singular Weierstrass models as a subset $\mathcal{S}_k \subseteq \mathcal{W}$ of the full space of elliptic fibrations. For Calabi-Yau threefolds, these results follow for any k from the result of Nakayama [24] and Grassi [25] that any elliptically fibered Calabi-Yau threefold with section has a realization as a Weierstrass model that is also Calabi-Yau; as in the case of the statements mentioned earlier concerning Jacobian fibrations of Calabi-Yau genus-one fibrations, this statement has not been mathematically proven for Calabi-Yau fourfolds, but there are no known examples to the contrary. Furthermore, we have

$$(2) \quad \mathcal{S}_k \subseteq \mathcal{J}^k \subseteq \mathcal{W},$$

meaning that the set of models with k independent sections can be viewed as a subset of the larger set of models with a k -fold multisection.

We give explicit formulae describing these inclusions in the case $k = 2$ in the next section, but the inclusion $\mathcal{S}_k \subseteq \mathcal{J}^k$ clearly holds for any k since having k independent sections is a special case of having a k -fold multisection where the k sections can be given distinct global labels. In particular, we can think of the multisection of an $(n + 1)$ -fold $X_{n+1} \in \mathcal{J}^k$ as a branched cover of the base; the multisection breaks into k distinct global sections on a subspace of moduli space where the branch points coalesce in such a way as to give trivial monodromy among the branches. In this picture, going from a model in \mathcal{S}_k to a model in \mathcal{J}^k can be interpreted physically as a partial Higgsing, where Higgsing some charged matter fields breaks $U(1)^{k-1}$ to a discrete subgroup, under which the remaining fields parameterizing \mathcal{J}^k carry discrete charges. In the

²The discrete part of a gauge group corresponds to the set of connected components of the group; a purely discrete gauge group is a finite group such as \mathbb{Z}_n

case $k = 2$, for example, we can have matter fields with various integer-valued $U(1)$ charges; if we Higgs matter fields with charge Q , we break $U(1)$ to \mathbb{Z}_Q .

In the case $k = 2$, we can also further analyze any model containing a $U(1)$ by considering the explicit form of a Weierstrass model with nonzero Mordell-Weil rank. From this point of view we can demonstrate that every $U(1)$ is associated geometrically with a nonabelian $SU(2)$ (or larger) symmetry arising from a Kodaira type I_2 singularity along a divisor on the base. Starting with such an $SU(2)$ having both adjoint and fundamental matter, there are several possible Higgsing steps: the first leaves us with a $U(1)$ under which the remnant of the adjoint matter has charge³ 2 and the remnant of the fundamental matter has charge 1; the second Higgsing (of matter fields of charge 2) leaves us with gauge group \mathbb{Z}_2 under which the remnant of the original fundamental matter is charged; a final Higgsing of the fields originally carrying charge 1 breaks the residual discrete gauge group and moves the model out of \mathcal{J}^k and into the moduli space \mathcal{W} of generic Weierstrass models.

In §2 we describe the general framework for this geometrical picture explicitly in the case $k = 2$, for a general base manifold B_n . In §3, we show explicitly in 6D how any $U(1)$ gauge field in an F-theory model can be associated with an $SU(2)$ gauge group that has been Higgsed by an adjoint matter field, and we look at several explicit examples. §4 contains some general remarks about the implications of this picture for 6D and 4D supergravity theories, and some comments on further directions for related research.

2. GENERAL FRAMEWORK

2.1. Calabi-Yau manifolds with bisections and with two different sections. In [6], an exercise in Galois theory provides an equation for the Jacobian of a genus-one fibration with a bisection

$$(3) \quad y^2 = x^3 - e_2 x^2 z^2 + (e_1 e_3 - 4e_0 e_4) x z^4 - (e_1^2 e_4 + e_0 e_3^2 - 4e_0 e_2 e_4) z^6,$$

where e_0, \dots, e_4 are sections of various line bundles over the base B_n (to be determined below). Completing the cube, changing variables, and setting $z = 1$ puts this in Weierstrass form

$$(4) \quad y^2 = x^3 + (e_1 e_3 - \frac{1}{3} e_2^2 - 4e_0 e_4) x + (-e_0 e_3^2 + \frac{1}{3} e_1 e_2 e_3 - \frac{2}{27} e_2^3 + \frac{8}{3} e_0 e_2 e_4 - e_1^2 e_4).$$

This parameterizes the set of all Jacobians of genus-one fibrations over B_n with bisections, represented through the Weierstrass models (of the Jacobian fibrations). In particular, this describes how $\mathcal{J}^2 \subseteq \mathcal{W}$ for any base B_n .

This class of Weierstrass models is closely related to the Weierstrass form for elliptically fibered Calabi-Yau $(n+1)$ -folds on B_n with two (different) sections. Elliptically fibered Calabi-Yau manifolds with two sections can be described as models with a non-Weierstrass presentation (like the E_7 models of [26, 27]) that are smooth for generic moduli. All such $(n+1)$ -folds, however, also have a (possibly singular) description as Weierstrass models. In [16], the general form of such a Weierstrass model was given as⁴

$$(5) \quad y^2 = x^3 + (e_1 e_3 - \frac{1}{3} e_2^2 - b^2 e_0) x + (-e_0 e_3^2 + \frac{1}{3} e_1 e_2 e_3 - \frac{2}{27} e_2^3 + \frac{2}{3} b^2 e_0 e_2 - \frac{1}{4} b^2 e_1^2).$$

³A field is said to have “charge n ” under a $U(1)$ gauge symmetry if it transforms as $e^{in\theta}$ under a gauge transformation $e^{i\theta} \in U(1)$.

⁴We have modified eq. (5.35) of [16] by using a scaling $(f, g) \mapsto (i^4 f, i^6 g)$ to change the sign of g , and by changing e_j in that paper to e_j here ($j = 0, 1, 2, 3$).

Note that this equation is equivalent to (4) under the replacement $b^2 \rightarrow 4e_4$. The interpretation of this analysis is that, as stated in the introduction,

$$(6) \quad \mathcal{S}_2 \subseteq \mathcal{J}^2 \subseteq \mathcal{W},$$

The condition $e_4 = b^2/4$ is precisely the condition that the branching loci of the bisection associated with a genus-one fibration in \mathcal{J}^k coalesce in pairs so that the total structure is that of an elliptic fibration with two sections.

In [6, 16], it was shown that for both an elliptic fibration with two sections, and for a genus-one fibration with a bisection, there is a natural model with a quartic equation of the general form

$$(7) \quad w^2 = e_0u^4 + e_1u^3v + e_2u^2v^2 + e_3uv^3 + e_4v^4.$$

If $e_4 = b^2/4$, the equation can be rewritten

$$(8) \quad \left(w + \frac{1}{2}bv^2\right)\left(w - \frac{1}{2}bv^2\right) = u(e_0u^3 + e_1u^2v + e_2uv^2 + e_3v^3),$$

which makes the two sections manifest: they are given by $u = w \pm \frac{1}{2}bv^2 = 0$. In general, when there are two sections one might need to make a linear redefinition of the variables u, v before (7) can be rewritten in the form (8), but after such a linear redefinition it can always be done.

From the condition that f, g in (1) are sections of the line bundles associated with $-4K, -6K$, we can characterize the line bundles of which the e_i and b are sections. Focusing on the e_i 's, we have

$$(9) \quad -4K = 2[e_2] = [e_1] + [e_3] = [e_0] + [e_4],$$

$$(10) \quad -6K = 2[e_1] + [e_4] = [e_0] + 2[e_3].$$

From $2[e_2] = -4K$, we have $[e_2] = -2K$. We also note that $[e_0] = -6K - 2[e_3]$ must be an even divisor class. Choosing $[e_0] \equiv 2L$, with L the class of an arbitrary line bundle, we have

$$(11) \quad [e_0] = 2L$$

$$(12) \quad [e_1] = -K + L$$

$$(13) \quad [e_2] = -2K$$

$$(14) \quad [e_3] = -3K - L$$

$$(15) \quad [e_4] = -4K - 2L$$

$$(16) \quad [b] = -2K - L.$$

For any given base, L can be chosen subject to the conditions that $[e_1], [e_3]$ are effective divisors (if this condition is not satisfied, then the only non-vanishing terms in the Weierstrass model are those proportional to powers of e_2 , and the discriminant vanishes identically). This constrains the range of possibilities to a finite set of possible strata in the moduli space. The consequences when $[e_4]$ and/or $[e_0]$ fail to be effective are discussed in §2.4.

This analysis shows that for any Calabi-Yau manifold X_{n+1} that is a genus-one fibration lacking a global section but having a bisection, there is a Jacobian fibration J_{n+1} , which has a description as a Weierstrass model through (4). Taking the limit $e_4 \rightarrow b^2/4$ gives a Weierstrass model for an elliptically fibered Calabi-Yau $(n+1)$ -fold with two sections, which therefore has a Mordell-Weil group of nonzero rank. In terms of the physical language of F-theory, as we describe in more detail in the following sections, this corresponds to the reverse of a process in which a $U(1)$ gauge symmetry is broken by matter fields of charge 2, leaving a discrete \mathbb{Z}_2 symmetry. In §3 we describe several explicit examples of this setup in 6D F-theory constructions.

