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TOPOLOGICAL CLASSIFICATION OF CIRCLE–VALUED SIMPLE MORSE–BOTT FUNCTIONS

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Abstract. In this work, we investigate the classification of Morse-Bott functions from S^2 to S^1, up to topological conjugacy. We give a complete topological invariant of simple Morse-Bott functions $f : S^2 \to S^1$. The invariant is based on the generalized Reeb graph associated to f (called here MB-Reeb graph). Moreover, a realization theorem is obtained.

1. Introduction

Let $f, g : \mathbb{M}^n \to \mathbb{R}$ be two smooth functions defined in a n-manifold $\mathbb{M}$. We say that f and g are *topologically equivalent* if there exist homeomorphisms $h : \mathbb{M}^n \to \mathbb{M}^n$ and $k : \mathbb{R} \to \mathbb{R}$ such that

$$f = k \circ g \circ h^{-1}.$$

The topological classification problem of smooth functions is a classical subject in Topology and Singularity theory. However, global results and global invariants are difficult to obtain. Then some restrictions in the manifold $\mathbb{M}^n$ or in the function f can be considered. For instance, Fukuda [11] shows that there exist a finite number of topological equivalence classes if considered the space of all polynomials $f : \mathbb{M}^n = \mathbb{R}^n \to \mathbb{R}$ of limited degree. Prishlyak [18] gives a topological classification of smooth functions on a closed surface $\mathbb{M}^2$ with isolated critical points. For Morse functions on surfaces such topological classification was obtained by Sharko [20, 21] and Kulinich [13]. There is a classification for functions on surfaces with unique critical level (except minimum and maximum) that is embedded graph [14]. In [2], Arnold presents results on the number of topological equivalence classes for Morse functions.

A special case to consider is when $\mathbb{M}^n = S^n \subset \mathbb{R}^{n+1}$ (the standard sphere). For $\mathbb{M}^2 = S^2$ it is known that the ordered Reeb graph is the only topological invariant of $f : S^2 \to \mathbb{R}$ up to diffeomorphisms and all regularly ordered graphs are realizable (see [2]). When $\mathbb{M}^n = S^n$, $n > 1$, such graphs were studied by Kronrod and Vitushkin (see [4]). Notice that the Reeb graph associated to a Morse function $f : S^n \to \mathbb{R}$, $n > 1$, is a tree. Arnold also studies the topological classification problem when $\mathbb{M}$ is a torus $\mathbb{T}^2$ (see [4]).

Recently, the first and second named authors studied the topological classification of stable maps from $\mathbb{M}^2 \subset \mathbb{R}^3$ to S^1 where $\mathbb{M}^2$ is a surface with boundary (see [7]) or $\mathbb{M}^2 = S^2$ (see [6]). In fact, in [6] it was introduced a generalization of classical Reeb graph, so-called *generalized Reeb graph* and proved that it is a complete topological invariant. As a consequence of this topological classification of stable maps from $\mathbb{M}^2 \subset \mathbb{R}^3 \to S^1$, the authors obtained in [6, 7] a topological classification of map germs from $(\mathbb{R}^3, 0)$ to $(\mathbb{R}^2, 0)$.

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Considering Morse-Bott functions defined from an orientable closed surfaces $\mathbb{M}^2$ to $\mathbb{R}$, the third named author obtained a classification of these functions up to topological conjugacy, constructing an invariant which is based on the classical Reeb graph of the function and the topological type of the singular level sets (see [16]).

In this paper our goal is to investigate the classification of Morse-Bott functions from S^2 to S^1 up to topological conjugacy, in the following sense:

Definition 1. *Two Morse-Bott functions $f, g : S^2 \to S^1$ are said to be* topologically conjugated *if there exist homeomorphisms $h : S^2 \to S^2$ and $k : S^1 \to S^1$ such that k preserves orientation, $f = k \circ g \circ h^{-1}$ and h sends singular fibers of g to singular fibers of f.*

If the Morse-Bott function $f : S^2 \to S^1$ is not surjective then the classification problem can be reduced to the case from S^2 to $\mathbb{R}$ (this study was done in [16]). The purpose of this paper is to investigate surjective Morse-Bott functions from S^2 to S^1.

The paper is organized as follows. In Sections 2 and 3, we give definitions and examples of Morse-Bott functions taking values in $\mathbb{R}$ and S^1, respectively. In Section 4 we construct a complete topological invariant, called here *MB-Reeb graph*, which classifies surjective simple Morse-Bott functions $f : S^2 \to S^1$ (Theorems 16 and 17), up to topological conjugacy. In Section 5, we state and prove a realization theorem for a given graph to be the MB-Reeb graph associated to a simple Morse-Bott function from S^2 to S^1 (Theorem 20).

2. Morse-Bott functions

Denote by $\mathbb{M}^n$ a smooth closed manifold of dimension n embedded in an open subset U of some Euclidean space.

Classical Morse theory deals only with functions all of whose critical points are nondegenerate; in particular, the critical points must all be isolated points. In many situations, however, the critical points form submanifolds of $\mathbb{M}^n$. One of Bott's first insights was to see how to extend the Morse theory to this situation. In [8], Bott introduced the notion of a nondegenerate critical submanifold: a critical submanifold $\mathbb{N} \subset \mathbb{M}$ is nondegenerate if at any point p in $\mathbb{N}$ the Hessian of f restricted to the normal space to $\mathbb{N}$ is nonsingular.

Let f be a smooth function from $\mathbb{M}^n$ to $\mathbb{R}$. The term smooth will mean "at least three times continuously differentiable" throughout the paper. A point $p \in \mathbb{M}^n$ is called a *singular point* of f if $rank(\mathrm{d}f(p))$ is not maximum, where $\mathrm{d}f(p)$ denotes the differential of f in $p \in \mathbb{M}^n$. Otherwise, p is called a *regular point* of f. A point $b \in \mathbb{R}$ is called a *singular value* of f if $f^{-1}(b)$ contains a singular point of f. The *singular set* of f, denoted by $Sing(f)$, is the set of all singular points of f. The image of $Sing(f)$ by f is called *discriminant set* of f, denoted by Δ_f.

