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EULER CHARACTERISTIC RECIPROCITY FOR CHROMATIC, FLOW AND ORDER POLYNOMIALS

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Abstract. The Euler characteristic of a semialgebraic set can be considered as a generalization of the cardinality of a finite set. An advantage of semialgebraic sets is that we can define "negative sets" to be the sets with negative Euler characteristics. Applying this idea to posets, we introduce the notion of semialgebraic posets. Using "negative posets", we establish Stanley's reciprocity theorems for order polynomials at the level of Euler characteristics. We also formulate the Euler characteristic reciprocities for chromatic and flow polynomials.

1. Introduction

Let P be a finite poset. The *order polynomial* $\mathcal{O}^{\leq}(P,t)\in\mathbb{Q}[t]$ and the *strict order polynomial* $\mathcal{O}^{<}(P,t)\in\mathbb{Q}[t]$ are polynomials which satisfy

$$\begin{aligned}\mathcal{O}^{\leq}(P,n)&=\#\operatorname{Hom}^{\leq}(P,[n]),\\ \mathcal{O}^{<}(P,n)&=\#\operatorname{Hom}^{<}(P,[n]),\end{aligned}\tag{1}$$

where $[n]=\{1,\dots,n\}$ with the usual ordering and

$$\operatorname{Hom}^{\leq(<)}(P,[n])=\{f:P\longrightarrow[n]\mid x<y\Longrightarrow f(x)\leq(<)f(y)\}$$

is the set of increasing (resp. strictly increasing) maps.

These two polynomials are related to each other by the following reciprocity theorem proved by Stanley ([10, 11], see also [1, 3, 4] for recent surveys).

$$\mathcal{O}^{<}(P,t)=(-1)^{\#P}\cdot\mathcal{O}^{\leq}(P,-t).\tag{2}$$

By putting $t=n$, the formula (2) can be informally presented as follows.

$$\text{“}\,\#\operatorname{Hom}^{<}(P,[n])=(-1)^{\#P}\cdot\#\operatorname{Hom}^{\leq}(P,[-n]).\,\text{”}\tag{3}$$

It is a natural problem to extend the above reciprocity to homomorphisms between arbitrary (finite) posets P and Q. We may expect a formula of the following type.

$$\text{“}\,\#\operatorname{Hom}^{<}(P,Q)=(-1)^{\#P}\cdot\#\operatorname{Hom}^{\leq}(P,-Q).\,\text{”}\tag{4}$$

Of course this is not a mathematically justified formula. In fact, we do not have the notion of a "negative poset $-Q$."

In [9], Schanuel discussed what "negative sets" should be. A possible answer is that a negative set is nothing but a semialgebraic set which has a negative Euler characteristic (Table 1). For

Finite set	Semialgebraic set
Cardinality	Euler characteristic

Table 1. Negative sets

example, the open simplex

$$\overset{\circ}{\sigma}_d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid 0 < x_1 < \cdots < x_d < 1\}$$

has the Euler characteristic $e(\overset{\circ}{\sigma}_d) = (-1)^d$, and the closed simplex

$$\sigma_d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid 0 \le x_1 \le \cdots \le x_d \le 1\}$$

has $e(\sigma_d) = 1$. Thus we have the following "reciprocity"

$$e(\overset{\circ}{\sigma}_d) = (-1)^d \cdot e(\sigma_d). \tag{5}$$

This formula looks like Stanley's reciprocity (2). This analogy would indicate that (2) could be explained via the computations of Euler characteristic of certain semialgebraic sets.

In this paper, by introducing the notion of *semialgebraic posets*, we settle Euler characteristic reciprocity theorems for poset homomorphisms. Semialgebraic posets also provide a rigorous formulation for the reciprocity (4). A similar idea works also for reciprocities of chromatic and flow polynomials.

Briefly, a semialgebraic poset P is a semialgebraic set with poset structure such that the ordering is defined semialgebraically (see Definition 2.2). Finite posets and the open interval $(0, 1) \subset \mathbb{R}$ are examples of semialgebraic posets. A semialgebraic poset P has the Euler characteristic $e(P) \in \mathbb{Z}$ which is an invariant of semialgebraic structure of P (see §2.1). In particular, if P is a finite poset, then $e(P) = \#P$, and if P is the open interval $(0, 1)$, then $e((0, 1)) = -1$.

The philosophy presented in the literature [9] leads one to consider the "moduli space" $\mathrm{Hom}^{\le(<)}(P, Q)$ of poset homomorphisms from a finite poset P to a semialgebraic poset Q, and then to compute the Euler characteristic of the moduli space instead of counting the number of maps.

Considering the space $\mathrm{Hom}^{\le(<)}(P, Q)$ itself and its Euler characteristic is not a new idea for the chromatic theory of finite graphs. For example, in [8], the Euler characteristic of the space of colorings is explored, and in [14] the functorial aspects of colorings are studied. The essential reasons why the Euler characteristic works well in these situations are its additivity properties and its consistency with the inclusion-exclusion principle.

The point of the present paper is to introduce the negative of a poset Q in the category of semialgebraic posets. We define $-Q := Q \times (0, 1)$ (See Definition 3.1). Then we have $e(-Q) = -e(Q)$. Furthermore, we have the following result.

Theorem 1.1 (Proposition 2.8 and Theorems 3.3, 3.7)**.** *Let P be a finite poset and Q be a semialgebraic poset.*

(i) $\mathrm{Hom}^{\le}(P, Q)$ *and* $\mathrm{Hom}^{<}(P, Q)$ *possess the structure of semialgebraic sets.*

(ii) *The following reciprocity of Euler characteristics holds,*

$$e(\mathrm{Hom}^{<}(P, \pm Q)) = (-1)^{\#P} \cdot e(\mathrm{Hom}^{\le}(P, \mp Q)).$$

(iii) *Let T be a semialgebraic totally ordered set. Then*

$$e(\mathrm{Hom}^{\le}(P, T)) = \mathcal{O}^{\le}(P, e(T)),$$
$$e(\mathrm{Hom}^{<}(P, T)) = \mathcal{O}^{<}(P, e(T)).$$

The most important result is the second assertion (ii) which is a rigorous formulation of the reciprocity (4). It should be emphasized that (ii) is a substantially new result since Q need not be a totally ordered set. When we specialize to the totally ordered sets $Q = [n]$ and $T = [n] \times (0, 1)$, our (ii) and (iii) recover Stanley's reciprocity (2) for order polynomials (see §3.3).

Similar Euler characteristic reciprocities are obtained also for Stanley's chromatic polynomials reciprocity [12] and for Breuer and Sanyal's flow polynomials reciprocity [6].