2.2. Singular fibers of type I_2 in codimension two. One of the key features of the quartic models is the presence of singular fibers in codimension two of Kodaira type I_2 , observed in [16] in the $U(1)$ case, and in [6] in the bisection case. When there is a $U(1)$, these singular fibers determine matter hypermultiplets that are charged under the $U(1)$, and there can be different charges: [16] focussed on the case when the charges are 1 and 2 only and found distinct geometrical interpretations for each of these. The geometric construction of I_2 fibers of charge 1 under $U(1)$ extends to the case of a bisection (in the deformation from \mathcal{S}_2 to \mathcal{J}^2), as we will now show explicitly. As explained above, the corresponding matter fields will be charged under the discrete \mathbb{Z}_2 gauge symmetry. Both the bisection and $U(1)$ cases have a description in terms of the quartic model (7). We begin by considering the I_2 fibers in the genus one (bisection) case where e_4 is generic, and then consider the limit where $e_4 = b^2$ is a perfect square, corresponding to the $U(1)$ model.

The curves of genus one in the quartic model are double covers of \mathbb{P}^1 branched in 4 points, as illustrated in the left half of Figure 1. When the 4 branch points come together in pairs, the resulting double cover splits into two curves of genus zero meeting in those two double branch points, as illustrated in the right half of Figure 1. Such fibers in the family have type I_2 in the Kodaira classification.

Thus, to find such a fiber of type I_2 in the quartic model, we seek points on the base B_n for which the right-hand side of the equation (7) takes the form of a perfect square. As we explain in appendix A, we can assume that e_4 does not vanish at such points on the base (if the model is sufficiently generic) and so we write our condition in the form

$$(17) \quad e_0u^4 + e_1u^3v + e_2u^2v^2 + e_3uv^3 + e_4v^4 = e_4(\alpha u^2 + \beta uv + v^2)^2,$$

for some unknown α and β . Multiplying out and equating coefficients, it is easy to solve

$$\beta = e_3/2e_4, \quad \alpha = (4e_2e_4 - e_3^2)/8e_4^2$$

and then determine the remaining conditions, which are:

$$(18) \quad e_3^4 - 8e_2e_3^2e_4 + 16e_2^2e_4^2 - 64e_0e_4^3 = 0$$

$$(19) \quad e_3^3 - 4e_2e_3e_4 + 8e_1e_4^2 = 0$$

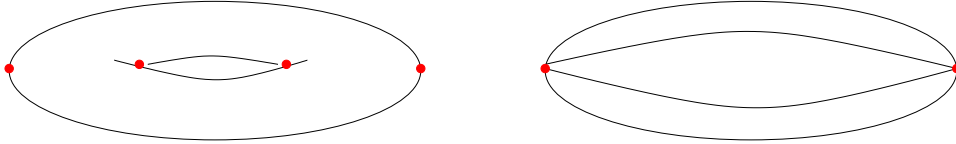


FIGURE 1. Fiber of type I_2 as a degenerate branched cover

To study the solutions of these equations, we introduce an auxiliary variable p and rewrite the equations as

$$(20) \quad p^4 - 8e_2e_4p^2 + 16e_2^2e_4^2 - 64e_0e_4^3 = 0$$

$$(21) \quad p^3 - 4e_2e_4p + 8e_1e_4^2 = 0$$

In appendix A we explain how to determine the condition

$$(22) \quad (4e_0e_2 - e_1^2)^2 = 64e_0^3e_4$$

for these equations to have a common root, and why that root is

$$(23) \quad p = \frac{8e_0e_1e_4}{4e_0e_2 - e_1^2}$$

when all of the coefficient functions e_0, \dots, e_4 are generic. The points we seek can be described as solutions to (22) which also satisfy $e_3 = p$.

We now show how to count the solutions (i.e., the number of I_2 fibers of this type), modifying an argument from [16]. Let us take a limit, replacing e_4 with $\epsilon^2 e_4$ and then taking ϵ very small (with both e_0 and e_1 of order 1). Condition (22) then shows that $4e_0e_2 - e_1^2$ has order ϵ , and (23) shows that p has order $\epsilon^2/\epsilon = \epsilon$. It follows that any simultaneous solution to (22) and $e_3 = p$ can be deformed to a simultaneous solution to (22) and $e_3 = 0$. That is, the isolated I_2 fibers are in one-to-one correspondence with the set

$$(24) \quad \{e_1^4 - 8e_0e_1^2e_2 + 16e_0^2e_2^2 - 64e_0^3e_4 = 0\} \cap \{e_3 = 0\}.$$

It follows that the number of I_2 fibers is

$$(25) \quad [4e_1] \cdot [e_3] = 4(-K + L) \cdot (-3K - L),$$

since $e_1^4 - 8e_0e_1^2e_2 + 16e_0^2e_2^2 - 64e_0^3e_4$ is in class $[4e_1]$.

When $e_4 = b^2/4$ so that we have a $U(1)$, the analysis above reproduces the count of I_2 fibers found in [16] which correspond to matter of charge 1 under the $U(1)$ gauge group. It was also observed there (and will be mentioned again below) that when $U(1)$ is further enhanced to $SU(2)$, this matter comes from matter in the fundamental representation of $SU(2)$.

On the other hand, the description of the matter of charge 2 in [16] is a bit different: it occurs where b and e_3 both vanish, and from (8) and (3) we see that both the quartic model and the Jacobian fibration have conifold singularities over each common zero of b and e_3 . When we partially Higgs by relaxing the condition $e_4 = b^2/4$, we do a complex structure deformation of that conifold singularity, giving a mass to the gauge field (as is standard in a conifold transition [28]).⁵ It would be interesting to find a more geometric interpretation of this massive gauge field, perhaps along the lines of [34, 35].⁶

The Weierstrass model of the Jacobian fibration also has a conifold singularity corresponding to each I_2 . For models with two sections, these conifold singularities have a (simultaneous) small resolution, as shown explicitly in [16] by blowing up the second section in the Weierstrass model. However, for Jacobians of models with a bisection, the conifold singularities (i.e., the deformations of those singularities whose corresponding hypermultiplet had charge 1 before Higgsing) have no Calabi–Yau resolution, which led to the question raised in [6] of whether these are genuinely new F-theory models.

2.3. Generalizations and geometry. In principle, our explicit analysis of bisections could be extended to the spaces \mathcal{J}^k of Jacobian fibrations associated with genus-one fibered Calabi-Yau manifolds with k -sections and \mathcal{S}_k of elliptically fibered Calabi-Yau manifolds with rank $r = k - 1$ Mordell-Weil group in a similar explicit fashion, at least for $k \leq 4$. Explicit formulae for $\mathcal{S}_3, \mathcal{S}_4$, the generic forms of elliptic fibrations with three and four sections respectively, were worked out in [20, 21] and [23], and the analogous formulae for \mathcal{J}^k are known [37] (although unwieldy to manipulate). For $k = 3, 4$, the points in \mathcal{S}_k correspond to singular Weierstrass presentations of Calabi-Yau $(n + 1)$ -folds with 3, 4 independent sections, which have smooth descriptions similar to the E_6 and D_4 fibrations of [26, 27].

⁵The distinction between conifold singularities which admit a Kähler small resolution and those which do not, and the relation to massive gauge fields, has appeared a number of times in the literature [29, 30, 31, 32, 33].

⁶We thank Volker Braun for emphasizing the crucial role which must be played by massive gauge fields in these models [36].

Even without an explicit description of the general form of a Jacobian fibration with a k -section, it is clear that the framework described in the previous section should generalize. In particular, we expect that any Jacobian fibration J_{n+1} with a multisection will have a discrete gauge group Γ in the corresponding F-theory picture, and that this will match the Tate-Shafarevich group $\Gamma = \text{III}(J_{n+1})$. There is a simple and natural geometric interpretation of this structure in the M-theory picture. When an F-theory model on J_{n+1} is compactified on a circle S^1 , it gives a 5D supergravity theory that can also be described by a compactification of M-theory on a Calabi-Yau $(n+1)$ -fold Y_{n+1} . When there is a discrete gauge group Γ in the 6D F-theory model, a nontrivial gauge transformation (Wilson line) around the complex direction gives a set of $|\Gamma|$ distinct 5D vacua associated with J_{n+1} . In the M-theory picture this corresponds precisely to the compactification on the set of distinct genus-one fibered Calabi-Yau manifolds in the Tate-Shafarevich group $\text{III}(J_{n+1})$.

We can get a clear picture of the meaning of the multiple Calabi-Yau manifolds with the same Jacobian fibration by considering the moduli space for the compactified theory on a circle, which can be analyzed using M-theory. We illustrate this in Figure 2, in which the moduli space \mathcal{W} of Weierstrass models (shown in blue) contains the subset \mathcal{J}^2 of Jacobians of models with a bisection, and this in turn contains the subset \mathcal{S}_2 of Jacobians of models with two sections. When there are two sections, the second Betti number of the Calabi-Yau increases and there is an additional dimension in the Kähler moduli space, which becomes a modulus in the compactified theory. We have illustrated this extra dimension as a red line emerging from the \mathcal{S}_2 .

