Journal *of* Singularities
Volume 18 (2018), 36-49

Special volume in honor of the life and mathematics of Egbert Brieskorn
DOI: 10.5427/jsing.2018.18d

ON THE b-EXPONENTS OF GENERIC ISOLATED PLANE CURVE SINGULARITIES

E. ARTAL BARTOLO[1], PI. CASSOU-NOGUÈS[2], I. LUENGO[3], AND A. MELLE-HERNÁNDEZ[3]

Dedicated to the memory of Egbert Brieskorn with great admiration

Abstract. In 1982, Tamaki Yano proposed a conjecture predicting how is the set of b-exponents of an irreducible plane curve singularity germ which is generic in its equisingularity class. In 1986, Pi. Cassou-Noguès proved the conjecture for the one Puiseux pair case in [9]. In [1] the authors proved the conjecture for two Puiseux pairs germs whose complex algebraic monodromy has distinct eigenvalues. A natural problem induced by Yano's conjecture is, for a generic equisingular deformation of an isolated plane curve singularity germ to study how the set of b-exponents depends on the topology of the singularity. The natural generalization suggested by Yano's approach holds in suitable examples (for the case of isolated singularites which are Newton non-degenerated, commode and whose set of spectral numbers are all distincts). Morevover we show with an example that this natural generalization is not correct. We restrict to germs whose complex algebraic monodromy has distinct eigenvalues such that the embedded resolution graph has vertices of valency at most 3 and we discuss some examples with multiple eigenvalues.

Introduction

Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a germ of a complex analytic function whose zero locus

$$(f^{-1}(0),0)\subset(\mathbb{C}^n,0)$$

defines an isolated hypersurface singularity germ, that is the Minor number of f at 0,

$$\mu(f,0):=\dim_{\mathbb{C}}\frac{\mathbb{C}\{z_1,\dots,z_n\}}{\left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n}\right)}$$

is finite. A *Milnor fibration* was constructed in [19] as follows. Set $B_\varepsilon=\{z\in\mathbb{C}^n:|z|<\epsilon\}$ and $S_\epsilon=\{z\in\mathbb{C}^n:|z|=\epsilon\}$, one can choose ϵ_0 such that for all $0<\epsilon\le\epsilon_0$, $f^{-1}(0)$ is transverse to S_ϵ. For $0<\eta\ll\epsilon_0$ and $D_\eta=\{t\in\mathbb{C}:|t|<\eta\}$, let $X(t)=f^{-1}(t)\cap B_{\epsilon_0/2}$ and $X=f^{-1}(D_\eta)\cap B_{\epsilon_0/2}$. By Milnor, for such suitable ϵ and η, the mapping $X\setminus f^{-1}(0)\to D_\eta\setminus\{0\}$ is a C^∞-locally trivial fibration whose general fibre $F_{f,0}$, called *Milnor fibre*, has the homotopy type of a bouquet of exactly $\mu(f,0)$ of $(n-1)$-dimensional spheres.

The geometric monodromy $h_{F_{f,0}}:F_{f,0}\to F_{f,0}$ of the Milnor fibration is the monodromy transformation of the Milnor fibration over the loop $c\exp(2\pi t), t\in[0,1]$ and c small enough. The geometric monodromy induces the complex algebraic monodromy $h^{a,j}:H^j(F_{f,0},\mathbb{C})\to H^j(F_{f,0},\mathbb{C})$

2010 *Mathematics Subject Classification.* Primary: 14F10, 32S40; Secondary: 32S05, 32A30.
Key words and phrases. Bernstein-Sato polynomial, b-exponents, Brieskorn lattice, improper integrals.
[1]Partially supported by the grant MTM2016-76868-C2-2-P and Grupo "Álgebra y Geometría" of Gobierno de Aragón/Fondo Social Europeo.
[2]Partially supported by MTM2016-76868-C2-1-P and MTM2016-76868-C2-2-P.
[3]Partially supported by the grant MTM2016-76868-C2-1-P and Grupo Singular UCM.

whose eigenvalues are roots of unity. Since the Milnor fibre is a connected bouquet of $(n-1)$-spheres, the only interesting algebraic monodromy is $h^{a,n-1} : H^{n-1}(F_{f,0}, \mathbb{C}) \to H^{n-1}(F_{f,0}, \mathbb{C})$, where $\dim_{\mathbb{C}} H^{n-1}(F_{f,0}, \mathbb{C}) = \mu(f, 0)$.