This paper is organized as follows. In §2, we introduce semialgebraic posets, semialgebraic abelian groups and Euler characteristics. In §3, we prove the main result, Theorem 1.1 (ii). The proof is based on topological (cut and paste) arguments. We also deduce Stanley's reciprocity (2) from the main theorem. In §4, we describe other Euler characteristic reciprocities for chromatic polynomials of simple graphs and flow polynomials of oriented graphs.

2. Semialgebraic posets and Euler characteristics

2.1. **Semialgebraic sets.** A subset $X \subset \mathbb{R}^n$ is said to be a semialgebraic set if it is expressed as a Boolean connection (i.e., a set expressed by a finite combination of $\cup, \cap$ and complements) of subsets of the form

$$\{x \in \mathbb{R}^n \mid p(x) > 0\},$$

where $p(x) \in \mathbb{R}[x_1, \dots, x_n]$ is a polynomial. Let $f\colon X \longrightarrow Y$ be a map (not necessarily continuous) between semialgebraic sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. It is called semialgebraic if the graph

$$\Gamma(f) = \{(x, f(x)) \mid x \in X\} \subset \mathbb{R}^{m+n}$$

is a semialgebraic set. If f is semialgebraic then the pull-back $f^{-1}(Y)$ and the image $f(X)$ are also semialgebraic sets (see [2, 5] for details).

Any semialgebraic set X has a finite partition into Nash cells (see [7] for details), namely, a partition $X = \bigsqcup_{\alpha=1}^{k} X_\alpha$ such that X_α is Nash diffeomorphic (that is a semialgebraic analytic diffeomorphism) to the open cell $(0,1)^{d_\alpha}$ for some $d_\alpha \geq 0$. Then the *Euler characteristic*

$$e(X) := \sum_{\alpha=1}^{k} (-1)^{d_\alpha} \tag{6}$$

is independent of the partition [7]. Moreover, the Euler characteristic satisfies

$$\begin{aligned} e(X \sqcup Y) &= e(X) + e(Y), \\ e(X \times Y) &= e(X) \times e(Y). \end{aligned}$$

Example 2.1. As mentioned in §1, the closed simplex σ_d and the open simplex $\mathring{\sigma}_d$ have $e(\sigma_d) = 1$ and $e(\mathring{\sigma}_d) = (-1)^d$.

2.2. **Semialgebraic posets.**

Definition 2.2. $(P, \leq)$ is called a *semialgebraic poset* if

(a) $(P, \leq)$ is a partially ordered set, and
(b) there is an injection $i\colon P \hookrightarrow \mathbb{R}^n$ ($n \geq 0$) such that the image $i(P)$ is a semialgebraic set and the image of
$$\{(x, y) \in P \times P \mid x \leq y\}$$
by the map $i \times i\colon P \times P \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$, is also a semialgebraic subset of $\mathbb{R}^n \times \mathbb{R}^n$.

Let P and Q be semialgebraic posets. The set of homomorphisms (strict homomorphisms) of semialgebraic posets is defined by

$$\mathrm{Hom}^{\leq(<)}(P, Q) = \left\{ f\colon P \longrightarrow Q \;\middle|\; \begin{array}{l} f \text{ is a semialgebraic map s.t.} \\ x < y \Longrightarrow f(x) \leq (<) f(y) \end{array} \right\}. \tag{7}$$

Example 2.3. (a) A finite poset $(P, \leq)$ admits the structure of a semialgebraic poset, since any finite subset in $\mathbb{R}^n$ is a semialgebraic set. A finite poset has the Euler characteristic $e(P) = \#P$.

(b) The open interval $(0,1)$ and the closed interval $[0,1]$ are semialgebraic posets with respect to the usual ordering induced from $\mathbb{R}$. Their Euler characteristics are $e((0,1)) = -1$ and $e([0,1]) = 1$, respectively.

In this paper, we always consider the following lexicographic ordering on the product $P \times Q$.

Definition 2.4. Let P and Q be posets. Define an ordering on $P \times Q$ by

$$(p_1, q_1) \leq (p_2, q_2) \Longleftrightarrow \begin{cases} p_1 < p_2, \text{ or} \\ p_1 = p_2 \text{ and } q_1 \leq q_2, \end{cases}$$

for $(p_i, q_i) \in P \times Q$.

Remark 2.5. There are several ways to define poset structures on the product $P \times Q$. However, the lexicographic ordering in Definition 2.4 seems to be the only one that works for our purposes. In particular, the decomposition (18) in §3.2 is crucial.

Proposition 2.6. Let P and Q be semialgebraic posets. Then the product poset $P \times Q$ (with lexicographic ordering) admits the structure of a semialgebraic poset.

Proof. Suppose $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$. Then

$$\begin{aligned}
&\{((p_1, q_1), (p_2, q_2)) \in (P \times Q)^2 \mid (p_1, q_1) \leq (p_2, q_2)\} \\
&= \{(p_1, q_1, p_2, q_2) \in (P \times Q)^2 \mid (p_1 < p_2) \text{ or } (p_1 = p_2 \text{ and } q_1 \leq q_2)\} \\
&\simeq \left(\{(p_1, p_2) \in P^2 \mid p_1 < p_2\} \times Q^2\right) \sqcup \left(P \times \{(q_1, q_2) \in Q^2 \mid q_1 \leq q_2\}\right)
\end{aligned}$$

is also semialgebraic since semialgebraicity is preserved by disjoint union, complement and Cartesian products. $\square$

Proposition 2.7. Let P and Q be semialgebraic posets. Then the projection onto the first factor $\pi\colon P \times Q \longrightarrow P$ is a homomorphism of semialgebraic posets.

Proof. This is straightforward from the definition of the lexicographic ordering. $\square$

The next result shows that the "moduli space" of homomorphisms from a finite poset to a semialgebraic poset has the structure of a semialgebraic set.

Proposition 2.8 (Theorem 1.1 (i))**.** Let P be a finite poset and Q be a semialgebraic poset. Then $\mathrm{Hom}^{\leq}(P, Q)$ and $\mathrm{Hom}^{<}(P, Q)$ have structures of semialgebraic sets.

Proof. Let us set $P = \{p_1, \ldots, p_n\}$ and $\mathcal{L} = \{(i,j) \mid p_i < p_j\}$. Since each element $f \in \mathrm{Hom}^{\leq}(P, Q)$ can be identified with the tuple $(f(p_1), \ldots, f(p_n)) \in Q^n$, we have the expression

$$\begin{aligned}
\mathrm{Hom}^{\leq}(P, Q) &\simeq \{(q_1, \ldots, q_n) \in Q^n \mid q_i \leq q_j \text{ for } (i,j) \in \mathcal{L}\} \\
&= \bigcap_{(i,j) \in \mathcal{L}} \{(q_1, \ldots, q_n) \in Q^n \mid q_i \leq q_j\}.
\end{aligned}$$

Clearly, the right-hand side is a semialgebraic set.