For each $a \in \mathbb{R}$ consider the level set $I_a(f) = f^{-1}(a)$. Notice $I_a(f)$ is a union of connected components, $i_a^k(f)$, $k = 1, \ldots, m(a)$, called *fibers*. A *singular fiber* is a connected component of a level set $I_a(f)$ which contains a singular point of f and it is denoted by $s_a(f)$.

If all nearby fibers around a singular fiber are homeomorphic to it then this fiber is called *reducible*. See [1] for details.

We say that $f : \mathbb{M}^n \to \mathbb{R}$ is *simple* if there is an unique connected component containg singular points in the singular level. It is contained in a singular fiber $s_a(f) \subset I_a(f)$ for each $a \in \mathbb{R}$.

Definition 2. *([8]) Let $f : \mathbb{M}^n \to \mathbb{R}$ be a smooth function. A smooth submanifold $S \subset Sing(f)$ is a* non-degenerate singular submanifold *of f if*

(i) *S has no boundary ($\partial S = \emptyset$);*
(ii) *S is compact and connected;*
(iii) *For all $s \in S$, the tangent space $T_sS = ker\ (Hess_s f)$, where $Hess_s f$ is the Hessian of f in s.*

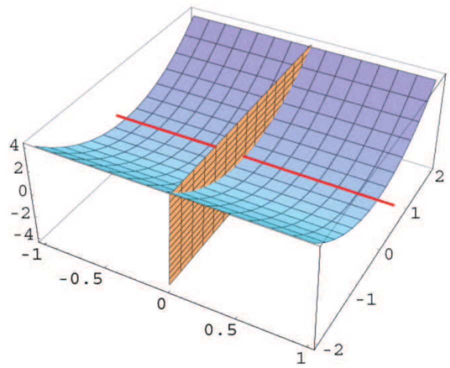

FIGURE 1. The graphic of $f(x,y) = x^2$. The red line is the $Sing(f)$.

The function f is called a Morse-Bott function *($\mathcal{MB}$ function from now on) if the set $Sing(f)$ consists of isolated points and non-degenerate singular submanifolds.*

Let $p \in Sing(f)$. By Morse-Bott Lemma (see [5]) there exists a local chart of $\mathbb{M}^n$ around p and a local splitting of the normal bundle of S, $N_pS = N_p^+S \oplus N_p^-S$ so that, if

$$p = (s, x, y),\ s \in S,\ x \in N_p^+S,\ y \in N_p^-S,$$

then

$$T_p\mathbb{M}^n = T_pS \oplus N_p^+S \oplus N_p^-S \qquad \text{and} \qquad f(p) = f(S) + |x|^2 - |y|^2.$$

The dimension of N_p^-S is the *index* of S and if p is not an isolated singularity of f then f is locally a Morse function on the image of N_pS under the exponential map.

It follows from Morse-Bott Lemma that Morse functions are $\mathcal{MB}$ functions with isolated singular points. Moreover, since $\mathbb{M}^n$ is compact then these functions have a finite number of isolated singular points.

Example 1. *Let $f : \mathbb{M}^2 = \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^2$. The function f is $\mathcal{MB}$ and the set $Sing(f)$ is the y-axis. See Figure 1.*

Another particular case of $\mathcal{MB}$ functions consists of the called *round Bott functions* (cf. [12]) where the set of all singular points is the disjoint union of circles, also called *critical loops*. An example of this kind of function is given in Example 2.

Example 2. *The parametric equation*

$$\phi(x_1, x_2) = ((R + r\cos x_1)\cos x_2, (R + r\cos x_1)\sin x_2, r\sin x_1),$$

with $(x_1, x_2) \in \mathbb{R}^2$, represents the 2-dimensional torus $\mathbb{T}^2$ in $\mathbb{R}^3$ obtained by the revolution around the z-axis of one circle of radius $r > 0$ in the plane-yz and its center is to $R > r$ units along the y-axis.

Let $f : \mathbb{T}^2 \longrightarrow \mathbb{R}$ be $f(\phi(x_1, x_2)) = r\sin x_1$. Then f is a round function (hence $\mathcal{MB}$ function). The $Sing(f)$ is the disjoint union of two circles in $\mathbb{T}^2$, of indexes 0 and 1, respectively. See Figure 2.

In [12] it was proved that there are no round functions on S^2.

Another interesting example of $\mathcal{MB}$ functions are the *Wigner's functions* (see [24]), which are quasi-probability distribution functions introduced in order to study quantum corrections

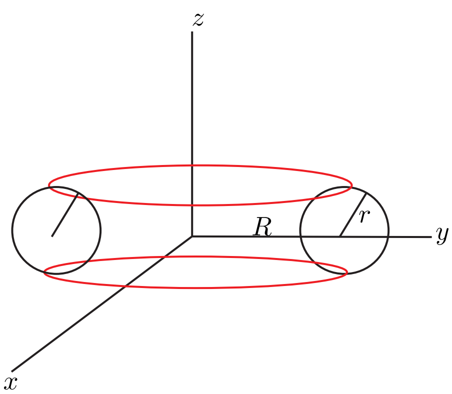

FIGURE 2. The function $f : \mathbb{T}^2 \longrightarrow \mathbb{R}$. The set $Sing(f)$ is the disjoint union of two circles, the red ones.

to classical statistical mechanics. According to [24], Wigner's functions have been useful in describing transport in quantum optics; nuclear physics; quantum computing, decoherence and chaos. The next example shows some particular cases of Wigner's functions.

Example 3. *According to [16], the functions*

(i) $f_0(x,p) = e^{(-p^2-x^2)}(2p^2+2x^2)/\pi$;
(ii) $f_1(x,p) = e^{(-p^2-x^2)}(-2p^2-2x^2+4p^4+8x^2p^2+4x^4)/\pi$;
(iii) $f_2(x,p) = e^{(-p^2-x^2)}(4p^6+12x^2p^4+12x^4p^2-8p^4-16x^2p^2+2p^2+4x^6-8x^4+2x^2)/\pi$,

where x denotes position and p the momentum, are three examples of special Wigner's functions.

These examples are $\mathcal{MB}$ functions having the origin as nondegenerate critical point. Also, each one of them has one, two and three nondegenerate critical submanifolds homeomorphic to circles, respectively (see Figures 3, 4 and 5). For more details see [16].