Let $\mathcal{O}$ be the ring of germs of holomorphic functions on $(\mathbb{C}^n, 0)$, let $\mathcal{D}$ be the ring of germs of holomorphic differential operators of finite order with coefficients in $\mathcal{O}$. Let s be an indeterminate commuting with the elements of $\mathcal{D}$ and set $\mathcal{D}[s] = \mathcal{D} \otimes_{\mathbb{C}} \mathbb{C}[s]$.

Given a holomorphic germ $f \in \mathcal{O}$, one considers $\mathcal{O}\left[\frac{1}{f}, s\right] \cdot f^s$ as a free $\mathcal{O}\left[\frac{1}{f}, s\right]$-module of rank 1 with the natural $\mathcal{D}[s]$-module structure. Then, there exits a non-zero polynomial $B(s) \in \mathbb{C}[s]$ and some differential operator $P = P(x, \frac{\partial}{\partial x}, s) \in \mathcal{D}[s]$, holomorphic in $x_1, \ldots, x_n$ and polynomial in $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, which satisfy the following functional equation in $\mathcal{O}\left[\frac{1}{f}, s\right] f^s$:

$$P(s, x, D) \cdot f(x)^{s+1} = B(s) \cdot f(x)^s. \tag{1}$$

The monic generator $b_{f,0}(s)$ of the ideal of such polynomials $B(s)$ is called the *Bernstein-Sato polynomial* (or b-function or Bernstein polynomial) of f at 0. The same result holds if we replace $\mathcal{O}$ by the ring of polynomials in a field $\mathbb{K}$ of zero characteristic with the obvious corrections, see e.g. [12, Section 10, Theorem 3.3].

This result was first obtained for f polynomial by Bernstein in [3] and in general by Björk [4]. One can prove that $b_{f,0}(s)$ is divisible by $s+1$, and we also consider the *reduced Bernstein-Sato polynomial*

$$\tilde{b}_{f,0}(s) := \frac{b_{f,0}(s)}{s+1}.$$

In the case where f defines an isolated singularity, one can consider the nowadays called *Brieskorn lattice* $H_0^{''} := \Omega^n / df \wedge d\Omega^{n-2}$ introduced by Brieskorn in [8], and its saturation

$$\tilde{H}_0^{''} = \sum_{k \geq 0} (\partial_t t)^k H_0^{''}.$$

Malgrange [18] showed that the reduced Bernstein polynomial $\tilde{b}_{f,0}(s)$ is the minimal polynomial of the endomorphism $-\partial_t t$ on the vector space $F := \tilde{H}_0^{''} / \partial_t^{-1} \tilde{H}_0^{''}$, whose dimension equals the Milnor number $\mu(f, 0)$ of f at 0. Following Malgrange [18], the set of *b-exponents* are the μ roots $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_\mu\}$ of the characteristic polynomial of the endomorphism $-\partial_t t$. Recall also that $\exp(-2i\pi\partial_t t)$ can be identified with the (complex) *algebraic monodromy* of the corresponding Milnor fibre $F_{f,0}$ of the singularity at the origin.

Kashiwara [15] expressed these ideas using differential operators and considered

$$\mathcal{M} := \mathcal{D}[s] f^s / \mathcal{D}[s] f^{s+1},$$

where s defines an endomorphism of $\mathcal{D}(s) f^s$ by multiplication. This morphism keeps invariant $\tilde{\mathcal{M}} := (s+1)\mathcal{M}$ and defines a linear endomorphism of $(\Omega^n \otimes_{\mathcal{D}} \tilde{\mathcal{M}})_0$ which is naturally identified with F and under this identification $-\partial_t t$ becomes the endomorphism defined by the multiplication by s.