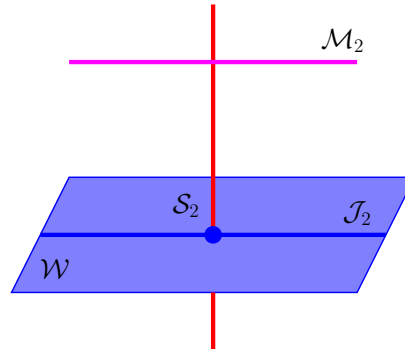


FIGURE 2. Moduli spaces for M-theory compactifications on Calabi-Yau threefolds with different structures of sections (described in text).

What initially seems puzzling is that while the Weierstrass models of Jacobians of genus one fibrations with two sections deform seamlessly to Jacobians of genus one fibrations with bisections (by relaxing the condition that e_4 be a square), and similarly the nonsingular fibrations with two sections deform seamless to genus one fibrations with a bisection, the conifold singularities in the Weierstrass model cannot be resolved in the bisection case. The key to understanding this is to remember that the extra divisor that is present when there are two sections (i.e., a $U(1)$) allows an additional Kähler degree of freedom which in particular allows us to specify the areas of the two components of an I_2 fiber independently. On the other hand, when there is only a bisection, the homology classes of those two components must each be one-half of the homology class of a smooth genus-one fiber; thus, the two components must have the same *area*.

The picture of the M-theory moduli space is thus completed by adding a new component \mathcal{M}_2 of smooth genus-one fibrations with a bisection, illustrated in purple in Figure 2. The new

component must emerge from the precise value of the additional Kähler classes (red line) at which the two components of an I_2 fiber in the $U(1)$ case have an identical area. (Generally, the red line can be viewed as parameterizing the difference of those areas.) The additional Kähler class in a $U(1)$ thus provides the connection between the Weierstrass models (in which the area of one of the two components is zero, corresponding to the conifold point without a Calabi-Yau resolution) and the bisection models (in which the areas of the two components are equal).

Let us reiterate the crucial point: away from the locus \mathcal{S}_2 , the complex structures on the Calabi-Yau manifolds represented by the spaces \mathcal{J}^2 and \mathcal{M}_2 are different (and not even birational to each other), and are only related by the “Jacobian fibration” construction. However, they determine the same underlying τ function, so the F-theory models are identical. Compactifying on a circle produces two distinct geometries for M-theory models, which is precisely what one expects for a discrete gauge symmetry. Moreover, the “extra” hypermultiplets have different but consistent explanations on the two components of the M-theory moduli space. Along \mathcal{M}_2 , they are seen as geometric I_2 fibers being wrapped by M2-branes, which were argued to have no continuous gauge charges in [6] (although we now see that they carry \mathbb{Z}_2 gauge charges). Along \mathcal{J}^2 , these same hypermultiplets are seen as complex structure moduli transverse to the \mathcal{J}^2 locus (moduli which are absent in \mathcal{M}_2).

One of the important lessons that we learn from this picture is that it is important not to discard an F-theory model just because all of the corresponding M-theory models after S^1 -compactification are singular. The lack of a nonsingular model means that the M-theory compactification cannot be studied in the supergravity approximation without some additional input to its structure, but such models must be included for a consistent overall picture of the moduli spaces.

2.4. Enhancement to $SU(2)$. For any elliptically fibered threefold with nonzero Mordell-Weil rank, we can carry the analysis of §2.1 further, and show that there is a limit in which an extra section in the Mordell-Weil group transforms into a “vertical” divisor class lying over a point in the base B_n . In the F-theory language this corresponds to an enhancement of the $U(1)$ gauge symmetry into a nonabelian gauge group with an \mathfrak{su}_2 gauge algebra (or in some special cases, a rank one enhancement of a larger nonabelian gauge group). At least at the level of geometry, this shows that any $U(1)$ gauge group factor in an F-theory construction can be found from the breaking of a nonabelian group containing an $SU(2)$ subgroup by Higgsing a field in the adjoint representation [38]. This fits into a very simple and general story associated with the Weierstrass form (4). Examples of situations where $U(1)$ ’s can be “unHiggsed” in this fashion were described in [16, 31]. In most situations the unHiggsed model with a vertical divisor is non-singular, though as we show explicitly in the following section, in some cases a singularity is present which can be interpreted either as a coupled superconformal theory, or as an indication that the unHiggsed model is at infinite distance from the interior of moduli.

If the classes associated with the coefficients e_0, e_4 in (4) are both effective, then all coefficients e_0, \dots, e_4 can generically be chosen to be nonzero, and we have a family of Weierstrass models that characterize Jacobian fibrations with a bisection, as discussed in §2.1. Let us consider what happens when e_0 and/or e_4 factorize or vanish either by tuning or because the associated divisors are not effective (in which case these coefficients would automatically vanish since there would be no sections of the associated line bundles).

As described in §2.1, if $e_4 = b^2/4$ is a perfect square, then the bisection becomes a pair of global sections, and the Mordell-Weil rank of the Jacobian fibration rises, which in the F-theory picture corresponds to the appearance of a $U(1)$ gauge factor. The equation is symmetric under $e_i \rightarrow e_{4-i}$, however, so we can also take $e_0 = a^2/4$ to produce a global section in another way.

In [16] it was observed that the zeros of b correspond to the intersection points of the two sections, and that the further tuning $b \rightarrow 0$, which naïvely would place the two sections on top of each other, in fact leads to a gauge symmetry enhancement to $SU(2)$. We can see this enhancement explicitly by choosing $e_4 = b^2/4 = 0$ so that the structure simplifies further, and the equation of the discriminant factorizes into the form

$$(26) \quad \Delta := 4f^3 + 27g^2 = e_3^2(-18e_0e_1e_2e_3 + 4e_1^3e_3 - e_1^2e_2^2 + 4e_0e_2^3 + 27e_0^2e_3^2).$$

Since neither f nor g are generically divisible by e_3 , this corresponds to a family of singular fibers⁷ of Kodaira type I_2 along the divisor $\{e_3 = 0\}$, associated with an \mathfrak{su}_2 Lie algebra component. (In some special cases the \mathfrak{su}_2 can be part of a larger nonabelian algebra, but we focus here on the generic \mathfrak{su}_2 case for simplicity.) In the F-theory picture the transition from the model with $b = 0$ to the $U(1)$ model with $b \neq 0$ is described by the Higgsing of an $SU(2)$ gauge group by a matter field in the adjoint representation.

Again, because the equation is symmetric, we can tune $e_0 = a^2/4 = 0$ in a similar fashion, giving a second I_2 singularity on the divisor $\{e_1 = 0\}$. In the F-theory picture this gives a second nonabelian gauge group factor with an \mathfrak{su}_2 algebra.

This gives a very generic picture in which, when the divisor classes $-K + L$ and $-3K - L$ are effective, we have a class of models with two A_1 Kodaira singularities on the divisors e_1, e_3 . This corresponds in the F-theory picture to a theory with gauge algebra $\mathfrak{su}_2 \oplus \mathfrak{su}_2$. When the divisor classes L and $-4K - 2L$ are effective we can turn on terms $e_0 = a^2/4$ and/or $e_4 = b^2/4$ that turn the “vertical” A_1 Kodaira singularities into global sections (without changing $h^{1,1}(X_{n+1})$); this corresponds in F-theory to Higgsing one or both of the nonabelian gauge groups through an adjoint representation to the $U(1)$ Cartan generator. When $L, -4K - 2L$ are nonzero classes, we can choose e_0 and/or e_4 to be generically nonzero and non-square, which further breaks the $U(1)$ ’s to a discrete \mathbb{Z}_2 symmetry. Because the discrete \mathbb{Z}_2 symmetry in the generic bisection model (4) is naturally identified with the center of both $U(1)$ fields, we expect only one \mathbb{Z}_2 in the center of the original nonabelian gauge group.

This can be seen geometrically by analyzing the charged matter under each of the $SU(2)$ factors, following [39, 40]. For any F-theory model, the “virtual” or “index” spectrum of massless matter multiplets minus massless vector multiplets can be described in terms of an algebraic cycle of codimension 2 on the base B_n , to each component of which is associated a representation of the gauge group. (For 6D compactifications, one then just counts points in the 2-cycle to determine the multiplicity of the representation, but for 4D compactifications there is an additional quantization which must be performed on each component of the 2-cycle to determine the multiplicity [41, 42], which may depend on the G-flux; what is fixed by the geometry is the set of representations which can appear in the spectrum.) For an I_2 fiber located along a divisor Σ , the virtual adjoint representation in the matter spectrum is associated to the algebraic cycle $\Sigma \cdot \frac{1}{2}(K + \Sigma)$, while the fundamental representation is associated to the cycle⁸ $\Sigma \cdot (-8K - 2\Sigma)$. Since we have bifundamental matter at the intersection of $[e_1]$ and $[e_3]$, which counts as $2(-K + L) \cdot (-3K - L)$ fundamentals for each of the $SU(2)$ factors, these bifundamentals account for all of the fundamental matter.⁹ Neither adjoints nor bifundamentals transform nontrivially under the diagonal \mathbb{Z}_2 in the combined gauge group. Since in the M-theory picture

⁷Note that when I_2 fibers occur in codimension one on the base, we associate them to \mathfrak{su}_2 , but when they occur in codimension two on the base, they are responsible for (charged) matter only and no additional gauge symmetry.