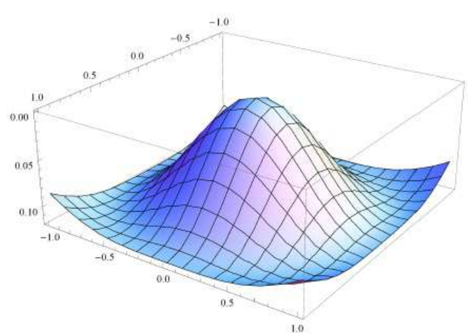

FIGURE 3. The function $f_0(x,p)$.

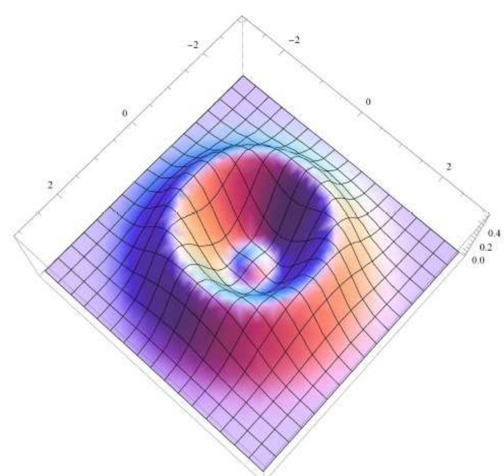

FIGURE 4. The function $f_1(x,p)$.

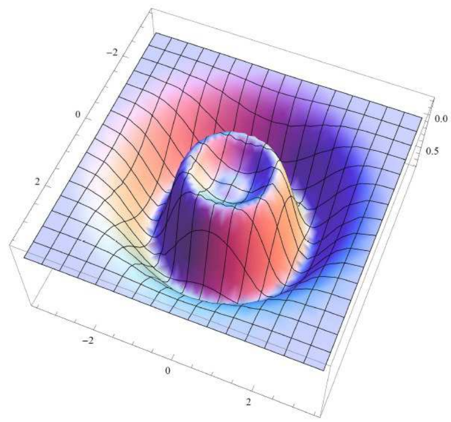

FIGURE 5. The function $f_2(x, p)$.

If $f : \mathbb{M}^2 \to \mathbb{R}$ is a $\mathcal{MB}$ function with isolated singular values, we can associate to f a classical graph, called *Reeb graph*. Reeb graphs were introduced by Reeb to study Morse functions from $\mathbb{M}^2$ to $\mathbb{R}$. The classical Reeb graph of f is the graph obtained by contracting each fiber $I_a(f)$ to a point, where the vertices correspond to the singular fibers $s_a(f)$ of f (see [19]). The Reeb graph allows us to study the evolution and to express the connectivity of the level sets of f. It has many interesting applications in Mathematics as well as in other areas such as Computational Geometry, Computer Graphics, etc. For instance, in Morse theory, it is well-known that the Reeb graph is a complete topological invariant for Morse functions from S^2 to $\mathbb{R}$ (see [3, 20]). Recently, a generalization of Reeb graph, so-called *generalized Reeb graph*, was introduced in [6, 7]. The generalized Reeb graph has extra additional information than classical Reeb graph.

Considering Morse-Bott functions $f : \mathbb{M}^2 \to \mathbb{R}$ defined on orientable closed surface, a topological classification (up to conjugacy equivalence) was done in [16]. In order to obtain this classification, it was also constructed an invariant based on the classical Reeb graph and the topological type of the singular level sets. Moreover, it was shown how the topological type of the singular level sets can be related with the order of the vertices of the Reeb graph associated to the $\mathcal{MB}$ function $f : \mathbb{M}^2 \to \mathbb{R}$ and induced by the values of f.

In this case, considering the dimension of the singular submanifolds and its index, the set $Sing(f)$ of a $\mathcal{MB}$ function $f : \mathbb{M}^2 \to \mathbb{R}$ can be subdivided in three subsets:

(i) Points in singular submanifolds which are homeomorphic to S^1. On these circles the function assumes extremal values. Such singular submanifolds are called *singular circles*.
(ii) Isolated singular points which are extremum points of f (maximum/minimum).
(iii) Isolated singularities of index 1 of f (saddle points).

3. Circle-valued Morse-Bott functions

In this section we will investigate Morse-Bott functions defined in an orientable closed surface $\mathbb{M}^2$ but now taking values in S^1 instead of $\mathbb{R}$. A similar approach was done for simple Morse functions from $\mathbb{M}^2$ to S^1 in [6, 7].

A smooth function $f : \mathbb{M}^2 \to S^1$ is called *circle-valued function*, where $\mathbb{M}^2$ is an orientable closed surface. Since circle-valued functions may be seen locally as a real-valued, all the local notions of Morse-Bott theory are carried over immediately to the framework of circle-valued functions.

Definition 3. *A circle-valued function $f : \mathbb{M}^2 \to S^1$ is called a* circle-valued Morse-Bott function *(or just* circle-valued $\mathcal{MB}$ function*) if for any $x \in \mathbb{M}^2$ we can choose a neighborhood V of $f(x) \in S^1$, and a diffeomorphism $\phi : V \to \mathbb{R}$, such that $\phi \circ (f|_U)$, where $U = f^{-1}(V)$, is a real-valued Morse-Bott function.*

We say that $f : \mathbb{M}^2 \to S^1$ is a *simple* circle-valued $\mathcal{MB}$ function if f is a $\mathcal{MB}$ function with up to one nondegenerate critical submanifold for each level curve. From now on, it will always be considered simple circle-valued $\mathcal{MB}$ functions.

Example 4. *Let $f : S^2 \to S^1$ be the radial projection as in Figure 6. Then f is a $\mathcal{MB}$ function.*

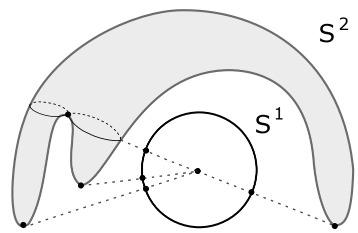

FIGURE 6. Circle-valued Morse-Bott function from sphere.