In [18], Malgrange proved that the set $R_{f,0}$ of roots of the Bernstein-Sato polynomial is contained in $\mathbb{Q}_{<0}$, see also Kashiwara [15], who also restricts the set of candidate roots. The number $-\alpha_{f,0} := \max R_{f,0}$ is the opposite of the *log canonical threshold* of the singularity and Saito [21, Theorem 0.4] proved that

$$R_{f,0} \subset [\alpha_{f,0} - n, -\alpha_{f,0}]. \tag{2}$$

Also Saito in [20] showed that the local moduli of μ-constant deformation is determined by the *Brieskorn lattice* if the μ-constant stratum is smooth, as in the case of germs of plane curves where he gave in [20, p. 30] a more simple formula describing the reduced Bernstein-Sato. There are many papers devoted to study Bernstein-Sato polynomial but it would be worthwhile to refer to the existence of a relative Bernstein-Sato polynomial in [5], by Briançon et al., and for results on the computation of the roots of Bernstein-Sato polynomial for functions with isolated singularity, even if the methods used in [6] are different. In [7], Briançon et al. gave a multiple of the Bernstein-Sato polynomial for any two variables function with isolated singularities. Some general properties of μ-constant deformations are also given by Varchenko in [24].

There is another set which is important too, the set of *exponents of the monodromy* (or spectral numbers, up to the shift by one, in the terminology of Varchenko [25]). This notion was first introduced by Steenbrink [22].

Let $f:(\mathbb{C}^n,0)\longrightarrow(\mathbb{C},0)$ be a germ of a holomorphic function with isolated singularity. In [22] Steenbrink constructed a mixed Hodge structure on $H^{n-1}(F_{f,0},\mathbb{C})$. Let

$$H^{n-1}(F_{f,0},\mathbb{C})_\lambda=\ker(T_s-\lambda:H^{n-1}(F_{f,0},\mathbb{C})\longrightarrow H^{n-1}(F_{f,0},\mathbb{C}));$$

where T_u,T_s are, respectively, the unipotent and semi-simple factors of the Jordan decomposition of the monodromy h^{n-1}.

The set $\operatorname{Spec}(f)$ of spectral numbers are μ rational numbers

$$0<\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_\mu<n$$

which are defined by the following condition:

$$\#\{j:\exp(-2\pi i\alpha_j)=\lambda,\lfloor\alpha_j\rfloor=n-p-1\}=\dim_{\mathbb{C}}\operatorname{Gr}_F^p H^{n-1}(F_{f,0},\mathbb{C})_\lambda,\qquad\lambda\neq1$$
$$\#\{j:\alpha_j=n-p\}=\dim_{\mathbb{C}}\operatorname{Gr}_F^p H^{n-1}(F_{f,0},\mathbb{C})_1.$$

The set $\operatorname{Spec}(f)$ of spectral numbers is symmetric, that is $\alpha_i+\alpha_{\mu-(i-1)}=n$. It is known that this set is constant under μ-constant deformation of f, see [25].

As it is well-known, neither the Bernstein-Sato polynomial nor the b-exponents are constant along μ-constant deformation. Given an equisingular type, a generic set of b-exponents or a generic Bernstein-Sato polynomial are expected. In [27], Yano proposed a formula (see next section) for the generic b-exponents for irreducible germs of curves (combined with the Jordan form of the monodromy, this also yields to a formula for the generic Bernstein polynomial). This formula was proved for one-Puiseux pair germs by the second named author in [10] and reproved by M. Saito in [20].

In [1], the conjecture was proved for irreducible singularities with two Puiseux pairs and monodromy without multiple eigenvalues. In this paper, we discuss how to extend the formula for reducible germs of singularities. There is a natural interpretation of Yano's formula in terms of the resolution graph of the singularity, see (5). We are going to prove in this paper that this formula holds for singularities with vertices of valency at most 3 (and at most two vertices of valency 3) and monodromy without multiple eigenvalues (distinct from 1) (in fact, the correct hypothesis may be distinct exponents of the monodromy, besides 1).