The semialgebraicity of $\mathrm{Hom}^{<}(P, Q)$ is proved similarly. $\square$

2.3. **Semialgebraic abelian groups.** An abelian group $(\mathcal{A}, +)$ is called a semialgebraic abelian group if there exists an injection $i\colon \mathcal{A} \hookrightarrow \mathbb{R}^n$ ($n \geq 0$) such that the image $i(\mathcal{A})$ is a semialgebraic set and the maps

$$+\colon i(\mathcal{A}) \times i(\mathcal{A}) \longrightarrow i(\mathcal{A}),\ (i(x), i(y)) \longmapsto i(x+y)$$
$$(-1)\colon i(\mathcal{A}) \longrightarrow i(\mathcal{A}),\ i(x) \longmapsto i(-x)$$

are semialgebraic maps. Finite abelian groups and the set of all real numbers $\mathbb{R}$ are semialgebraic abelian groups.

It is easy to see that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are semialgebraic abelian groups, then so is the product $\mathcal{A}_1 \times \mathcal{A}_2$.

3. Euler characteristic reciprocity

3.1. **The main result.**

Definition 3.1. For a semialgebraic poset Q, let us define the negative by $-Q := Q \times (0,1)$. (Recall that we consider the lexicographic ordering on $-Q$.)

Remark 3.2. Note that since $-(-Q) = (Q \times (0,1)) \times (0,1)$, $-(-Q)$ is not equal to Q.

The main theorem of this paper is the following.

Theorem 3.3 (Theorem 1.1 (ii))**.** *Let P be a finite poset and Q be a semialgebraic poset. Then*

$$e(\mathrm{Hom}^{<}(P, \pm Q)) = (-1)^{\#P} \cdot e(\mathrm{Hom}^{\leq}(P, \mp Q)).$$

In other words,

$$e(\mathrm{Hom}^{<}(P, Q)) = (-1)^{\#P} \cdot e(\mathrm{Hom}^{\leq}(P, Q \times (0,1))) \tag{8}$$

and

$$e(\mathrm{Hom}^{<}(P, Q \times (0,1))) = (-1)^{\#P} \cdot e(\mathrm{Hom}^{\leq}(P, Q)) \tag{9}$$

hold.

Note that since $-(-Q) \neq Q$ (Remark 3.2), two formulas (8) and (9) are not equivalent.

Before the proof of Theorem 3.3, we present an example which illustrates the main idea of the proof.

Example 3.4. Let $P = Q = \{1, 2\}$ with $1 < 2$. Clearly we have

$$\mathrm{Hom}^{<}(P, Q) = \{\mathrm{id}\}.$$

Let us describe $\mathrm{Hom}^{\leq}(P, Q \times (0,1))$. Note that $Q \times (0,1)$ is isomorphic to the semialgebraic totally ordered set $(1, \frac{3}{2}) \sqcup (2, \frac{5}{2})$ by the isomorphism

$$\varphi\colon Q \times (0,1) \longrightarrow \left(1, \frac{3}{2}\right) \sqcup \left(2, \frac{5}{2}\right), (a, t) \longmapsto a + \frac{t}{2}.$$

A homomorphism $f \in \mathrm{Hom}^{\leq}(P, Q \times (0,1))$ is described by the two values $f(1) = (a_1, t_1)$ and $f(2) = (a_2, t_2) \in Q \times (0,1)$. The condition imposed on a_1, a_2, t_1 and t_2 (by the inequality $f(1) \leq f(2)$) is

$$(a_1 < a_2),\ \text{or } (a_1 = a_2 \text{ and } t_1 \leq t_2),$$

which is equivalent to $a_1 + \frac{t_1}{2} \leq a_2 + \frac{t_2}{2}$. Therefore, the semialgebraic set $\mathrm{Hom}^{\leq}(P, Q \times (0,1))$ can be described as in Figure 1. Each diagonal triangle in Figure 1 has a stratification $\mathring{\sigma}_2 \sqcup \mathring{\sigma}_1$. Therefore the Euler characteristic is $e(\mathring{\sigma}_2 \sqcup \mathring{\sigma}_1) = e(\mathring{\sigma}_2) + e(\mathring{\sigma}_1) = (-1)^2 + (-1)^1 = 0$. On the

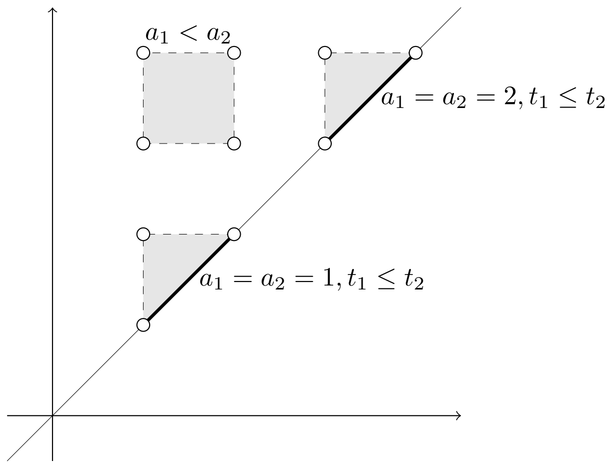

FIGURE 1. $f(1) \leq f(2)$.

other hand, the square region corresponding to $a_1 < a_2$ has the Euler characteristic $(-1)^2 = 1$. Hence we have

$$e(\mathrm{Hom}^{\leq}(P, Q \times (0,1))) = 1 = e(\mathrm{Hom}^{<}(P, Q)).$$

The following lemma will be used in the proof of Theorem 3.3.

Lemma 3.5. *Let $P \subset \mathbb{R}^n$ be a d-dimensional polytope which has a hyperplane description*

$$P = \{\alpha_1 \geq 0\} \cap \cdots \cap \{\alpha_N \geq 0\}$$

of P where α_i are affine maps from $\mathbb{R}^n$ to $\mathbb{R}$ (see [16]). For a given $x_0 \in P$, define the associated locally closed subset P_{x_0} of P (see Figure 2) by

$$P_{x_0} = \bigcap_{\alpha_i(x_0)=0} \{\alpha_i \geq 0\} \cap \bigcap_{\alpha_i(x_0)>0} \{\alpha_i > 0\}.$$

Then the Euler characteristic is

$$e(P_{x_0}) = \begin{cases} (-1)^d, & \text{if } x_0 \in \mathring{P} \\ 0, & \text{otherwise } (x_0 \in \partial P), \end{cases}$$

where $\mathring{P}$ is the relative interior of P and $\partial P = P \smallsetminus \mathring{P}$.