⁸More precisely, these cycles are determined by the intersections with $\{f = 0\}$ and $\{g = 0\}$ in the Weierstrass model, as described in [40].

⁹Here we are using the fact that when $\Sigma = -K + L$, we have $-8K - 2\Sigma = 2(-3K - L)$ and when $\Sigma' = -3K - L$, we have $-8K - 2\Sigma' = 2(-K + L)$.

the set of divisors must be dual to the set of curves in the Calabi-Yau, the diagonal \mathbb{Z}_2 is not part of the gauge group unless some field (associated with a curve in the resolved threefold) transforms under it [43, 44]. Thus, the full gauge group in the model with $e_0 = e_4 = 0$ will be $(SU(2) \times SU(2))/\mathbb{Z}_2$ where the discrete quotient is taken by the diagonal \mathbb{Z}_2 . Note that if either $[e_0]$ or $[e_4]$ is not effective then the corresponding A_1 (on $[e_1]$ or $[e_3]$) cannot be deformed away in the Weierstrass model while preserving the element of $h^{1,1}(X)$ in the form of a section; in the F-theory picture this corresponds to an $SU(2)$ that does not have massless matter in the adjoint representation.

We can confirm this analysis by exhibiting an explicit element of the Mordell-Weil group of order 2, as predicted by [45] (see also [44]). Namely, when $e_0 = e_4 = 0$ the Weierstrass equation (4) takes the form

$$(27) \quad y^2 = x^3 + \left(-\frac{1}{3}e_2^2 + e_1e_3\right)x + \left(\frac{1}{3}e_1e_2e_3 - \frac{2}{27}e_2^3\right) = \left(x + \frac{1}{3}e_2\right)\left(x^2 - \frac{1}{3}e_2x - \frac{2}{9}e_2^2 + e_1e_3\right).$$

The factorization of the right side of the equation corresponds to a point of order two on each elliptic fiber, and since this factorization is uniform over the base, the locus $\{x = -\frac{1}{3}e_2, y = 0\}$ defines a section which has order two in the Mordell-Weil group. Note that either on the locus $e_1 = 0$ or on the locus $e_3 = 0$, the Weierstrass equation (27) takes the form

$$(28) \quad y^2 = x^3 + \left(-\frac{1}{3}e_2^2 + e_1e_3\right)x + \left(\frac{1}{3}e_1e_2e_3 - \frac{2}{27}e_2^3\right) = \left(x + \frac{1}{3}e_2\right)^2\left(x - \frac{2}{3}e_2\right),$$

showing that the A_1 singularity of the singular fiber is located at $\{x = -\frac{1}{3}e_2, y = 0\}$ in each case, i.e., exactly at the section of order two. This implies that the \mathbb{Z}_2 quotient is nontrivial on each $SU(2)$, and thus that the global structure of the group must be $(SU(2) \times SU(2))/\mathbb{Z}_2$ using the diagonal \mathbb{Z}_2 .

One additional complication that can arise in this picture is when $-4K$ contains a divisor A as an irreducible effective component. In this case, there may be an automatic vanishing of f, g over A giving a nonabelian gauge group, such as in 6D for the non-Higgsable clusters of [46]. In this case, this component must be subtracted out from $-4K$ in computing the complementary divisors on which the $SU(2)$ factors reside, and some of the matter fields may transform under the gauge group living on A as well as one of the $SU(2)$ factors. We describe this mechanism further in the 6D context in the following section.

The upshot of this analysis is that for any elliptically fibered Calabi-Yau manifold with a nonzero rank Mordell-Weil group, a global section can be associated with a divisor class $D = [e_3]$ to which the section can be moved as an A_1 (or higher) Kodaira type singularity. In the language of F-theory geometry (without considering effects such as G-flux relevant in four dimensions, §4.2) this means that any $U(1)$ gauge symmetry can be seen as arising from a broken nonabelian symmetry. Furthermore, there is an intriguing structure in which for each such divisor class D there is a complementary divisor class

$$(29) \quad D' = [e_1] = -4K - D,$$

that can (and in some cases must) also be tuned to support an A_1 singularity, which may be associated with a second independent section. In situations where $-4K$ has a base locus over which f, g have enforced vanishing associated with Kodaira singularities giving nontrivial gauge groups, the base locus must also be subtracted out in (29). In the next section we give several explicit examples of how this works in some 6D models.

3. 6D EXAMPLES

The arguments given up to this point have been very general, and in principle apply to elliptically fibered Calabi-Yau manifolds in all dimensions where a suitable Weierstrass model is

available. In this section we consider some simple explicit examples of 6D F-theory models to illustrate some of the general points. We begin by describing explicitly the way in which any $U(1)$ in 6D can be seen as arising from an $SU(2)$ factor that has been Higgsed by turning on a vacuum expectation value for an adjoint hypermultiplets. We then describe some general aspects of models with bisections in this context, and conclude by explicitly analyzing various possible ways in which the “unHiggsing” to $SU(2)$ may encounter problems with singularities. While such singularities do not arise in most cases, we identify one situation where such a singularity arises, which can only be removed by blowing up the F-theory base manifold.

Before beginning, let us recall how the various moduli spaces of Weierstrass models are linked together through transitions involving coupling to 6D superconformal field theories [47], sometimes called “tensionless string transitions” [48, 49]. As we tune the coefficients of a Weierstrass model over a fixed base B_2 , various singularities are encountered that have explanations in terms of nonabelian gauge symmetry or the massless matter spectrum. However, if a singular point P is encountered at which f has multiplicity at least 4, and g has multiplicity at least 6, the model has a superconformal field theory sector and another branch emerges in which a tensor multiplet is activated [3, 50]. The other branch consists of Weierstrass models over the blowup $\text{Bl}_P(B_2)$ of B_2 at P , and the area of the exceptional curve of the blowup serves as the expectation value of the scalar in the new tensor multiplet. We generally refer to such points as “(4, 6) points.”

Even more special is the case in which either (f, g) have multiplicities at least (8, 12) at a point, or have multiplicities of at least (4, 6) along a curve. In this case, the total space of the fibration is not Calabi-Yau, and in fact any resolution of the space in algebraic geometry has no nonzero holomorphic 3-forms. It is known that the points in the moduli space of Weierstrass models at which such singularities occur are boundary points of moduli at infinite distance from the interior of the moduli space [51, 52].

3.1. $U(1)$ from a Higgsed $SU(2)$ in 6D. In six dimensions, we can demonstrate explicitly that in most situations a $U(1)$ can be enhanced to an $SU(2)$ in a conventional F-theory model on the same base (i.e., one not involving a superconformal theory or at infinite distance from the interior of the moduli space) by considering general classes of acceptable $U(1)$ model in which tuning $b^2 \rightarrow 0$ in (5) need not introduce a (4, 6) point. We can also identify some situations in which this tuning does necessarily lead to such a singularity. A forced (4, 6) point can in principle occur in one of two ways: first, if $[e_3]$ contains a curve of negative self intersection over which f, g are required to vanish to high degree, and second if $[e_3]$ has nonzero intersection with another curve C or combination of curves A, B, \dots over which f, g vanish to high enough degree to force a (4, 6) vanishing at an intersection point. We outline the general structure of the analysis here, and describe some special cases in the later parts of this section.

First, let us consider the case where C is a curve in the class $[e_3]$, C is irreducible, and $[e_1] = -4K - [e_3]$ does not contain as irreducible components any curves of self-intersection below -2 . We consider the Weierstrass model of the form (5), and take the limit as $b^2 \rightarrow 0$, which produces an $SU(2)$ over C with matter in the adjoint representation. For the enhancement to $SU(2)$ with an adjoint or higher representation to occur on a curve C , the curve must have genus $g > 0$. This follows from the general result [53] that every representation of $SU(N)$ with a Young diagram having more than one column makes a positive contribution to the genus of the curve through the anomaly equations. It was shown in [46] that a curve of positive genus cannot have negative self-intersection without forcing a (4, 6) vanishing all along the curve. So $C \cdot C \geq 0$ and f, g cannot be required to vanish on the irreducible curve for a generic Weierstrass model over the given base. To see where there are enhanced singularities at points on C in the $SU(2)$ model, we can use the 6D anomaly cancellation conditions [54, 55, 56, 57, 58, 59]. For a generic curve C in the class $[e_3]$, where $SU(2)$ matter is only in A hypermultiplets that transform

under the adjoint (symmetric) representation and x fundamental hypermultiplets, the anomaly conditions read

$$(30) \quad K \cdot C = \frac{1}{6} [4(1 - A) - x]$$

$$(31) \quad C \cdot C = -\frac{1}{3} [8(1 - A) - x/2] .$$

Solving these equations gives

$$(32) \quad A = g = 1 + \frac{1}{2}(K \cdot C + C \cdot C)$$

and

$$(33) \quad x = 2C \cdot (-4K - C) = 2[e_3] \cdot [e_1] .$$

When $e_0 = a^2/4 = 0$ and there are $SU(2)$ gauge factors supported on both e_1 and e_3 , this shows that all matter fields – and hence all enhanced singularities – arise at the intersection points between these two curves. Note that this is the same conclusion about the matter spectrum that we reached in §2.4 in a different way. Note also that the location of the singularities associated with the matter charged under the $SU(2)$ on C is the same whether or not we take the $a \rightarrow 0$ limit, and that this matter corresponds to the extra charged matter fields found on the I_2 locus in §2.2.