The restriction on the number 3-valency vertices comes from technical reason but it is most probably avoidable; for example, the second named author proved it in [11] for singularities with non-degenerate and commode Newton polygon (and distinct exponents for the monodromy). The other two conditions seem to be more important, since we will give examples where it does not hold in at least two cases: germs where the vertices have valencies at most 3 but there are multiple exponents, and germs with vertices with valency greater than 3. We will discuss also other examples and we will introduce the needed results about improper integrals.

1. Extended Yano's problem

Let $f:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ be a germ of a non-zero holomorphic function such that its zero locus defines an isolated singularity germ.

Extended Yano's Problem ([27])**.** *For a generic equisingular deformation of an isolated plane curve singularity germ $f:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ and Milnor number μ, to study how the set of b-exponents $\{\tilde{\beta}_1,\dots,\tilde{\beta}_\mu\}$ depends on the topology of f.*

The local Bernstein-Sato polynomial $b_{f,0}(s)$ of a singularity germ is a powerful analytic invariant, but it is, in general, extremely hard to compute, even in the case of irreducible plane curve singularities. It is well-known that the Bernstein-Sato polynomial varies in families in the (non-singular) μ-constant stratum $\Sigma_{\mu(f,0)}$ of f at 0. Since, for plane curves this stratum is irreducible, it is conceivable that a *generic* Bernstein-Sato polynomial exists, i.e., the Bernstein-Sato polynomial of a germ f with the same topology as f, depends on f, but there is a *generic* Bernstein-Sato polynomial $b^{\text{gen}}_{\Sigma_{\mu(f,0)}}(s)$: for every μ-constant deformation of such an f, there is a Zariski dense open set $\mathcal{U}$ on which the Bernstein-Sato polynomial of any germ in $\mathcal{U}$ equals $b^{\text{gen}}_{\Sigma_{\mu(f,0)}}(s)$.

1.1. The original Yano's conjecture: the irreducible case.

Let f be an irreducible germ of plane curve. In 1982, Tamaki Yano [27] made a conjecture concerning the b-exponents of such germs. Let $(n,b_1,b_2,\dots,b_g)$ be the characteristic sequence of f, see e.g. [26, Section 3.1]. Recall that this means that $f(x,y)=0$ has as root (say over x) a Puiseux expansion

$$x=\cdots+a_1y^{\frac{b_1}{n}}+\cdots+a_gy^{\frac{b_g}{n}}+\dots$$

with exactly g characteristic monomials. Denote $b_0:=n$ and define recursively

$$e^{(k)}:=\begin{cases} n & \text{if } k=0,\\ \gcd(e^{(k-1)},b_k) & \text{if } 1\le k\le g.\end{cases}$$

We define the following numbers for $1\le k\le g$:

$$R_k:=\frac{1}{e^{(k)}}\left(b_ke^{(k-1)}+\sum_{j=0}^{k-2}b_{j+1}\left(e^{(j)}-e^{(j+1)}\right)\right),\qquad r_k:=\frac{b_k+n}{e^{(k)}}.$$

Note that R_k admits the following recursive formula:

$$R_k:=\begin{cases} n & \text{if } k=0,\\ \frac{e^{(k-1)}}{e^{(k)}}\left(R_{k-1}+b_k-b_{k-1}\right) & \text{if } 1\le k\le g.\end{cases}$$

We end with the following definitions $R'_0:=n$, $r'_0:=2$ and for $1\le k\le g$:

$$R'_k:=\frac{R_ke^{(k)}}{e^{(k-1)}},\quad r'_k:=\left\lfloor r_ke^{(k)}/e^{(k-1)}\right\rfloor+1.$$

Yano defined the following polynomial with fractional powers in t

$$R(n,b_1,\dots,b_g;t):=t+\sum_{k=1}^{g}t^{\frac{r_k}{R_k}}\frac{1-t}{1-t^{\frac{1}{R_k}}}-\sum_{k=0}^{g}t^{\frac{r'_k}{R'_k}}\frac{1-t}{1-t^{\frac{1}{R'_k}}},\tag{3}$$

and he proved that $R(n,b_1,\dots,b_g;t)$ has non-negative coefficients.