Example 5. *Let $f : \mathbb{M}^2 \to S^1$ be the radial projection as in Figure 7, where $\mathbb{M}^2$ is a bitorus. Then f is a $\mathcal{MB}$ function with two saddle points.*

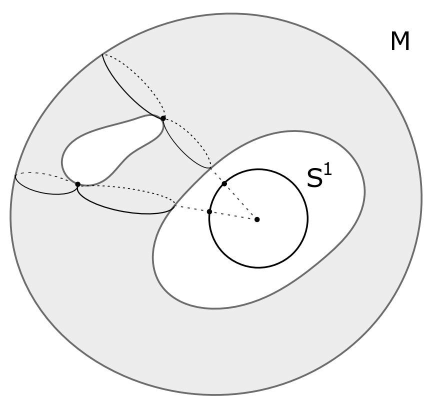

FIGURE 7. Circle-valued Morse-Bott function from bitorus.

Proposition 4. *Let $f : S^2 \to S^1$ be a simple $\mathcal{MB}$ function. Then f is not a regular map.*

Proof. Suppose f is a regular map, then $f(S^2) \subset S^1$ would be an open set. Since $f(S^2)$ is also closed, we get $f(S^2) = S^1$ and hence, f is surjective. By Ehresmann's fibration theorem [9], f is a locally trivial fibration. In particular, if F is a fiber we have that

$$2 = \chi(S^2) = \chi(S^1)\chi(F) = 0,$$

which is an inconsistency. $\square$

Proposition 5. *Let $f : S^2 \to S^1$ be a simple $\mathcal{MB}$ function. Then the nondegenerate critical submanifolds of f are homemorphic either S^1 or points. Moreover, there exists a finite number of them on S^2.*

Proof. The critical submanifolds of f in S^2 can be 0, 1 or 2-dimensional. Therefore, up to a diffeomorphism, in the case 0-dimensional the non degenerate critical submanifolds of f are critical points of f; in the case 1-dimensional, we can have each nondegenerate critical submanifold diffeomorphic to S^1 or a straight line. But, since S^2 is a compact the only possibility is a circle S^1. Finally, it can be observed that the 2-dimensional case does not hold because the existence of a normal subspace in each point is not possible. Moreover, since S^2 is compact we have only a finite number of critical submanifolds of f in S^2. □

Follows from Proposition 5, that the set $Sing(f)$ associated to a simple $\mathcal{MB}$ function $f : S^2 \to S^1$ can also be divided in three subsets, according to dimension of the singular submanifolds and its index. The possibilities are: singular circles (homeomorphic to S^1), maximum/minimum points or saddle points.

4. The MB-Reeb graph of a simple circle-valued $\mathcal{MB}$ function

It is well-known that the Reeb graph is a powerful tool to study the topological classification of functions. In fact, Arnold [2], Kulinich [13] and Sharko [22] classified Morse functions on surfaces using Reeb graphs with some additional information and Prishyak [18] classified smooth functions with isolated critical points on closed surfaces.

Here our goal is to extend the concept of generalized Reeb graph introduced in [6] for simple circle-valued $\mathcal{MB}$ functions from S^2 to S^1 and to study the classification problem of them, up to topological conjugacy.

Given a simple $\mathcal{MB}$ function $f : S^2 \to S^1$, we consider the following equivalence relation on S^2:

$x \sim y$ if and only if $f(x) = f(y)$ and x and y are in the same connected component of $f^{-1}(f(x))$.

Proposition 6. *Let $f : S^2 \to S^1$ be a simple $\mathcal{MB}$ function. Then the quotient space $S^2/\sim$ admits the structure of a connected graph in the following way:*

(1) *the vertices are the connected components of level curves $f^{-1}(v)$, where $v \in S^1$ is a critical value;*
(2) *each edge is formed by points that correspond to connected components of level curves $f^{-1}(v)$, where $v \in S^1$ is a regular value.*

Proof. Since f is a simple $\mathcal{MB}$ function we have a finite number of critical values $v_1, \dots, v_r$ and for each $i = 1, \dots, r$, $f^{-1}(v_i)$ has a finite number of connected components. Then,

$$f|S^2 - f^{-1}(\{v_1, \dots, v_r\}) : S^2 - f^{-1}(\{v_1, \dots, v_r\}) \to S^1 - \{v_1, \dots, v_r\}$$

is regular, and the induced map

$$\tilde{f} : (S^2 - f^{-1}(\{v_1, \dots, v_r\}))/\sim \to S^1 - \{v_1, \dots, v_r\}$$

is a local homeomorphism. Each connected component of $S^1 - \{v_1, \dots, v_r\}$ is homeomorphic to an open interval, so each connected component of $(S^2 - f^{-1}(\{v_1, \dots, v_r\}))/\sim$ is also homeomorphic to an open interval. □

Each vertex of the graph obtained as in Proposition 6 can be of four types, depending on if the connected component has a saddle point, maximum/minimum critical point, regular points which images are critical values or singular circles. The vertices corresponding to critical circles will be denoted by white vertices " $\circ$ ". Then, the possible incidence rules of edges and vertices are given in Figure 8.

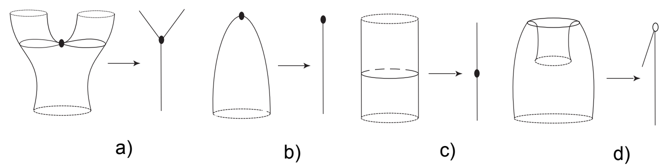

FIGURE 8. Incidence rules of edges and vertices.

Let $v_1, \ldots, v_r \in S^1$ be the critical values of f. We choose a base point $v_0 \in S^1$ and an orientation. We can reorder the critical values such that $v_0 \leq v_1 < \ldots < v_r$ and label each vertex with the index $i \in \{1, \ldots, r\}$, if it corresponds to the critical value v_i.