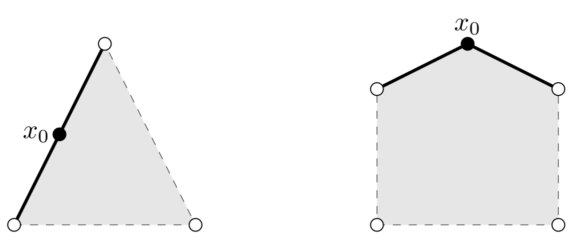

FIGURE 2. P_{x_0}.

Proof. If $x_0 \in \overset{\circ}{P}$, then $P_{x_0} = \overset{\circ}{P}$. The Euler characteristic is $e(\overset{\circ}{P}) = (-1)^d$.

Suppose $x_0 \in \partial P$. Then P_{x_0} can be expressed as

$$P_{x_0} = \bigsqcup_{F \ni x_0} \overset{\circ}{F}, \tag{10}$$

where F runs over the faces of P containing x_0 and $\overset{\circ}{F}$ denotes its relative interior. Then we obtain the decomposition

$$P_{x_0} = \overset{\circ}{P} \sqcup \bigsqcup_{F \ni x_0, F \subset \partial P} \overset{\circ}{F}.$$

We look at the structure of the second component $Z := \bigsqcup_{F \ni x_0, F \subset \partial P} \overset{\circ}{F}$. For any point $y \in Z$, the segment $[x_0, y]$ is contained in Z. Hence Z is contractible open subset of ∂P, which is homeomorphic to the $(d-1)$-dimensional open disk. The Euler characteristic is computed as

$$\begin{aligned} e(P_{x_0}) &= e(\overset{\circ}{P}) + e(Z) \\ &= (-1)^d + (-1)^{d-1} \\ &= 0. \end{aligned}$$

□

3.2. **Proof of the main result.** Now we prove Theorem 3.3. The strategy is to decompose the space $\mathrm{Hom}^{\leq}(P, -Q)$ into appropriate semialgebraic subsets, and then to apply Lemma 3.5 to compute the Euler characteristics.

We first prove (8). Let $\varphi \in \mathrm{Hom}^{<}(P, Q \times (0,1))$. Then φ is a pair of maps

$$\varphi = (f, g),$$

where $f\colon P \longrightarrow Q$ and $g\colon P \longrightarrow (0,1)$. Let $\pi_1\colon Q \times (0,1) \longrightarrow Q$ be the projection onto the first factor. Since π_1 is order-preserving (Proposition 2.7), so is $f = \pi_1 \circ \varphi$, and hence $f \in \mathrm{Hom}^{\leq}(P, Q)$.

In order to compute the Euler characteristics, we consider the map

$$\pi_{1*}\colon \mathrm{Hom}^{\leq}(P, Q \times (0,1)) \longrightarrow \mathrm{Hom}^{\leq}(P, Q),\ \varphi \longmapsto \pi_1 \circ \varphi = f. \tag{11}$$

Let us set

$$\begin{aligned} M :=& \mathrm{Hom}^{\leq}(P, Q) \smallsetminus \mathrm{Hom}^{<}(P, Q) \\ =& \{f \in \mathrm{Hom}^{\leq}(P, Q) \mid \exists x < y \in P \text{ s.t. } f(x) = f(y)\}. \end{aligned} \tag{12}$$

Then obviously, we have

$$\mathrm{Hom}^{\leq}(P, Q) = \mathrm{Hom}^{<}(P, Q) \sqcup M. \tag{13}$$

This decomposition induces that of $\mathrm{Hom}^{\leq}(P, Q \times (0,1))$,

$$\mathrm{Hom}^{\leq}(P, Q \times (0,1)) = \pi_{1*}^{-1}\left(\mathrm{Hom}^{<}(P, Q)\right) \sqcup \pi_{1*}^{-1}(M). \tag{14}$$

By the additivity of the Euler characteristics, we obtain

$$e\left(\mathrm{Hom}^{\leq}(P, Q \times (0,1))\right) = e\left(\pi_{1*}^{-1}\left(\mathrm{Hom}^{<}(P, Q)\right)\right) + e(\pi_{1*}^{-1}(M)). \tag{15}$$

We claim the following two equalities which are sufficient for the proof of (8).

$$e\left(\pi_{1*}^{-1}\left(\mathrm{Hom}^{<}(P, Q)\right)\right) = (-1)^{\#P} \cdot e\left(\mathrm{Hom}^{<}(P, Q)\right), \tag{16}$$

$$e(\pi_{1*}^{-1}(M)) = 0. \tag{17}$$

We first prove (16). Let $\varphi \in \pi_{1*}^{-1}\left(\mathrm{Hom}^{<}(P,Q)\right)$, that is $\varphi = (f,g)$ with $f \in \mathrm{Hom}^{<}(P,Q)$. By the definition of the ordering of $Q \times (0,1)$, for every $g\colon P \longrightarrow (0,1)$ the pair (f,g) is contained in $\pi_{1*}^{-1}\left(\mathrm{Hom}^{<}(P,Q)\right)$. This implies

$$\pi_{1*}^{-1}\left(\mathrm{Hom}^{<}(P,Q)\right) \simeq \mathrm{Hom}^{<}(P,Q) \times (0,1)^{\#P}, \tag{18}$$

which yields (16).

The proof of (17) requires further stratification of M. Let

$$\mathcal{L}(P) := \{(p_1,p_2) \in P \times P \mid p_1 < p_2\}.$$

For given $f \in M$, consider the set of collapsing pairs,

$$K(f) := \{(p_1,p_2) \in \mathcal{L}(P) \mid f(p_1) = f(p_2)\}.$$

Note that $f \in M$ if and only if $K(f) \neq \emptyset$. We decompose M according to $K(f)$. Namely, for any nonempty subset $X \subset \mathcal{L}(P)$ define a subset $M_X \subset M$ by

$$M_X := \{f \in M \mid K(f) = X\}.$$

Since $\mathcal{L}(P)$ is a finite set,

$$M = \bigsqcup_{\substack{X \subset \mathcal{L}(P)\\ X \neq \emptyset}} M_X \tag{19}$$

is a decomposition of M into finitely many semialgebraic sets. Therefore, we obtain

$$e(\pi_{1*}^{-1}(M)) = \sum_{\substack{X \subset \mathcal{L}(P)\\ X \neq \emptyset}} e(\pi_{1*}^{-1}(M_X)).$$

Thus it is enough to show $e(\pi_{1*}^{-1}(M_X)) = 0$ for all $X \subset \mathcal{L}(P)$ as long as $\pi_{1*}^{-1}(M_X) \neq \emptyset$ (note that $\pi_{1*}^{-1}(M_X) = \emptyset$ can occur for a nonempty X e.g. when $\#Q = 1$).