Additional complications can arise if e_1 or e_3 are reducible, particularly when either or both contain irreducible factors that carry nontrivial Kodaira singularities. In such cases, f, g will vanish on the associated curve A , with an extra nonabelian gauge group factor, according to the classification of non-Higgsable clusters in [46]. To show when a $U(1)$ that arises in a Weierstrass form (5) can be associated with a broken $SU(2)$ in a conventional F-theory model without changing the base, we need to prove that in these cases a $(4, 6)$ point cannot be introduced by taking the $b \rightarrow 0$ limit. There can also be more complicated singularities introduced if the curve C is not a generic curve in the class $[e_3]$ and itself has singularities. There is not yet a complete dictionary relating codimension two singularities of this type to matter representations, though there has been some recent progress in this direction [60, 61, 62]. We do not consider such cases here in any detail, though an example is discussed in §4.1; here we assume that the curve C is taken to be generic in the class $[e_3]$, so the statement that a $U(1)$ can be viewed as a Higgsing of an $SU(2)$ model should be understood as involving the Higgsing of an $SU(2)$ model with a generic C given $[e_3]$, with further tuning of C carried out as necessary to achieve the given $U(1)$ model of the form (5).

If $[e_3]$ intersects a curve A in $[e_1]$ that carries a nonabelian gauge group G_A (again, assuming A is a generic curve in its class), some of the matter charged under the $SU(2)$ living on the curve C will also be charged under G_A . This must occur in such a way that setting $b^2 \rightarrow 0$ does not increase the degree of vanishing of f, g on A , or the spectrum of fields charged under the $U(1)$ would not match the spectrum of fields charged under the $SU(2)$ in the $b \rightarrow 0$ limit determined as above by the anomaly conditions. Indeed, explicit analysis of the possibilities shows that such an intersection can occur only when A is a -3 or -4 curve. In these cases, when $[e_3] \cdot A \neq 0$, the degrees of vanishing of f, g on A are increased above the minimal Kodaira levels, and G_A carries an enhanced gauge group with charged matter that also carries charges under the $U(1)$ or $SU(2)$ on C in a consistent fashion. When A is a -5 (or less) curve, there is a $(4, 6)$ point on A even in the $U(1)$ model (5), so no such conventional $U(1)$ theory can be constructed. We consider some explicit examples of these cases in the subsequent sections and demonstrate the unconventional presence of a superconformal theory explicitly for -5 curves in §3.6.

In a similar fashion, we can analyze the special cases where e_3 contains a curve D of negative self-intersection as an irreducible component. Note that if $d|e_3$ and also $d|b$, then we can move the factor of d from e_3 into e_1 (with two factors of d extracted from b^2 and moved into e_0). Thus, if $[e_3]$ contains $[d]$ as a component, and $b = -2K - L$ is such that $[b] - [d]$ is effective, we can tune $b = db'$ and the analysis becomes that of the previous case. So we need only consider situations where e_3 contains an irreducible component D that is not a component of b . It turns out this is possible for curves of self-intersection $-3, -4, -5$, and -6 ; in each of these cases there are configurations where e_3 contains such curves as a component but b does not. When the parameter b is tuned to vanish, the enhancement to $SU(2)$ is combined with an enhancement of the gauge group over D in a way that is consistent with anomaly cancellation and does not introduce $(4, 6)$ points. For curves of self-intersection -7 or below, the $U(1)$ model already has $(4, 6)$ singularities, so there are no conventional models. We give some examples of these kinds of configurations in the subsequent parts of this section.

Although a single curve of negative self-intersection contained in e_3 does not lead to a problematic singularity, there are also situations where e_3 contains a more complicated configuration of intersecting negative self-intersection curves. In particular, there exist non-Higgsable clusters identified in [46] that contain intersecting -3 and -2 curves. In such a situation, as we show explicitly below, a $(4, 6)$ point can arise at the intersection between these curves when a $U(1)$ is unHiggsed to $SU(2)$ by taking the $b \rightarrow 0$ limit. This is the one situation we have clearly identified in which such a singularity can arise.

This argument shows that a $U(1)$ gauge factor in a 6D F-theory model over any base can be viewed as arising from an $SU(2)$ gauge group supported on a corresponding effective irreducible divisor class $[e_3]$, after Higgsing a matter hypermultiplet in the adjoint representation; in a wide range of situations the unHiggsing results in a conventional F-theory model with reduced Mordell-Weil rank, though in certain special cases the unHiggsing either gives rise to a model which is coupled to a superconformal theory or is at infinite distance from the interior of moduli space. This general framework gives strong restrictions on the ways in which $U(1)$ factors can arise in 6D F-theory models, and illuminates the structure of the Mordell-Weil group for elliptically fibered Calabi-Yau threefolds over general bases.

3.2. 6D theories on \mathbb{P}^2 with two sections or a bisection. As a simple specific example of a class of 6D theories that illustrate the general structure of models with bisection, two sections ($U(1)$) and enhanced $(SU(2) \times SU(2))/\mathbb{Z}_2$ gauge group, we consider the case of 6D F-theory compactifications on the simplest base surface $B_2 = \mathbb{P}^2$. Models of this type with $U(1)$ fields were considered from the point of view of supergravity and anomaly equations in [14], and an explicit F-theory analysis and Calabi-Yau constructions were given in [16]. In this case, $-K = 3H$, where H is the hyperplane (line) divisor with $H \cdot H = 1$. Tuning an I_2 singularity along a degree d curve C in \mathbb{P}^2 by adjusting the degrees of vanishing of f, g, Δ along C to be $0, 0, 2$, respectively, gives an F-theory model with gauge group $SU(2)$. A generic curve of degree d has genus $g = (d-1)(d-2)/2$, and the associated $SU(2)$ gauge group has a matter content consisting of g massless hypermultiplets in the adjoint representation and $24d - 2d^2$ multiplets in the fundamental representation (note that for $SU(2)$, unlike $SU(N)$ for $N > 2$, the antisymmetric representation is trivial). By tuning higher order singularities in the curve C , some of the adjoint matter fields can be transformed into higher-dimensional matter fields, with a simple relation between the matter representations and contribution to the arithmetic genus of C , as described in [53, 60].

To describe the class of Calabi-Yau threefolds on \mathbb{P}^2 associated with a Jacobian fibration with a bisection, we consider Weierstrass equations of the form (4), where the classes of the e_i are given in (11–15). We parameterize the set of models of interest by $[e_3] = -3K - L = mH$, where

m corresponds to the degree of a curve in the class $[e_3]$. For any m in the range $0 \leq m \leq 12$ there is a class of Weierstrass models of the form (4) that give “good” F-theory models without (4, 6) points (points which, if present, would involve coupling to superconformal field theories or would violate the Calabi-Yau condition). The generic model in each of these classes corresponds to a Jacobian fibration, and in the F-theory picture there is a discrete \mathbb{Z}_2 gauge group, with a number of charged matter hypermultiplets. For $3 \leq m \leq 9$, both $[e_0]$ and $[e_4]$ are effective; in this range of models, there is a subset of models with $e_4 = b^2/4$ a perfect square, giving an extra section contributing to the Mordell-Weil rank, which is associated in the F-theory picture with a $U(1)$ gauge factor, and there is also a (partially overlapping) subset of models with $e_0 = a^2/4$ with another $U(1)$ factor. Either or both of these $U(1)$ factors can be further enhanced to an $SU(2)$ by fixing $b^2 = 0$ or $a^2 = 0$. When both factors are enhanced ($e_0 = e_4 = 0$) the total gauge group is $(SU(2) \times SU(2))/\mathbb{Z}_2$. When $m < 3$ or $m > 9$ the story is similar but one of the two $SU(2)$ factors is automatically imposed by the non effectiveness of the divisor $[e_4]$ or $[e_0]$; in these cases there is only one possible $U(1)$ factor.