Definition 7. *The graph given by $S^2/\sim$ together with the labels and colors of the vertices, as previously defined, will be called here the* MB-Reeb graph *associated to a simple $\mathcal{MB}$ function $f : S^2 \to S^1$. The MB-Reeb graph associated to f will be denoted by Γ_f.*

Remark 8. *Since the nomenclatures "Reeb graph" and "generalized Reeb graph" are already established in the literature (cf. [3, 6, 7, 14, 18, 19, 20, 21, 22]), it was chosen here the expression "MB-Reeb graph" to designate the graph associated to a simple $\mathcal{MB}$ function $f : S^2 \to S^1$.*

Definition 9. *Let Γ_f be the MB-Reeb graph associated to a simple $\mathcal{MB}$ function $f : S^2 \to S^1$. The vertices of degree 2 in Γ_f with indegree 1 and outdegree 1 are called* regular vertices *of Γ_f. The other vertices of Γ_f are called* critical vertices. *Denote by V_f the set of critical vertices of Γ_f and by $\sharp V_f$ the number of elements of V_f.*

Proposition 10. *Let $f : S^2 \to S^1$ be a simple $\mathcal{MB}$ function. Then the MB-Reeb graph of f is a tree.*

Proof. Let Γ_f be the MB-Reeb graph of f. Since Γ_f is connected, in order to show that Γ_f is a tree, we only need to prove that its Euler characteristic is $\chi(\Gamma_f) = 1$. We have that $\chi(\Gamma_f) = V_f - E_f$, where V_f, E_f denote the number of vertices and edges of Γ_f, respectively.

On the one hand, $V_f = M_f + S_f + I_f + C_f$ where M_f, S_f, I_f, C_f denote the numbers of vertices of each type: maximum/minimum, saddle, regular or singular circles, respectively. Note that $V_f \neq 0$ by Proposition 4.

Since the set of Morse functions is dense in the set of $\mathcal{MB}$ functions, there exists a Morse function $g : S^2 \to S^1$ such that $M_g = M_f$ and $S_g = S_f$. Then from the Morse formula follows that $M_f - S_f = M_g - S_g = \chi(S^2) = 2$.

By Euler's formula $E_f = \frac{1}{2}\sum \deg(v_i)$ where v_i are the vertices of Γ_f and $\deg(v_i)$ is the degree of v_i, that is, the number of edges adjacent to v_i. Since f is a simple $\mathcal{MB}$ function, the degree of each vertex of maximum/minimum type is 1, while of regular type is 2 and of saddle type is 3 (see again Figure 8). Hence,

$$\chi(\Gamma_f) \;=\; V_f - E_f = M_f + S_f + I_f + C_f - \frac{1}{2}(M_f + 2I_f + 2C_f + 3S_f) \;=\; \frac{M_f - S_f}{2} = 1.$$

$\square$

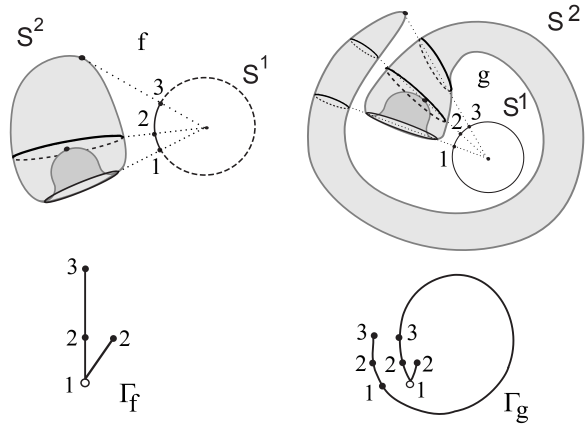

FIGURE 9. MB-Reeb graphs associated to f and g.

Example 6. *Let $f,\, g : S^2 \to S^1$ be two simple $\mathcal{MB}$ functions given by the radial projections as in Figure 9. Then their respective MB-Reeb graphs are given as appear in Figure 9.*

According to the Example 6, both $\mathcal{MB}$ functions share the same classical Reeb graph, but their MB-Reeb graphs are distinct. The function f is a no surjective map, whilst g is surjective. Therefore, f and g are not topologically equivalent. This shows that the classical Reeb graph is not enough to distinguish between these two simple $\mathcal{MB}$ functions.

Remark 11.
1. *The MB-Reeb graph was inspired on the invariants used in the works [6, 16]. However, the MB-Reeb graph contains some extra information. In fact, it has vertices corresponding to the regular connected components of $f^{-1}(v)$, where v is a critical value, and white vertices corresponding to singular circles.*
2. *If $f : S^2 \to S^1$ is not a surjective $\mathcal{MB}$ function, then f may be regarded as a $\mathcal{MB}$ function from S^2 to $\mathbb{R}$ (via stereographic projection) and we can apply the results of [16] to the MB-Reeb graphs.*
3. *The regular vertices in MB-Reeb graph distinguish between surjective and no surjective case.*

Example 7. *Let $h : S^2 \to S^1$ be a simple $\mathcal{MB}$ function given by the radial projection as in Figure 10. Notice that the function h has the same type of singularities as the function g in Example 6, but their MB-graphs differ by three regular vertices.*

It is obvious that the labeling of vertices of the MB-Reeb graph is not uniquely determined, since it depends on the chosen orientations and the base points on each S^1. Different choices will produce either a cyclic permutation or a reversion of the labeling in the MB-Reeb graph. This leads us to the following definition of equivalent MB-Reeb graphs.

Let $f, g : S^2 \to S^1$ be two simple $\mathcal{MB}$ functions. Let Γ_f and Γ_g be their respective MB-Reeb graphs. Consider the induced quotient maps $\bar{f} : \Gamma_f \to S^1_f$ and $\bar{g} : \Gamma_g \to S^1_g$, where S^1_f, S^1_g denote S^1 with the graph structure whose vertices are the critical values of f, g, respectively.