Yano's Conjecture ([27])**.** *For almost all irreducible plane curve singularity germs $f:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ with characteristic sequence $(n,b_1,b_2,\dots,b_g)$, the b-exponents $\{\tilde{\beta}_1,\dots,\tilde{\beta}_\mu\}$ are given by the generating series*

$$\sum_{i=1}^{\mu}t^{\tilde{\beta}_i}=R(n,b_1,\dots,b_g;t).$$

For almost all means for an open dense subset in the μ-constant strata in a deformation space.

Yano's conjecture holds for $g = 1$ as it was proved by Pi. Cassou-Noguès in [10] making explicitly a relation between two variables improper integrals and the Bernstein-Sato polynomial of f, see also [9].

In [1], the authors, with the same ideas, were interested in the case $g = 2$. For $g = 2$, the characteristic sequence (n, b_1, b_2) can be written as $(n_1 n_2, mn_2, mn_2+q)$ where $n_1, m, n_2, q \in \mathbb{Z}_{>0}$ satisfying

$$\gcd(n_1, m) = \gcd(n_2, q) = 1.$$

In [1] we solve Yano's conjecture for the case

$$\gcd(q, n_1) = 1 \text{ or } \gcd(q, m) = 1. \tag{4}$$

The above condition is equivalent to ask for the algebraic monodromy to have distinct eigenvalues. In that case, the μ b-exponents are all distinct and they coincide with the opposite of roots of the reduced Bernstein-Sato polynomial (which turns out to be of degree μ).

To encode the topology of a germ of an irreducible plane curve singularity

$$(C = f^{-1}\{0\}, 0) \subset (\mathbb{C}^2, 0)$$

several sets of invariants can be used: Puiseux characteristic exponents, Puiseux pairs, Newton pairs, (minimal) embedded resolution graph, Eisenbud-Neumann splice diagram, semigroup $\Gamma_{(C,0)} \subset \mathbb{N}$ generated by all the possible intersection multiplicities $i(\{h = 0\}, C)$ at 0 for all $h \in \mathcal{O}_{(\mathbb{C}^2,0)}$, etc.

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a germ of a non-zero holomorphic function f. Let B be an open ball centered at the origin. Let $\pi : X \to B$ be an embedded resolution of $(f^{-1}\{0\}, 0)$. We denote by $E_i, i \in J$, the irreducible components of $\pi^{-1}(f^{-1}\{0\})_{\text{red}}$. For every $i \in J$, let N_i and $\nu_i - 1$ be the multiplicities of E_i in the divisor of respectively $f \circ \pi$ and $\pi^*(dx \wedge dy)$ on X. One has that N_i and ν_i belong to $\mathbb{N}^*$ and if E_i is an irreducible component of the strict transform of $f^{-1}\{0\}$ then $\nu_i = 1$. Denote also $\mathring{E}_i := E_i \setminus (\cup_{j \neq i} E_j)$ for $i \in J$. Then one has the following interpretation of the $R(n, b_1, \ldots, b_g; t)$

$$R(n, b_1, \ldots, b_g; t) = t - \sum_{i \in J,\, E_i \neq \tilde{C}} \chi(\mathring{E}_i) t^{\nu_i/N_i} \frac{1-t}{1-t^{1/N_i}}$$

where $\tilde{C}$ is the unique strict transform of $f^{-1}\{0\}$. For a vertex i of the minimal embedded resolution graph its valency δ_i is the number of adjacent vertices to it. A vertex is called a *rupture vertex* if its valency is at least 3. Most of the vertices in the resolution graph have valency 2 and since the corresponding exceptional divisors E_i are rational curves $\chi(\mathring{E}_i) = 0$. Furthermore in this case the valency of the vertex are either 1, 2 or 3.