Now we fix $X \subset \mathcal{L}(P)$ such that $\pi_{1*}^{-1}(M_X) \neq \emptyset$. Then we can show that $\pi_{1*}^{-1}(M_X) \longrightarrow M_X$ is a trivial fibration. Indeed, for any $f \in M_X$, the condition imposed on g by

$$(f,g) \in \mathrm{Hom}^{\leq}(P, Q \times (0,1))$$

is

$$(p_1,p_2) \in X \Longrightarrow g(p_1) \leq g(p_2).$$

Hence the fiber $\pi_{1*}^{-1}(f)$ is independent of $f \in M_X$ and isomorphic to

$$F_X := \{(t_p)_{p\in P} \in (0,1)^P \mid (p_1,p_2) \in X \Longrightarrow t_{p_1} \leq t_{p_2}\}, \tag{20}$$

and we have

$$\pi_{1*}^{-1}(M_X) \simeq M_X \times F_X. \tag{21}$$

The fiber F_X is a locally closed polytope defined by the following inequalities.

$$0 < t_p < 1, t_{p_1} \leq t_{p_2} \text{ for } (p_1,p_2) \in X.$$

The closure $\overline{F_X}$ is defined by

$$\overline{F_X} = \{(t_p)_{p\in P} \in [0,1]^P \mid t_{p_1} \leq t_{p_2} \text{ for } (p_1,p_2) \in X\}.$$

Then F_X is equal to the locally closed polytope $(\overline{F_X})_{x_0}$ associated to the point

$$x_0 = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in \partial\overline{F_X}.$$

Since $X \neq \emptyset$, x_0 is not contained in the interior of $\overline{F_X}$. By Lemma 3.5, $e(F_X) = 0$. Together with (21), we conclude $e(\pi_{1*}^{-1}(M_X)) = 0$. This completes the proof of (8) of Theorem 3.3.

The proof of the other formula (9) is similar to and actually simpler than that of (8) since we do not need Lemma 3.5. Again the first projection $\pi_1 : Q \times (0,1) \longmapsto Q$ induces the map

$$\pi_{1*} : \mathrm{Hom}^{<}(P, Q \times (0,1)) \longrightarrow \mathrm{Hom}^{\leq}(P, Q).$$

We can prove that this map is surjective and each fiber of $\pi_{1*}^{-1}(M_X)$ (now $X = \emptyset$ is allowed) is isomorphic to

$$\mathring{F}_X = \{(t_p)_{p \in P} \in (0,1)^P \mid t_{p_1} < t_{p_2} \text{ for all } (p_1, p_2) \in X\}.$$

This fiber is an open polytope of dimension $\#P$ and hence is isomorphic to $(0,1)^{\#P}$ whose Euler characteristic is $(-1)^{\#P}$. Thus we obtain

$$\begin{aligned} e(\mathrm{Hom}^{<}(P, Q \times (0,1))) &= \sum_{X \subset \mathcal{L}(P)} e(\pi_{1*}^{-1}(M_X)) = \sum_{X \subset \mathcal{L}(P)} e(M_X \times \mathring{F}_X) \\ &= \sum_{X \subset \mathcal{L}(P)} e(M_X) \cdot (-1)^{\#P} = (-1)^{\#P} \cdot e\left(\bigsqcup_{X \subset \mathcal{L}(P)} M_X \right) \\ &= (-1)^{\#P} \cdot e(\mathrm{Hom}^{\leq}(P, Q)). \end{aligned}$$

This completes the proof.

3.3. **Stanley's reciprocity for order polynomials.** In this section, we deduce Stanley's reciprocity (2) from Theorem 3.3. The idea is to take semialgebraic totally ordered posets as the target posets.

Example 3.6. Any semialgebraic set $X \subset \mathbb{R}$ with the induced ordering is a semialgebraic totally ordered set. Furthermore, since $\mathbb{R}^n$ is totally ordered by the lexicographic ordering, any semialgebraic set $X \subset \mathbb{R}^n$ admits the structure of a semialgebraic totally ordered set.

The Euler characteristic of $\mathrm{Hom}^{\leq}(P, T)$, with T a semialgebraic totally ordered set, can be computed by using the order polynomial $\mathcal{O}^{\leq(<)}(P, t)$.

Theorem 3.7 (Theorem 1.1 (iii))**.** *Let P be a finite poset and T be a semialgebraic totally ordered set. Then*

$$e(\mathrm{Hom}^{\leq}(P, T)) = \mathcal{O}^{\leq}(P, e(T)), \tag{22}$$

$$e(\mathrm{Hom}^{<}(P, T)) = \mathcal{O}^{<}(P, e(T)). \tag{23}$$

Before proving Theorem 3.7, we need several lemmas on the Euler characteristics of configuration spaces.

Definition 3.8. Let X be a semialgebraic set. The ordered configuration space of n-points on X, denoted by $C_n(X)$, is defined by

$$C_n(X) = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}.$$

Lemma 3.9. $e(C_n(X)) = e(X) \cdot (e(X) - 1) \cdots (e(X) - n + 1)$.

Proof. It is proved by induction. When $n = 1$, it is obvious from $C_1(X) = X$. Suppose $n > 1$. Consider the projection

$$\pi : C_n(X) \longrightarrow C_{n-1}(X), (x_1, \ldots, x_n) \longmapsto (x_1, \ldots, x_{n-1}).$$

Then the fiber of π at the point $(x_1, \ldots, x_{n-1}) \in C_{n-1}(X)$ is

$$X \setminus \{x_1, \ldots, x_{n-1}\},$$

which has the Euler characteristic

$$e(X \smallsetminus \{x_1, \dots, x_{n-1}\}) = e(X) - (n-1).$$

Therefore, from the inductive assumption, we have

$$\begin{aligned} e(C_n(X)) &= e(C_{n-1}(X)) \cdot (e(X) - n + 1) \\ &= e(X) \cdot (e(X) - 1) \cdots (e(X) - n + 1). \end{aligned}$$

□

Remark 3.10. We will give a stronger result later (Theorem 4.2 and Corollary 4.3).