This class of models can be understood most easily in the F-theory picture starting from the locus $e_0 = e_4 = 0$ where the gauge algebra is $\mathfrak{su}_2 \oplus \mathfrak{su}_2$. In this case, the two \mathfrak{su}_2 summands are associated with 7-branes wrapped on divisors D, D' given by curves of degrees m and $12-m$ in the classes $[e_3], [e_1]$. The spectrum of the theory consists of $m(12-m)$ bifundamental hypermultiplets (associated with the intersection points of D, D'), and $(m-1)(m-2)/2, (11-m)(10-m)/2$ fields in the adjoint representation of each $SU(2)$. The limiting cases $m = 0, 12$ correspond to situations with only a single $SU(2)$ factor and no fundamental hypermultiplets. In all cases, an $SU(2)$ on a curve of degree $d \geq 3$ has adjoint hypermultiplets, of which one can be used to Higgs the nonabelian gauge group to a $U(1)$. Under this Higgsing, the remaining adjoints become scalar fields of charge 2 under the resulting $U(1)$, while fundamentals acquire a charge of 1. When $3 \leq m \leq 9$, such Higgsing to abelian factors is possible for both $SU(2)$ factors; for other values only one of the groups can be Higgsed. Once one or both of the nonabelian factors are Higgsed to $U(1)$ fields, a further breaking can be done by making e_0 or e_4 a generic non-square. This corresponds to using the charge 2 fields to Higgs the $U(1)$ to a discrete gauge group \mathbb{Z}_2 . Under this Higgsing, the charge 1 fields retain a charge under the discrete gauge group. It is straightforward to check that the numbers of fields in each of these models satisfies the gravitational anomaly cancellation condition $H - V = 273 - 29T$, and matches with the results of [14, 16, 6] for the various component theories.

In particular, note that for $m = 3$ the $SU(2)$ gauge group on $D = [e_3]$ only has a single adjoint field, so after breaking to $U(1)$ there are only charge 1 hypermultiplets. Thus, in this case there is no way of breaking to a model with a bisection and residual discrete gauge group. Note also that by tuning a non-generic singularity on the curve C carrying an $SU(2)$ factor, it should be possible to construct higher dimensional representations of $SU(2)$, which will correspond to larger charges $Q \geq 3$ after breaking to $U(1)$, and which can give rise to higher order discrete gauge groups \mathbb{Z}_Q . We return to this issue in §4.1. In Table 1, we provide an explicit list of the charges that arise for the $SU(2)$ and $U(1)$ factors in the various relevant components of the Weierstrass moduli space $\mathcal{S}_2, \mathcal{J}^2$

3.3. 6D theories on $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. A similar structure will hold on any base B_2 that supports an elliptically fibered Calabi-Yau threefold; a classification of such bases was given in [46], and a complete list of toric bases was given in [65]. As another example we consider the Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

For \mathbb{F}_0 , a basis of $h^{1,1}$ is given by S, F with $S \cdot S = F \cdot F = 0, S \cdot F = 1$. A divisor $D = aS + bF$ is effective if $a, b \geq 0$, and the anticanonical class is $-K = 2S + 2F$. The genus of a curve in the

m	n_a	n_f	n_2	n_1
1	0	22	–	–
2	0	40	–	–
3	1	54	0	108
4	3	64	4	128
5	6	70	10	140
6	10	72	18	144
7	15	70	28	140
8	21	64	40	128
9	28	54	54	108
10	36	40	70	80
11	45	22	88	44
12	55	0	108	0

TABLE 1. Table of $SU(2)$ charges in adjoint and fundamental, and $U(1)$ charges in associated theory, when e_3 describes a curve of degree m in \mathbb{P}^2 . Note that $SU(2)$ and $U(1)$ charges match with Higgsing description ($n_2 = 2(n_a - 1), n_1 = 2n_f$) as well as with charges computed in [14, 16]. Note also that n_1 matches for $m, 12 - m$, in agreement with the general picture that all charged matter lies on intersection points of $[e_1], [e_3]$ for the $(SU(2) \times SU(2))/\mathbb{Z}_2$ theory.

class $C = aS + bF$ can be computed as

$$(34) \quad (K + C) \cdot C = 2g - 2 = 2(ab - a - b).$$

The genus is nonzero iff $2 \leq a, b$.

The range of possible models (4) with a bisection is thus given by $[e_3] = aS + bF$ with $0 \leq a, b \leq 8$. The values of a, b for which the curves $[e_3], [e_1]$ both have nonzero genus and associated $SU(2)$ s can be broken is $2 \leq a, b \leq 6$. Within this range we have the full set of possible enhancements of a model of type (4); there is a model with $(SU(2) \times SU(2))/\mathbb{Z}_2$ symmetry, where either or both $SU(2)$'s can be broken to $U(1)$ or further to the discrete \mathbb{Z}_2 symmetry. Again, counting charged multiplets confirms that anomaly cancellation in both the nonabelian and abelian theories matches with the Higgsing process. The spectrum of charged matter fields for an $SU(2)$ tuned on a divisor $aS + bF$ consists of $g = ab - a - b + 1$ adjoints and $16(a + b) - 4ab$ fundamentals. As in the \mathbb{P}^2 case, the number of fundamental fields is symmetric under $a \leftrightarrow 8 - a, b \leftrightarrow 8 - b$ ($e_1 \leftrightarrow e_3$), corresponding to the fact that all charged matter in the overall $(SU(2) \times SU(2))/\mathbb{Z}_2$ theory is contained in the adjoints and $8(a + b) - 2ab$ bifundamental fields.

3.4. 6D theories on \mathbb{F}_3 . Some interesting points are illuminated by examples on the Hirzebruch surface \mathbb{F}_3 . Here we have a basis of curves S, F with $S \cdot S = -3, S \cdot F = 1, F \cdot F = 0$. The canonical class is $-K = 2S + 5F$, and there is an automatic vanishing of f, g, Δ to degrees 2, 2, 4 giving an $SU(3)$ gauge group supported on the divisor S in a generic elliptic fibration.

The simplest irreducible curve e_3 that can give rise to a $U(1)$ factor is $C = 2\tilde{S} = 2S + 6F$, since e_4 must be effective; a generic curve in this class C is irreducible and has genus 2. Choosing

$$(35) \quad [e_3] = 2S + 6F, \Rightarrow [e_1] = 6S + 14F.$$

We note that $[e_1] \cdot S = -4$, so $[e_1]$ contains S as an irreducible component with multiplicity at least 2. There is an $SU(3)$ over S , but this does not cause any problems since $S \cdot [e_3] = 0$ so there is no matter charged under the $SU(3)$ that interacts with the $SU(2)$ supported on C or

the corresponding $U(1)$ when $e_4 = b^2/4 \neq 0$ or discrete group $\mathbb{Z}/2$ when e_4 is non-square. So this case works like the others above, with $[e_1] \cdot [e_3] = 28$ bifundamental matter fields.

The next case of interest is

$$(36) \quad [e_3] = 2S + 7F, \Rightarrow [e_1] = 6S + 13F.$$

In this case the curve C defined by the vanishing locus of e_3 is generically a smooth irreducible curve of genus 3. In this case, $C \cdot S = 1$, so there is matter charged under the gauge group lying over S . To analyze this explicitly, we see that $[e_0] = 8S + 16F$, $[e_1] = 6S + 13F$, $[e_2] = 4S + 10F$ contain the irreducible component S with multiplicities 3, 2, and 1 respectively. From (5), (26), this shows that f, g, Δ vanish to degrees 2, 3, 6 at generic points over S , and to degrees 2, 3, 8 at points of intersection $[e_3] \cdot S$. As discussed in §3.1, in this situation the gauge group over the -3 curve has an algebra that is larger than the minimal \mathfrak{su}_3 for a generic model over \mathbb{F}_3 . The configuration in this case is similar to the $(-3, -2)$ non-Higgsable cluster [46], in which a -3 curve carries a \mathfrak{g}_2 algebra, and there are matter fields charged under both this algebra and an \mathfrak{su}_2 on a curve that intersects the -3 curve.

We can also analyze the case where $C = -K = 2S + 5F$, where C is reducible and contains S as a component in a similar fashion, which also gives a $(2, 3, 6)$ vanishing on S , with a similar interpretation

3.5. \mathbb{F}_4 . The analysis in the case of a -4 curve in \mathbb{F}_4 is similar to \mathbb{F}_3 . The curve S has $S \cdot S = -4$. For the minimal irreducible case $[e_3] = 2S + 8F$, there is an $SU(2)$ with adjoint matter that does not intersect S . For $[e_3] = 2S + 9F$, we have

$$(37) \quad [e_2] = 4S + 12F = S + X_{\text{eff}}^{(2)}$$

$$(38) \quad [e_1] = 6S + 16F = 2S + X_{\text{eff}}^{(1)}$$

$$(39) \quad [e_0] = 8S + 20F = 3S + X_{\text{eff}}^{(0)}$$

where $X_{\text{eff}}^{(a)}$ are effective divisors that contain no further components of S . We can read off the order of vanishing of f, g, Δ from (5) and (26) as $(2, 3, 6)$ on S , enhanced to $(2, 3, 8)$ on $[e_3] \cdot S$, so again we have hypermultiplets charged under the gauge group on S as well as the $SU(2)$ on $[e_3]$. For curves such as $[e_3] = -K = 2S + 6F$, where $[e_3]$ contains S as an irreducible component, a similar analysis holds.