Definition 12. *An* isomorphism *between two graphs Γ_1 and Γ_2 is a bijection f from $V(\Gamma_1)$ to $V(\Gamma_2)$, where $V(\Gamma_i) = \{$vertices of $\Gamma_i\}$, such that two vertices v and w are adjacent in Γ_1 if and only if $f(v)$ and $f(w)$ are adjacent in Γ_2.*

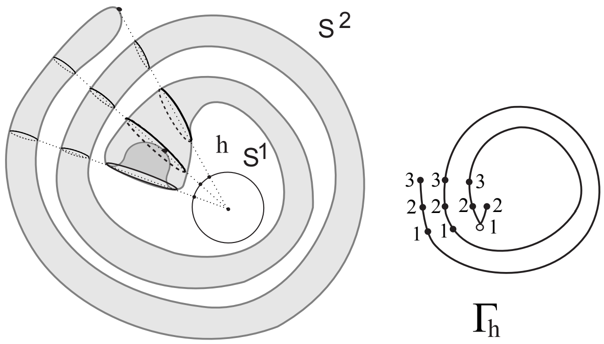

FIGURE 10. MB-Reeb graph associated to a surjective simple $\mathcal{MB}$ function

Definition 13. *We say that* Γ_f is equivalent to Γ_g *and we denote it by* $\Gamma_f \sim \Gamma_g$*, if there exist graph isomorphisms* $j : \Gamma_f \to \Gamma_g$ *and* $l : S^1_f \to S^1_g$*, such that the following diagram is commutative:*

$$\begin{tikzcd} V_f \arrow[r, "\bar{f}|V_f"] \arrow[d, "j|_{V_f}"'] & \Delta_f \arrow[d, "l|_{\Delta_f}"] \\ V_g \arrow[r, "\bar{g}|V_g"] & \Delta_g \end{tikzcd}$$

where $V_f = \{\textit{vertices of}\, \Gamma_f\}$, $V_g = \{\textit{vertices of}\, \Gamma_g\}$ *and* Δ_f *and* Δ_g *are their respective discriminant sets.*

Definition 14. *Two* $\mathcal{MB}$ *functions* $f, g : S^2 \to S^1$ *are said to be* topologically conjugated *if there exist homeomorphisms* $h : S^2 \to S^2$, $k : S^1 \to S^1$ *such that* k *preserves orientation,* $f = k \circ g \circ h^{-1}$ *and* h *sends singular fibers of* g *to singular fibers of* f*.*

For simple $\mathcal{MB}$ functions we get the trivial following result:

Proposition 15. *If* f *and* g *are two topologically conjugated simple* $\mathcal{MB}$ *functions, then the singular fibers* $s_a(f)$ *and* $s_{k(a)}(g)$ *are homeomorphic.*

Theorem 16. *Let* $f, g : S^2 \to S^1$ *be two simple* $\mathcal{MB}$ *functions. If* f *and* g *are topologically conjugated then their respective MB-Reeb graphs are equivalent.*

Proof. Since f and g are topologically conjugated there exist homeomorphisms $h : S^2 \to S^2$ and $k : S^1 \to S^1$ such that $f = k \circ g \circ h^{-1}$. Then h maps singular fibers into singular fibers and k maps critical values into critical values. Hence h induces a graph isomorphism from Γ_f to Γ_g and k induces a graph isomorphism from S^1_f to S^1_g which give the equivalence between the MB-Reeb graphs. □

Theorem 17. *Let* $f, g : S^2 \to S^1$ *be simple* $\mathcal{MB}$ *functions and* Γ_f *and* Γ_g *their respective MB-Reeb graphs. If* Γ_f *and* Γ_g *are equivalent then* f *and* g *are topologically conjugated.*

Proof. Since $\Gamma_f \sim \Gamma_g$, there exist graph isomorphisms $j : \Gamma_f \to \Gamma_g$ and $l : S^1_f \to S^1_g$ as in Definition 13. We choose a homeomorphism $h : \Gamma_f \to \Gamma_g$ and a diffeomorphism $k : S^1_f \to S^1_g$ which realize the graph isomorphisms j, l respectively and such that $\bar{g} \circ h = k \circ \bar{f}$.

Since $k \circ f$ is topologically conjugated to f then by Theorem 16 we have $\Gamma_{k \circ f} \sim \Gamma_f$. Moreover, these graphs are the same because $k \circ \bar{f} = \overline{k \circ f}$. In other words the following diagram is commutative:

$$\begin{tikzcd} \Gamma_g \arrow[r, "\bar{g}"] & S^1 \\ \Gamma_f \arrow[u, "h"] \arrow[ur, "\overline{k\circ f}"'] & \end{tikzcd}$$

For simplicity, we write simply f instead of $k \circ f$. By construction $h(V_f) = V_g$, but now f and g have the same critical values $v_1, \ldots, v_n \in S^1$. We choose a base point and an orientation in S^1 and assume that $v_1 < v_2 < \ldots < v_n$.

Denote by $\mathrm{arc}(a, b)$ the oriented arc from a to b, where a and b are regular distinct values of f in S^1, and by $\overline{\mathrm{arc}(a, b)}$ its closure. Then, for both a and b we choose $\epsilon_a > 0$ and $\epsilon_b > 0$ such that $A := \overline{\mathrm{arc}(a + \epsilon_a, b - \epsilon_b)}$ and $B := \overline{\mathrm{arc}(b + \epsilon_b, a - \epsilon_a)}$ have only regular values of f. Notice that the restrictions maps

$$\begin{aligned} f_a &:= f\ : f^{-1}(A) \to A, & f_b &:= f\ : f^{-1}(B) \to B, \\ g_a &:= g\ : g^{-1}(A) \to A, & g_b &:= g\ : g^{-1}(B) \to B, \end{aligned}$$

can be considered $\mathcal{MB}$ functions with values in $\mathbb{R}$. Then by Theorem 3.17 of [16], there exist homeomorphisms $h_a : f^{-1}(A) \to g^{-1}(A)$ and $h_b : f^{-1}(B) \to g^{-1}(B)$ such that $f_a = g_a \circ h_a$ and $f_b = g_b \circ h_b$, because $\Gamma_{f_a} \sim \Gamma_{g_a}$ and $\Gamma_{f_b} \sim \Gamma_{g_b}$.

Notice that the boundary of the sets $f^{-1}(A)$, $f^{-1}(B)$, $g^{-1}(A)$, $g^{-1}(B)$ is formed by a finite number of disjoint closed curves. We can assume that the homeomorphisms h_a and h_b when restricted to the boundary preserve orientation. Then, we can extend the homeomorphisms h_a and h_b to $f^{-1}(\mathrm{arc}(a - \epsilon_a, a + \epsilon_a) \cup \mathrm{arc}(b - \epsilon_b, b + \epsilon_b))$ such that they coincide in

$$f^{-1}(\mathrm{arc}(a - \epsilon_a, a + \epsilon_a) \cup \mathrm{arc}(b - \epsilon_b, b + \epsilon_b))$$

(for details of extensions of homeomorphisms see [23]).