Lemma 3.11. *Let T be a semialgebraic totally ordered set. Then*

$$e(\mathrm{Hom}^{<}([n], T)) = \frac{e(T) \cdot (e(T) - 1) \cdots (e(T) - n + 1)}{n!}. \tag{24}$$

Proof. The set

$$\mathrm{Hom}^{<}([n], T) = \{(x_1, \dots, x_n) \in T^n \mid x_1 < \cdots < x_n\}$$

is obviously a subset of the configuration space $C_n(T)$. Moreover, using the natural action of the symmetric group $\mathfrak{S}_n$ on $C_n(T)$ and the fact that T is totally ordered, we have

$$C_n(T) = \bigsqcup_{\sigma \in \mathfrak{S}_n} \sigma(\mathrm{Hom}^{<}([n], T)).$$

Since the group action preserves the Euler characteristic, we obtain the following.

$$e(C_n(T)) = n! \cdot e(\mathrm{Hom}^{<}([n], T)).$$

□

Proof of Theorem 3.7. We fix $\varepsilon \in \{\leq, <\}$. Let $f \in \mathrm{Hom}^{\varepsilon}(P, T)$. Since P is a finite poset, the image $f(P) \subset T$ is a finite totally ordered set. Suppose $\# f(P) = k$. Then the map f is decomposed as

$$f \colon P \xrightarrow{\alpha} [k] \xrightarrow{\beta} T,$$

where $\alpha \colon P \longrightarrow [k]$ is surjective while $\beta \colon [k] \longrightarrow T$ is injective. Hence β can be considered as an element of $\mathrm{Hom}^{<}([k], T)$, and we have the following decomposition,

$$\mathrm{Hom}^{\varepsilon}(P, T) = \bigsqcup_{k \geq 1} \mathrm{Hom}^{\varepsilon, \mathrm{surj}}(P, [k]) \times \mathrm{Hom}^{<}([k], T), \tag{25}$$

where $\mathrm{Hom}^{\varepsilon, \mathrm{surj}}(P, [k])$ is the set of surjective maps in $\mathrm{Hom}^{\varepsilon}(P, [k])$. By putting $T = [n]$ and then extending n to real numbers t, we obtain the expression for the (strict) order polynomial,

$$\mathcal{O}^{\varepsilon}(P, t) = \sum_{k \geq 1} \# \mathrm{Hom}^{\varepsilon, \mathrm{surj}}(P, [k]) \cdot \frac{t(t-1) \cdots (t-k+1)}{k!}, \tag{26}$$

which was already obtained by Stanley [10, Theorem 1]. Using (25), Lemma 3.11 and (26), we have

$$\begin{aligned} e(\mathrm{Hom}^{\varepsilon}(P, T)) &= \sum_{k \geq 1} e(\mathrm{Hom}^{\varepsilon, \mathrm{surj}}(P, [k])) \cdot e(\mathrm{Hom}^{<}([k], T)) \\ &= \sum_{k \geq 1} \# \mathrm{Hom}^{\varepsilon, \mathrm{surj}}(P, [k]) \cdot \frac{e(T)(e(T) - 1) \cdots (e(T) - k + 1)}{k!} \\ &= \mathcal{O}^{\varepsilon}(P, e(T)). \end{aligned}$$

This completes the proof of Theorem 3.7. □

Corollary 3.12. *(Stanley's reciprocity [10]) Let P be a finite poset and $n \in \mathbb{N}$. Then*

$$\# \operatorname{Hom}^{<}(P,[n]) = (-1)^{\#P} \cdot \mathcal{O}^{\leq}(P,-n). \tag{27}$$

Proof. Since $\operatorname{Hom}^{<}(P,[n])$ is a finite poset, the cardinality is equal to the Euler characteristic: $\# \operatorname{Hom}^{<}(P,[n]) = e(\operatorname{Hom}^{<}(P,[n]))$. We apply the Euler characteristic reciprocity (Theorem 3.3),

$$e(\operatorname{Hom}^{<}(P,[n])) = (-1)^{\#P} \cdot e(\operatorname{Hom}^{\leq}(P,[n] \times (0,1))).$$

Note that $[n] \times (0,1)$ is a semialgebraic totally ordered set (with the lexicographic ordering) with the Euler characteristic $e([n] \times (0,1)) = -n$. Applying Theorem 3.7, we have

$$e(\operatorname{Hom}^{\leq}(P,[n] \times (0,1))) = \mathcal{O}^{\leq}(P,-n),$$

which implies (27). □

4. Chromatic and flow polynomials for finite graphs

In this section, we formulate Euler characteristic reciprocities for chromatic polynomials of finite simple graphs and for flow polynomials of finite oriented graphs.

4.1. **Chromatic polynomials.**

Let $G = (V,E)$ be a finite simple graph with vertex set V and (un-oriented) edge set E. The chromatic polynomial is a polynomial $\chi(G,t) \in \mathbb{Z}[t]$ which satisfies

$$\chi(G,n) = \#\{c \colon V \longrightarrow [n] \mid v_1 v_2 \in E \Longrightarrow c(v_1) \neq c(v_2)\},$$

for all $n > 0$. The chromatic polynomial is also characterized by the following properties:

- if $E = \emptyset$ then $\chi(G,t) = t^{\#V}$;
- if $e \in E$, then $\chi(G,t) = \chi(G-e,t) - \chi(G/e,t)$, where $G-e$ and G/e are the deletion and the contraction with respect to the edge e, respectively.

(See [15] for these terminologies and basic properties of chromatic polynomials.)

Definition 4.1. Given a set X, define the set of vertex coloring with X (or the graph configuration space) by

$$\underline{\chi}(G,X) = \{c \colon V \longrightarrow X \mid v_1 v_2 \in E \Longrightarrow c(v_1) \neq c(v_2)\}. \tag{28}$$

The assignment $X \longmapsto \underline{\chi}(G,X)$ can be considered as a functor [14]. The space $\underline{\chi}(G,X)$ is also called the graph (generalized) configuration space [8].

The chromatic polynomial $\chi(G,t) \in \mathbb{Z}[t]$ satisfies $\chi(G,n) = \#\underline{\chi}(G,[n])$ for all $n \in \mathbb{N}$.

In this section, we investigate the Euler characteristic aspects of the chromatic polynomial for a finite simple graph.

When X is a semialgebraic set, $\underline{\chi}(G,X)$ is also a semialgebraic set. The following result generalizes [8, Theorem 2], where the result is proved when X is a complex projective space.

Theorem 4.2. *Let $G = (V,E)$ be a finite simple graph and X be a semialgebraic set. Then*

$$e(\underline{\chi}(G,X)) = \chi(G,e(X)). \tag{29}$$

Proof. This result is proved by induction on $\#E$. When $E = \emptyset$,

$$e(\underline{\chi}(G,X)) = e(X^{\#V}) = e(X)^{\#V} = \chi(G,e(X)).$$

Suppose $e \in E$. Then we can prove

$$\underline{\chi}(G-e,X) \simeq \underline{\chi}(G,X) \sqcup \underline{\chi}(G/e,X). \tag{30}$$

Using the additivity of the Euler characteristic and the recursive relation for the chromatic polynomial, we obtain (29). □

Note that for the complete graph $G = K_n$, $\underline{\chi}(K_n, X)$ is identical to the configuration space $C_n(X)$ of n-points. Applying Theorem 4.2 to the complete graph K_n (which has the chromatic polynomial $\chi(K_n, t) = t(t-1)\cdots(t-n+1)$), we have the following.

Corollary 4.3. $e(C_n(X)) = e(X)(e(X)-1)\cdots(e(X)-n+1)$.

To formulate the reciprocity for chromatic polynomials, we recall the notion of acyclic orientations on a graph G. (See [3, 12] for details.)