3.6. \mathbb{F}_5 and -5 curves. Now let us consider a -5 curve, beginning with the case of \mathbb{F}_5 . As in the previous cases, for $[e_3] = 2S + 10F$, there is no intersection with S and the $SU(2)$ story is as above. For the next interesting case, however, we have

$$(40) \quad [e_3] = 2S + 11F$$

$$(41) \quad [e_2] = 4S + 14F = 2S + X_{\text{eff}}^{(2)}$$

$$(42) \quad [e_1] = 6S + 17F = 3S + X_{\text{eff}}^{(1)}$$

$$(43) \quad [e_0] = 8S + 21F = 4S + X_{\text{eff}}^{(0)}.$$

Now, analyzing (5) and (26) we find vanishing orders of f, g, Δ on S of $(3, 4, 9)$, enhanced to $(4, 6, 12)$ on $S \cdot [e_3]$, even when $b^2 \neq 0$. Thus, there cannot be a $U(1)$ model based on (5) using $e_3 = 2S + 11F$ (unless the intersection point is blown up, giving a model on a different base).

More generally, we can show that a $U(1)$ based on an extra section can never be constructed on any curve $[e_3] = C$ if $C \cdot A > 0$ for some curve A of self-intersection -5 or less. The argument basically follows exactly the same steps as above. In general, as described in [46], from $[e_2] = -2K$ it follows that e_2 vanishes to degree 2 on A just as in the \mathbb{F}_5 case. We have $-4K \cdot A = -12$ and $[e_3] \cdot A > 0$, so $[e_1] \cdot A = (-4K - [e_3]) \cdot A < -12$ and e_1 vanishes to degree 3

on A . From $[e_3] \cdot A > 0$, it follows that $L \cdot A \leq -10$, so $[e_0] \cdot A \leq -20$, and e_0 vanishes to order 4 on S . Thus, no $U(1)$ can be built using (5) on any curve e_3 that has positive intersection with a curve A of self-intersection -5 . The condition on each term is stronger as the self-intersection decreases further, so the same result holds for any curve of self-intersection < -5 .

Now, let us consider the case that e_3 itself has a -5 curve D as a component. For this to happen we must have $[e_3] \cdot D < 0$, but as argued in §3.1 we should also have $[e_4] \cdot D \geq 0$, or we could move the associated factor out of e_3 and into e_1 . This can lead to a conventional model when $[e_3] \cdot D = -2$ or -3 . In these cases, e_0, e_1, e_2, e_3 vanish to degrees 3, 2, 2, 1 on D , and f, g vanish to degrees 3, 4. In the limit $b^2 \rightarrow 0$, f, g vanish to degrees 3, 5 and the symmetry is enhanced to \mathfrak{e}_7 . Note that when $[e_3] \cdot D = -1$, e_0, e_1 vanish to degrees 4, 3 on D , giving multiplicities (4, 5) along D that are enhanced to (4, 6) at points of intersection with the remainder of e_3 , so such models are not conventional even before unHiggsing.

As an example of a conventional model of this type, consider on \mathbb{F}_5 the $U(1)$ model given by (5) with

$$(44) \quad [e_3] = 2S + 8F = S + X_{\text{eff}}^{(3)}$$

$$(45) \quad [e_2] = 4S + 14F = 2S + X_{\text{eff}}^{(2)}$$

$$(46) \quad [e_1] = 6S + 20F = 2S + X_{\text{eff}}^{(1)}$$

$$(47) \quad [e_0] = 8S + 26F = 3S + X_{\text{eff}}^{(0)}.$$

As discussed above, this gives (3, 4) vanishing on the -5 curve S in the $U(1)$ model, enhanced to (3, 5) at points of intersection with e_1 . When $b \rightarrow 0$, the group is enhanced to (3, 5) on the whole curve S , with further enhancement to (4, 5) at points of intersection with e_3 .

3.7. -6 curves. The situation for -6 curves is very similar to that for -5 curves. There is a coupled superconformal theory if $[e_3]$ has positive intersection with a -6 curve, but $[e_3]$ can contain a -6 curve D as a component if $[e_3] \cdot D = -4$, in which case e_0, e_1, e_2, e_3 vanish to orders 3, 2, 2, 1 on D and the story is similar to the above. In this case, however, e_1 does not intersect D , so there are no points where this intersection increases the degree of the singularity.

3.8. -7 curves. There are no conventional $U(1)$ configurations of the form (5) where e_3 either intersects or contains a curve D of self-intersection -7 or below. The closest to an acceptable configuration is when $[e_3] \cdot D = -5$, in which case e_0, e_1, e_2, e_3 vanish to orders 3, 3, 2, 1 on D . This leads to a (3, 5) vanishing of (f, g) on D , which is however enhanced to a (4, 6) vanishing at the point where $[e_0] - [D]$ intersects D (of which there is at least one since $[e_0] \cdot D = -20$). Any other combination of intersections leads to a similar singularity. A similar problem arises for curves of self-intersection -8 or below.

3.9. The $-3, -2$ non-Higgsable cluster. Finally, we consider the case where e_3 contains both a -3 curve A and a -2 curve B that intersects A transversely ($A \cdot B = 1$). In this case we find that, at least for some choice of L , a (4, 6) point is forced at the intersection point between A and B . In particular, we choose $L = -2K$, so that $[e_n] = (n - 4)K$. From the analysis in [46], we know that a section of $-4K$ must vanish on A, B to degrees 2, 1 respectively, so

$$(48) \quad [e_0] = 2A + B + X_{\text{eff}}^{(0)}.$$

It follows that each of the e_n must contain both A and B as irreducible components at least once, for $n < 4$,

$$(49) \quad [e_1] = A + B + X_{\text{eff}}^{(1)}$$

$$(50) \quad [e_2] = A + B + X_{\text{eff}}^{(2)}$$

$$(51) \quad [e_3] = A + B + X_{\text{eff}}^{(3)}$$

$$(52) \quad [b] = X_{\text{eff}}^{(4)}.$$

Now, consider the degrees of vanishing of the various terms in (5). While for $b \neq 0$, there are terms in f and g that only vanish to degrees (3, 5) at the intersection of A and B (namely those proportional to $e_0 b^2, e_0 e_2 b^2$), when we take $b \rightarrow 0$, all the remaining non-vanishing terms are of degrees at least (4, 6) at the intersection point.

This means that in such a situation, while there can exist a 6D F-theory model with a $U(1)$ gauge symmetry associated with a nontrivial Mordell-Weil rank, and the Weierstrass coefficients can be tuned to naively produce a nonabelian $SU(2)$ structure, the resulting model might have an isolated (4, 6) point and hence be coupled to a superconformal theory,¹⁰ or in other situations [64] might have (4, 6) singularities all along one or more curves after unHiggsing¹¹, which indicates that these models are at infinite distance from the interior of the moduli space. There are many known examples of base surfaces that contain $-3, -2$ non-Higgsable clusters; a variety of such examples were constructed in [65, 71]. It would be interesting to analyze in detail the structure of $U(1)$ symmetries that could be tuned over some of these bases.

4. IMPLICATIONS FOR 6D AND 4D F-THEORY MODELS

4.1. F-theory and supergravity in six dimensions. Six dimensions provides a rich but tractable context in which to study general aspects of string vacua and quantum supergravity theories. In six dimensions, F-theory seems to provide constructions for essentially all known string vacua, and the space of F-theory vacua matches closely with the set of potentially consistent quantum supergravity theories [66, 67, 68, 69, 59]. The class of 6D F-theory constructions based on Weierstrass models of elliptically fibered Calabi-Yau threefolds with section form a single moduli space of smooth components associated with different bases B_2 that are connected through tensionless string transitions [49, 3]; recent work has made progress in providing a global picture of this connected moduli space [59, 46, 70]. The results of [6] raised a question of whether genus-one fibrations without section might constitute a class of F-theory models that were disconnected from the rest of the F-theory moduli space. The picture outlined in this note makes it clear that in fact the Jacobian fibrations for threefolds without section fit neatly into the connected moduli space of Weierstrass models. Furthermore, this picture sheds light on how $U(1)$ gauge fields in 6D F-theory models may be understood in the context of the full moduli space of models.

In [6, 20, 21, 23], a systematic description was given of the general form for Weierstrass models containing one, two, and three $U(1)$ fields. It is known that 6D models can be constructed with up to eight or more $U(1)$ fields; for example, as described in [60] there are F-theory constructions on \mathbb{P}^2 with an $SU(9)$ tuned on a curve of genus one that contain an adjoint representation the breaking of which gives gauge factors $U(1)^8$, and in [71] a class of \mathbb{C}^* -bases B_2 were found with varying automatic ranks for the Mordell-Weil group for generic elliptic fibrations; the resulting threefolds are closely related to the Schoen manifold [64]. One such base in particular is a

¹⁰We would like to thank Jim Halverson for discussions on this point. Analogous curves in 4D F-theory models are identified and classified in [63].