We now define a map $H : S^2 \to S^2$ given by

$$H(x) = \begin{cases} h_a(x), & \text{if } x \in f^{-1}(\mathrm{arc}(a - \epsilon_a, b + \epsilon_b)), \\ h_b(x), & \text{if } x \in f^{-1}(\mathrm{arc}(b - \epsilon_b, a + \epsilon_a)). \end{cases}$$

By construction, $h_a = h_b$ on $f^{-1}(\mathrm{arc}(a - \epsilon_a, a + \epsilon_a) \cup \mathrm{arc}(b - \epsilon_b, b + \epsilon_b))$. Therefore, H is well defined. Moreover, $H : S^2 \to S^2$ is a homeomorphism which conjugates f and g. Hence, f and g are topologically conjugated. □

Remark 18. *In the Example 6, if we do not consider the regular vertices in the MB-Reeb graphs associated to f and g, then these graphs would become indistinguishable (and hence equivalent). However, the functions f and g are clearly not topologically conjugated.*

5. Realization theorem

In this Section we present the second main result of this paper: a realization theorem for circle-valued simple Morse-Bott functions on S^2. In fact, the Theorem 20 gives necessary and sufficient conditions for a finite connected directed tree $\mathcal{G}$ to be associated to a simple Morse-Bott function on S^2 with values in S^1. An analogous result was proved in [16] for a graph associated to a simple Morse-Bott function on an orientable closed surface with values in $\mathbb{R}$.

We recall some basic terminology from topological graph theory which did not appear in the previous sections (more details see [10]).

A *directed graph* $\mathcal{G}$ consists of a finite nonempty set V of points together with a prescribed collection E of ordered pairs of distinct points. The elements of V are called *vertices* and the elements of E are *directed edges* or *arcs*. Denote by $e = (u, v)$ an edge of $\mathcal{G}$. Then, the edge goes from vertex u to vertex v and it is incident with u and v. We also say that e is *adjacent* to v and v is adjacent from u. The *outdegree* of a vertex v, denoted by e_v^-, is the number of vertices adjacent from it, and the *indegree*, denoted by e_v^+, is the number of vertices adjacent to it. The sum of the indegree and the outdegree of v is called the *degree* of v. A *source vertex* is a vertex with indegree 0 and a *sink vertex* is a vertex with outdegree 0. Given two vertices u, v of $\mathcal{G}$ a *path* connecting u and v is denoted by $[u, v]$. Of course, if u and v are consecutive vertices of $\mathcal{G}$ then the path $[u, v]$ is an edge of $\mathcal{G}$. A *directed path* connecting u and v is a path from u to v.

Definition 19. *A graph is n-*partite *if it can be partitioned into n disjoint independent sets, called* partite sets*. Edges in a n-partite graph are interpartite, having endpoints in different partite sets.*

By Definition 19, a n-partite graph has no intrapartite edges, that is, edges between vertices in the same partite set.

Let $\mathcal{G}$ be any finite connected directed tree. Denote by $V_{\mathcal{G}}$ the set of all vertices of $\mathcal{G}$ having degree 1, degree 2 (of type sink or source) and degree 3.

Theorem 20. *Let $\mathcal{G}$ be a finite connected directed tree. Then $\mathcal{G}$ can be realized as the MB-graph associated to a simple $\mathcal{MB}$ function $f : S^2 \to S^1$ if and only if the following conditions are satisfied:*

i) *The vertices of $\mathcal{G}$ only have degree 1,2 or 3.*
ii) *The vertices of degree 3 have indegree 1 and outdegree 2 or indegree 2 and outdegree 1.*
iii) *The tree $\mathcal{G}$ is n-partite, where $n = \sharp V_{\mathcal{G}}$, and each partite set has one, and only one element of $V_{\mathcal{G}}$.*
iv) *The direction of $\mathcal{G}$ induces an order in the set $V_{\mathcal{G}}$ and a natural quotient map $\gamma : \mathcal{G} \to S^1$ that identifies each partite set with a point $w_1, \dots, w_n \in S^1$. Consequently, if in a directed path of $\mathcal{G}$ the only two elements of $V_{\mathcal{G}}$ are the initial and terminal vertices of the path, then the number of vertices with indegree 1 and outdegree 1 in the path is:*
   (a) *$\sharp V_{\mathcal{G}} k + j - i - 1$ if v_i is the initial vertex and v_j in the terminal vertex of the directed path connecting v_i and v_j.*
   (b) *$\sharp V_{\mathcal{G}}(1 + k) - (j - i + 1)$ if v_j is the initial vertex and v_i in the terminal vertex of the directed path connecting v_i and v_j.*

   where $1 \le i < j \le n$ and k is the number of vertices in the path that belong to the partite set associated to w_j.

Proof. ($\Rightarrow$) Let $\mathcal{G} = \Gamma_f$ be the MB-Reeb graph associated to a simple $\mathcal{MB}$ function $f : S^2 \to S^1$.

The conditions (i) and (ii) are consequence of the type of singularities in a $\mathcal{MB}$ function and from the definition of the MB-Reeb graph.

Let $w_1, \dots, w_n \in S^1$ be the critical values of f. We choose a base point $w_0 \in S^1$ and an orientation in S^1, reordering the critical values $w_0 \le w_1 < \dots < w_n$ and labeling the vertices of Γ_f by $i \in \{1, \dots, n\}$ if the vertex is associated to critical value w_i. Consider γ the natural quotient map induced by f mapping each vertex of label i of Γ_f to its respective critical value $w_i \in S^1$. Then Γ_f can be partitioned into r disjoint independent sets $\gamma^{-1}(w_1), \gamma^{-1}(w_2), \dots, \gamma^{-1}(w_n)$, and since f is a simple $\mathcal{MB}$ function each set contains only one critical vertex. Moreover, the edges of Γ_f correspond to the connected components of $\gamma^{-1}(\alpha_l)$, where $\alpha_1, \alpha_2, \dots, \alpha_n$ are the arcs of S^1 limited by two consecutive critical values of f. The map γ is continuous, then the edges of Γ_f have endpoints in different partite sets. Consequently Γ_f is a n-partite graph.