Let $G = (V, E)$ be a finite simple graph. The set of edges E can be considered as a subset of

$$(V \times V \smallsetminus \Delta)/\mathfrak{S}_2,$$

where $\Delta = \{(v, v) \mid v \in V\}$ is the diagonal subset and $\mathfrak{S}_2$ acts on $V \times V$ by transposition. There is a natural projection

$$\pi\colon V \times V \smallsetminus \Delta \longrightarrow (V \times V \smallsetminus \Delta)/\mathfrak{S}_2.$$

An edge orientation on G is a subset $\widetilde{E} \subset V \times V \smallsetminus \Delta$ such that $\pi|_{\widetilde{E}}\colon \widetilde{E} \xrightarrow{\simeq} E$ is a bijection. An orientation $\widetilde{E}$ is said to contain an oriented cycle, if there exists a cyclic sequence $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1) \in \widetilde{E}$ for some $n > 2$. The orientation $\widetilde{E}$ is called *acyclic* if it does not contain oriented cycles.

Definition 4.4. Let $G = (V, E)$ be a finite simple graph. Fix an acyclic orientation $\widetilde{E} \subset V \times V \smallsetminus \Delta$. Let T be a totally ordered set.

(a) A map $c\colon V \longrightarrow T$ is said to be *compatible* with $\widetilde{E}$ if

$$(v, v') \in \widetilde{E} \Longrightarrow c(v) \le c(v').$$

(b) A map $c\colon V \longrightarrow T$ is said to be *strictly compatible* with $\widetilde{E}$ if

$$(v, v') \in \widetilde{E} \Longrightarrow c(v) < c(v').$$

We denote the sets of all pairs of an acyclic orientation with a compatible map, and with a strictly compatible map, by

$$\mathcal{AOC}^{\le}(G, T) := \left\{ (\widetilde{E}, c) \;\middle|\; \begin{array}{l} \widetilde{E} \text{ is an acyclic orientation, and } c\colon V \to T \\ \text{is a map compatible with } \widetilde{E} \end{array} \right\},$$

and

$$\mathcal{AOC}^{<}(G, T) := \left\{ (\widetilde{E}, c) \;\middle|\; \begin{array}{l} \widetilde{E} \text{ is an acyclic orientation, and } c\colon V \to T \\ \text{is a map strictly compatible with } \widetilde{E} \end{array} \right\},$$

respectively.

If T is a semialgebraic totally ordered set, then these spaces possess the structure of semialgebraic sets. We will see a reciprocity between these two spaces from which Stanley's reciprocity for chromatic polynomials is deduced.

It is straightforward that $\mathcal{AOC}^{<}(G, T)$ can be identified with $\underline{\chi}(G, T)$. In particular, we have

$$e(\mathcal{AOC}^{<}(G, T)) = \chi(G, e(T)). \tag{31}$$

We formulate a reciprocity for chromatic polynomials in terms of Euler characteristics.

Theorem 4.5. *Let $G = (V, E)$ be a finite simple graph and T be a semialgebraic totally ordered set. Then*

$$e(\mathcal{AOC}^{\le}(G, T)) = (-1)^{\#V} \cdot e(\mathcal{AOC}^{<}(G, T \times (0, 1))), \tag{32}$$

$$e(\mathcal{AOC}^{<}(G, T)) = (-1)^{\#V} \cdot e(\mathcal{AOC}^{\le}(G, T \times (0, 1))). \tag{33}$$

To prove Theorem 4.5, we give alternative descriptions of $\mathcal{AOC}^{\leq(<)}(G,T)$ in terms of poset homomorphisms and graph configuration spaces. Let $\widetilde{E}$ be an acyclic orientation of $G=(V,E)$. Then $\widetilde{E}$ determines an ordering on V, called the transitive closure of $\widetilde{E}$, defined by

$$v < v' \Longleftrightarrow \exists v_0,\dots,v_n \in V \text{ s.t. } \begin{cases} v=v_0, v'=v_n, \text{ and} \\ (v_{i-1},v_i)\in \widetilde{E} \text{ for } 1\leq i\leq n. \end{cases}$$

This ordering defines a poset which we denote by $P(V,\widetilde{E})$.

A map $c\colon V \longrightarrow T$ is compatible with $\widetilde{E}$ if and only if c is an increasing map from $P(V,\widetilde{E})$ to T. Hence the set of maps compatible with $\widetilde{E}$ is identified with $\mathrm{Hom}^{\leq}(P(V,\widetilde{E}),T)$. We have the following decomposition.

$$\mathcal{AOC}^{\leq}(G,T) \simeq \bigsqcup_{\widetilde{E}\text{: acyclic ori.}} \mathrm{Hom}^{\leq}(P(V,\widetilde{E}),T). \tag{34}$$

Similarly, $\mathcal{AOC}^{<}(G,T)$ is decomposed as follows.

$$\mathcal{AOC}^{<}(G,T) \simeq \bigsqcup_{\widetilde{E}\text{: acyclic ori.}} \mathrm{Hom}^{<}(P(V,\widetilde{E}),T). \tag{35}$$

Proof of Theorem 4.5. We prove (32). Using the above decompositions (34) and (35) together with Theorem 3.3, we obtain

$$\begin{aligned} e(\mathcal{AOC}^{\leq}(G,T)) &= e\left(\bigsqcup_{\widetilde{E}\text{: acyclic ori.}} \mathrm{Hom}^{\leq}(P(V,\widetilde{E}),T)\right) \\ &= \sum_{\widetilde{E}\text{: acyclic ori.}} e\left(\mathrm{Hom}^{\leq}(P(V,\widetilde{E}),T)\right) \\ &= (-1)^{\#V}\cdot \sum_{\widetilde{E}\text{: acyclic ori.}} e\left(\mathrm{Hom}^{<}(P(V,\widetilde{E}),T\times(0,1))\right) \\ &= (-1)^{\#V}\cdot e\left(\bigsqcup_{\widetilde{E}\text{: acyclic ori.}} \mathrm{Hom}^{<}(P(V,\widetilde{E}),T\times(0,1))\right) \\ &= (-1)^{\#V}\cdot e(\mathcal{AOC}^{<}(G,T\times(0,1))). \end{aligned}$$

This completes the proof. The second formula (33) is proved similarly. □

We deduce Stanley's reciprocity on chromatic polynomials ([12]). Applying Theorem 4.5 and (31) shows that (note that $T\times(0,1)$ is also a semialgebraic totally ordered set)

$$\begin{aligned} e(\mathcal{AOC}^{\leq}(G,T)) &= (-1)^{\#V}\cdot e(\mathcal{AOC}^{<}(G,T\times(0,1))) \\ &= (-1)^{\#V}\cdot \chi(G,e(T\times(0,1)) \\ &= (-1)^{\#V}\cdot \chi(G,-e(T)). \end{aligned}$$

Putting $T=[n]$, we have the following Stanley's reciprocity.