¹¹We would like to thank D. Park for discussions on this point.

generalized del Pezzo nine over which the generic elliptic fibration has a rank 8 Mordell-Weil group, corresponding to gauge factors $U(1)$ ⁸. In [14], it was shown from 6D anomaly cancellation arguments that for a pure abelian theory in 6D with no tensor multiplets (corresponding to an F-theory model on \mathbb{P}^2) the number of $U(1)$ fields is bounded above by $r \leq 17$. The approach taken in this paper shows that for a single $U(1)$ factor, it is often possible to tune the model so that the $U(1)$ can be seen as arising from an $SU(2)$ or larger nonabelian factor that is Higgsed by VEVs for an adjoint field in a conventional F-theory model. It would be interesting to investigate the possible apparent exceptions to this construction, such as the ones we encountered with base contains a $-3, -2$ cluster, where the F-theory model becomes coupled to a superconformal theory. While in general the construction of higher rank Mordell-Weil models seems very challenging due to the global nature of the sections, it would be very interesting to explore when higher rank abelian models can arise from Higgsed nonabelian gauge symmetries. This would provide a powerful tool for the construction of general models with abelian gauge symmetries, since a systematic analysis of the nonabelian sector is much more straightforward, both in F-theory and 6D supergravity. It would also be interesting to explore in more detail the way in which the basic $SU(2) \rightarrow U(1) \rightarrow \mathbb{Z}_2$ Higgsing pattern interacts with other nonabelian gauge symmetries which may be present in a given model.

The existence of an underlying $SU(2)$ for many $U(1)$ gauge factors also greatly clarifies the set of possible spectra. The spectrum of $SU(2)$ theories is quite constrained by anomaly cancellation [53], which in turn places strong constraints on the spectrum of possible charges for abelian factors in the 6D supergravity gauge group. When an $SU(2)$ factor is tuned on a curve of genus g over a general base B_2 generically the model will include g symmetric (adjoint) representations and some number of fundamentals. After breaking to a $U(1)$, this gives charges 1 and 2, so these are the only charges expected in generic models. For specially tuned singular curves, however, higher representations of $SU(2)$ are possible.

For example, following the lines of [60], we expect that an $SU(2)$ on a quintic curve on P^2 can carry a 3-symmetric (4-dimensional) representation when the curve is tuned to have a triple point of self-intersection. Group theoretically, this should correspond to an embedding of $\mathfrak{su}_2 \oplus \mathfrak{su}_2 \oplus \mathfrak{su}_2$ in an \mathfrak{e}_7 singularity associated with the triple intersection point. After breaking the $SU(2)$ to $U(1)$ by an adjoint VEV, this would give rise to a massless scalar hypermultiplet of charge ± 3 under the $U(1)$. By the mechanism discussed in this paper, such fields could then be used to break the $U(1)$ to a discrete \mathbb{Z}_3 gauge symmetry, associated again with a Weierstrass model associated with the Jacobian of an elliptic fibration with a multisection. Exploring the range of possibilities of this type that may be possible for general representations of $SU(2)$ and higher rank nonabelian groups on arbitrary curves on general F-theory bases B_2 promises to provide a rich and interesting range of phenomena. The analysis here shows that there are strong constraints on the charge spectrum for $U(1)$ fields in many 6D F-theory models. These constraints are stronger than those imposed simply by 6D anomaly cancellation. In the spirit of [66], it would be interesting to understand if some of the F-theory constraints on charge structure could be seen as consistency conditions for the low-energy 6D supergravity theories with abelian gauge factors.

4.2. Four dimensions. At the level of geometry, the framework developed in this paper should be valid for Calabi-Yau manifolds of any dimension. It has not been shown, however, that all genus-one fibered Calabi-Yau $(n+1)$ -folds X_{n+1} that lack a global section have an associated Jacobian fibration J_{n+1} whose total space is Calabi-Yau when $n \geq 3$, so it is possible in principle that the analysis described here can only be applied in a subset of cases where there is a Jacobian fibration available. If so, the application to four-dimensional F-theory constructions would only be relevant in those cases. When a Jacobian fibration is available, however, the analysis of §2

should hold: the Jacobian fibrations of all Calabi-Yau fourfolds with a genus-one fibration but no global section should fit into the moduli space of Weierstrass models over complex threefold bases B_3 , with an explicit description of the form (4) when the Jacobian fibration has a bisection. When the section $e_4 = b^2/4$ is a perfect square, the bisection becomes a pair of global sections and the Mordell-Weil rank increases by one. When $b \rightarrow 0$, the extra section transforms into a vertical A_1 Kodaira type singularity without changing the total Hodge number $h^{1,1}(X_4)$. In many situations, the physics interpretation of this geometry through F-theory will be the same as in 6 dimensions: the bisection geometry will be associated with a discrete \mathbb{Z}_2 gauge symmetry that arises from a broken $U(1)$ gauge field, which in turn can be viewed as coming from an $SU(2)$ gauge group broken by an adjoint VEV. Wrapping the 4D theory on a circle will give distinct vacua, again associated with the Tate-Shafarevich group and in the M-theory picture with a discrete choice of Calabi-Yau fourfold with a genus-one fibration but no section.

We also expect a similar story to hold for higher degree multisections and elliptic fibrations with higher rank Mordell-Weil group. In four dimensions, however, there is additional structure beyond the geometry that can modify this story. In particular, G-flux, associated with 4-form flux of the antisymmetric 3-form potential in the dual M-theory picture, produces a superpotential that gives masses to many of the scalar moduli of the Calabi-Yau geometry. This mechanism can modify the gauge group and matter spectrum of the theory from that described purely by the geometry. At this point a complete understanding of the role of G-flux in F-theory is still lacking, despite some recent progress in this direction [72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 42]. We leave the analysis of how the results in this note are affected by G-flux and the 4D superpotential to further work. The implications of the generic appearance of an $SU(2)$ (or larger) nonabelian enhancement for most $U(1)$ vector fields are, however, a question of obvious phenomenological interest.

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APPENDIX A. SOLVING THE EQUATIONS DETERMINING I_2 FIBERS

In this appendix, we will explain how to solve the equations (18), (19) which determine the location of the codimension two I_2 fibers by finding the conditions for the quartic equation to be a square, as in (17). The first observation is that when the coefficient functions e_0, e_1, \dots, e_4 are generic, none of them will vanish at any of the solutions to (17).

In the case of e_4 , if e_4 vanishes at a solution then $u = 0$ is one of the double roots so e_3 must also vanish. For the remaining root to be double, we also need $e_1^2 = 4e_0e_2$ to vanish, but now we have three conditions on the base and the solutions are in codimension two. The case of e_0 is similar: if it vanishes, then e_1 and $e_3^2 - e_2e_4$ would both also have to vanish.

In the case of e_3 vanishing, β would need to vanish and then the equation would take the form $e_4((e_2/2e_4)u^2 + v^2)^2$. Again we get three conditions: $e_3 = 0$, $e_1 = 0$, and $e_2^2 = 4e_0e_4$ which is of too large a codimension to be generic. The case of e_1 is similar.

Finally, in the case of e_2 vanishing, we have an equation of the form

$$e_4(-(e_3^2/8e_4^2)u^2 + (e_3/2e_4)uv + v^2)^2,$$

and this implies the additional conditions $e_3^3 = -8e_4^2e_1$ and $e_3^4 = 64e_4^3e_0$. Once again we have three conditions and this is not possible.

Now we turn to the solution of (18), (19). As in §2.2, the first step is to introduce an auxiliary variable p , and to express the solutions as the common zeros of two auxiliary polynomials

$$(53) \quad \Phi_1 := p^4 - 8e_2e_4p^2 + 16e_2^2e_4^2 - 64e_0e_4^3$$

$$(54) \quad \Phi_2 := p^3 - 4e_2e_4p + 8e_1e_4^2$$

(together with the equation $e_3 = p$). From this, we can form additional polynomials which must vanish on the solution, roughly following the Gröbner basis algorithm (but allowing division by e_1 , e_2 or e_4 , which are known not to vanish on solutions). This gives the following sequence of polynomials:

$$(55) \quad \Phi_3 := (-\Phi_1 + p\Phi_2)/4e_4 = e_2p^2 + 2e_1e_4p - 4e_2^2e_4 + 16e_0e_4^2$$

$$(56) \quad \Phi_4 := (-e_2\Phi_2 + p\Phi_3)/2e_4 = e_1p^2 + 8e_0e_4p - 4e_1e_2e_4$$

$$(57) \quad \Phi_5 := (e_1\Phi_3 - e_2\Phi_4)/2e_4 = (e_1^2 - 4e_0e_2)p + 8e_0e_1e_4$$

$$(58) \quad \Phi_6 := ((4e_0e_2 - e_1^2)\Phi_4 + pe_1\Phi_5)/4e_2e_4 = 8e_0^2p - e_1(4e_0e_2 - e_1^2)$$

$$(59) \quad \Phi_7 := (8e_0^2\Phi_5 + (4e_0e_2 - e_1^2)\Phi_6)/e_1 = 64e_0^3e_4 - (4e_0e_2 - e_1^2)^2.$$

The variable p has been eliminated from Φ_7 , so the equation $\Phi_7 = 0$ gives the condition for a p to exist (this is (22)). The equation $\Phi_5 = 0$ can then be solved for p ; this gives (23).

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