Since Γ_f is a tree, given any two vertices i and j of Γ_f, there exists an unique path connecting i and j, denoted here by $[i, j]$. Moreover, regular vertices have degree 2, then all regular vertices of Γ_f belong to some path connecting critical vertices.

Let i and j be two any critical vertices of Γ_f, with $i < j$. Suppose that $[i, j]$ does not have other critical vertices. Then, the points of $[i, j]$ are associated to the values of f takes in $arc(w_i, w_j)$ or in $arc(w_j, w_i)$ of S^1 (see Figure 11).

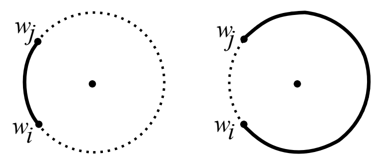

FIGURE 11. Possible arcs connecting w_i and w_j.

Of course $arc(w_i, w_j)$ contains $j-i-1$ critical values in its interior. By other hand, $arc(w_j, w_i)$ contains $\sharp V_f - (j - i - 1 + 2) = \sharp V_f - (j - i + 1)$ critical values in its interior.

Since each vertex is associated to a critical value of f and taking into account that f can be a surjective function, we have two possibilities for the number of regular vertices contained in $[i, j]$:

i) $\sharp V_f k + j - i - 1$, if $[i, j]$ is associated to $arc(w_i, w_j)$,
ii) $\sharp V_f(1 + k) - (j - i + 1)$, if $[i, j]$ is associated to $arc(w_j, w_i)$,

where $k \in \mathbb{Z}_+$ is the number of regular vertices associated to the critical value w_j which are in $[i, j]$.

($\Leftarrow$) Let $\mathcal{G}$ be any finite connected directed tree satisfying the conditions (i)-(iv) in Theorem 20.

By (iv), the direction of $\mathcal{G}$ induces an ordering in the set $V_{\mathcal{G}}$ and a natural quotient map $\gamma : \mathcal{G} \to S^1$ that identifies each partite set with a point $w_1, \ldots, w_n \in S^1$. Let $\beta_1, \beta_2, \ldots, \beta_n$ be arcs of S^1 containing $w_1, \ldots w_n$, respectively, such that each w_i belongs only to the arc β_i, $i = 1, \ldots, n$.

For each $i = 1, \ldots, n$, consider $\gamma^{-1}(\beta_i)$. By condition (iii), each set $\gamma^{-1}(\beta_i)$ has one and only one element of $V_{\mathcal{G}}$. Let v_i be such critical vertex in $\gamma^{-1}(\beta_i)$. By condition (i) and (ii) and from Definition of $V_{\mathcal{G}}$, the vertex v_i has degree 1 or degree 2 (of type sink or source) or degree 3. By condition (iv), each $\gamma^{-1}(\beta_i)$ can also has vertices of degree 2 with indegree 1 and outdegree 1. According to the degree of critical vertex $v_i \in \gamma^{-1}(\beta_i)$, we can constructed a neighborhood N_i in S^2 as follows:

If $e^-_{v_i}$ and $e^+_{v_i}$ are the outdegree and indegree of v_i, respectively, then N_i is the compact surface obtained from S^2 without $e^-_{v_i} + e^+_{v_i}$-disks. However, N_i may not be connected. The Figure (12) illustrates how these neighborhoods can be constructed.

Using the same arguments and ideas by Matsumoto and Saeki (see more details in the proof of Theorem 5.1 in [17]), we can consider $\mathcal{MB}$ functions $f|_{N_i} : N_i \to \mathbb{R}$, $i = 1, \ldots, n$.

Let us now constructed a desired simple Morse-Bott function $f : S^2 \to S^1$ by gluing the above-constructed $\mathcal{MB}$ functions $f|_{N_i} : N_i \to \mathbb{R}$, $i = 1, \ldots, n$, according to the ordering given in condition (iv). Hence, it follows that $\mathcal{G}$ is isomorphic to the MB-Reeb graph of $f : S^2 \to S^1$, as required. $\square$

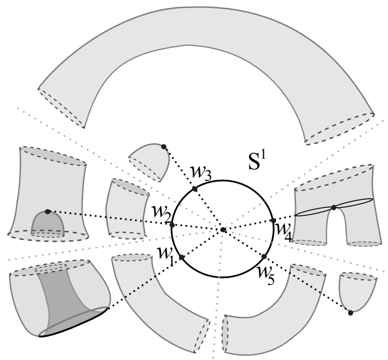

FIGURE 12. Examples of neighborhoods N_i in S^2, according to the degree of critical vertex $v_i \in \gamma^{-1}(\beta_i)$.

Remark 21. *Given the neighborhoods of critical vertex N_i as in the Theorem 20, consider standard embeddings $\phi_i : N_i \to \mathbb{R}^3$, $i = 1, \ldots, n$.*

Once more, we can apply the same arguments and ideas by Masumoto and Saeki (cf. [17]) to construct an embedding $\phi : S^2 \to I\!R^3$ by gluing the above-embeddings ϕ_i, $i = 1, \ldots, n$, according to the ordering given in condition (iv). Notice that the quotient map $\gamma : \mathcal{G} \to S^1$ induces in a natural way a radial projection $\pi : \phi(S^2) \to S^1$. As a consequence, the function f constructed in the Theorem 20 can in fact be seen as $f = \pi \circ \phi$.

In this way, we show that any finite connected directed tree satisfying the conditions (i) - (iv) of Theorem 20 can be realized as the MB-Reeb graph of a radial projection associated with an embedding of S^2 into $\mathbb{R}^3$. In other words, we have the following result:

Corollary 22. *Let $\mathcal{G}$ be a finite connected directed tree satisfying the conditions (i) - (iv) as in the Theorem 20. Then, there exist an embedding $\phi : S^2 \to \mathbb{R}^3$ and a radial projection $\pi : \phi(S^2) \to S^1$ such that $f = \pi \circ \phi$ is a simple Morse-Bott function whose MB-Reeb graph is isomorphic to $\mathcal{G}$.*

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