Corollary 4.6. *Let $G=(V,E)$ be a finite simple graph and $n\in\mathbb{N}$. Then*

$$\#\,\mathcal{AOC}^{\leq}(G,[n]) = (-1)^{\#V}\cdot \chi(G,-n).$$

4.2. **Flow polynomials.** This section treats finite oriented graphs that are allowed to have distinguished multiple edges and loops. Our object is a tuple $G = (V, E, h, t)$ where V and E are finite sets and $h\colon E \longrightarrow V$ and $t\colon E \longrightarrow V$ are maps. An element of V is called a vertex and an element of E is called an edge. For an edge $e \in E$, $h(e) \in V$ is called the head and $t(e) \in V$ is called the tail. An edge $e \in E$ is a loop if $h(e) = t(e)$. In Figure 3, the oriented graph G has five edges $e_1, \ldots, e_5$ and their orientations are described by $h(e_1) = h(e_2) = t(e_3) = x$, $t(e_1) = t(e_2) = h(e_3) = h(e_4) = y$ and $t(e_4) = h(e_5) = t(e_5) = z$.

An oriented graph G can also be seen as a 1-dimensional CW-complex. The number of connected components and the 1-st Betti numbers are denoted by $b_0(G)$ and $b_1(G)$, respectively. Note that $b_0(G) - b_1(G) = \#V - \#E$. An edge $e \in E$ is called a coloop if $b_0(G \smallsetminus e) = b_0(G) + 1$. The graph in Figure 3 has the unique coloop e_4.

Let $\mathcal{A}$ be an abelian group. The map $f\colon E \longrightarrow \mathcal{A}$ is called an $\mathcal{A}$-flow if f satisfies

$$\sum_{e:h(e)=v} f(e) = \sum_{e:t(e)=v} f(e) \tag{36}$$

for all $v \in V$ (see [6, 15] more on the notion of flow and flow polynomials). Let f be an $\mathcal{A}$-flow. Denote $\operatorname{Supp}(f) = \{e \in E \mid f(e) \neq 0\}$. An $\mathcal{A}$-flow is called nowhere zero if $\operatorname{Supp}(f) = E$. The set of all $\mathcal{A}$-flows and nowhere zero $\mathcal{A}$-flows are denoted by $\mathcal{F}(G, \mathcal{A})$ and $\mathcal{F}^0(G, \mathcal{A})$, respectively.

Let $\mathcal{A}$ be a semialgebraic abelian group. Then clearly $\mathcal{F}^0(G, \mathcal{A})$ possesses a structure of a semialgebraic set.

The flow polynomial is a polynomial $\phi_G(t) \in \mathbb{Z}[t]$ which satisfies

$$\phi_G(k) = \#\mathcal{F}^0(G, \mathbb{Z}/k\mathbb{Z}),$$

for all $k > 0$. The flow polynomial is also characterized by the following properties:

- if $E = \emptyset$, then $\phi_G(t) = 1$;
- if $e \in E$ is a loop, then $\phi_G(t) = (t-1)\phi_{G\smallsetminus e}(t)$;
- if $e \in E$ is a coloop, then $\phi_G(t) = 0$;
- if $e \in E$ is neither a loop nor a coloop, then $\phi_G(t) = \phi_{G/e}(t) - \phi_{G\smallsetminus e}(t)$.

Proposition 4.7. Let G be a finite oriented graph, and $\mathcal{A}$ be a semialgebraic abelian group.

(a) If $e \in E$ is a loop, then $\mathcal{F}^0(G, \mathcal{A}) \simeq (\mathcal{A} \smallsetminus \{0\}) \times \mathcal{F}^0(G \smallsetminus e, \mathcal{A})$.
(b) If $e \in E$ is a coloop, then $\mathcal{F}^0(G, \mathcal{A}) = \emptyset$.
(c) If $e \in E$ is neither a loop nor a coloop, then $\mathcal{F}^0(G/e, \mathcal{A}) \simeq \mathcal{F}^0(G, \mathcal{A}) \sqcup \mathcal{F}^0(G \smallsetminus e, \mathcal{A})$.

Proof. Straightforward. □

Theorem 4.8. *Let G be a finite oriented graph and $\mathcal{A}$ be a semialgebraic abelian group. Then $e(\mathcal{F}^0(G, \mathcal{A})) = \phi_G(e(\mathcal{A}))$.*

Proof. Using Proposition 4.7, it is proved by induction on the number of edges. (See Theorem 4.2.) □

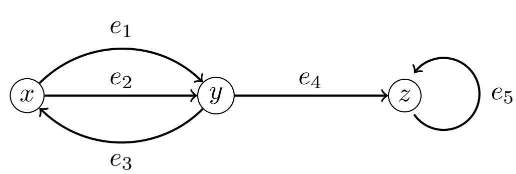

FIGURE 3. An oriented graph.

An oriented graph G is called totally cyclic if every edge is contained in an oriented cycle. Let $\sigma \subset E$ be a subset of edges and denote by ${}_{\sigma}G$ the reorientation of G along σ. A subset $\sigma \subset E$ is a totally cyclic reorientation if ${}_{\sigma}G$ is totally cyclic.

Let us denote by $\mathcal{FTC}(G, \mathcal{A})$ the set of all pairs (f, σ) of the flow f and totally cyclic reorientation $\sigma \subset E \smallsetminus \mathrm{Supp}(f)$. Namely,

$$\mathcal{FTC}(G, \mathcal{A}) = \left\{ (f, \sigma) \;\middle|\; \begin{array}{l} f \in \mathcal{F}(G, \mathcal{A}), \text{ and } \sigma \subset E \smallsetminus \mathrm{Supp}(f) \text{ is a} \\ \text{totally cyclic reorientation for } G_{/\,\mathrm{Supp}(f)} \end{array} \right\}.$$

For each subset $\sigma \subset E$, the set of all f with $(f, \sigma) \in \mathcal{FTC}(G, \mathcal{A})$ forms a semialgebraic subset of $\mathcal{F}(G, \mathcal{A})$. Therefore $\mathcal{FTC}(G, \mathcal{A})$ possesses a structure of semialgebraic set. Let us define $-\mathcal{A}$ by

$$-\mathcal{A} := \mathcal{A} \times \mathbb{R}.$$

The following is proved along the same lines of the proof presented in [6, Appendix A], which can be considered as a Breuer-Sanyal's reciprocity at the level of Euler characteristic.

Theorem 4.9. *Let G be a finite oriented graph and $\mathcal{A}$ be a semialgebraic abelian group. Then*

$$e(\mathcal{FTC}(G, \pm\mathcal{A})) = (-1)^{b_1(G)} e(\mathcal{F}^0(G, \mp\mathcal{A})).$$

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