The shape of the minimal embedded resolution graph in this case is the same as the Eisenbud-Neumann splice diagram (cf. [14, page 49]). If the germ $(C, 0)$ has g Newton pairs $\{(p_k, q_k)\}_{k=1}^g$ with $\gcd(p_k, q_k) = 1$ and $p_k \geq 2$ and $q_k \geq 1$ (and by convention, $q_1 > p_1$), define the integers $\{a_k\}_{k=1}^g$ by $a_1 := q_1$ and $a_{k+1} := q_{k+1} + p_{k+1} p_k a_k$ for $k \geq 1$. Then its Eisenbud-Neumann splice diagram decorated by the following splice data $\{(p_k, a_k)\}_{k=1}^g$ and has the following shape:

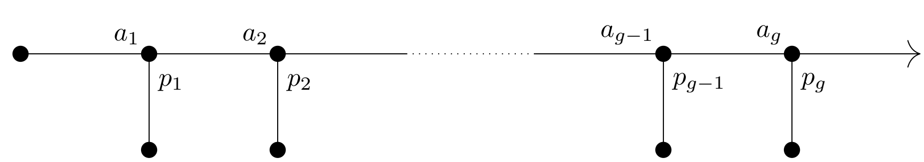


FIGURE 1.

The g rupture components $\tilde{E}_1, \dots, \tilde{E}_g$, ordered from the left to the right of the resolution graph are the same as in the splice diagram and their numerical data can be computed inductively from the

$$\begin{array}{ll} \tilde{N}_k := a_k \cdot p_k \cdot p_{k+1} \cdot \dots \cdot p_g & \text{for } 1 \leq k \leq g; \\ \tilde{\nu}_k := p_k \tilde{\nu}_{k-1} + q_k & \text{where } \tilde{\nu}_0 = 1, \end{array}$$

The numerical data associated to the components $g+1$ components of valency 1 $E_0, E_1, \dots, E_g$, here E_0 is the most left hand side vertex corresponding to the first blow-up and its numerical data is equal to $(N_0, \nu_0) = (n, 2)$ with $n = p_1 p_2 \cdots p_g$. The numerical data associated to other valency one components can be also computed from

$$\begin{array}{ll} N_k = a_k \cdot p_{k+1} \cdot \dots \cdot p_g & \text{for } 1 \leq k \leq g; \\ \nu_k = \tilde{\nu}_{k-1} + \lceil \frac{q_k}{p_k} \rceil & \text{for } 1 \leq k \leq g \end{array}$$

1.2. **Yano's conjecture for isolated germs of plane curves.**

A natural extension of the Yano conjecture for isolated plane curve singularity germ could be the following conjecture

Extended Yano's Conjecture. *For almost all isolated plane curve singularity germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with isolated singularity and Milnor number μ, the b-exponents $\{\tilde{\beta}_1, \dots, \tilde{\beta}_\mu\}$ are given by the generating series*

$$\sum_{i=1}^{\mu} t^{\tilde{\beta}_i} = t + \sum_i (\delta_i - 2) \left(t^{\nu_i / N_i} \frac{1-t}{1-t^{1/N_i}} \right), \tag{5}$$

showing how b-exponents depends on the topology of f.

Example 1.1. Let $f(x, y) = y^4 - x^6$ be a germ with two $\mathbb{A}_2$-singularities having intersection number equals 6. The minimal embeded resolution graph has 3 exceptional divisors E_1, E_2, E_3 with numerical data (N, ν, δ) given respectively by equals $(4, 2, 1)$, $(6, 3, 1)$ and $(12, 5, 4)$. Then (5) equals

$$t + 2 \left(t^{5/12} \frac{(1-t)}{(1-t^{1/12})} \right) - \left(t^{2/4} \frac{(1-t)}{(1-t^{1/4})} + t^{3/6} \frac{(1-t)}{(1-t^{1/6})} \right)$$

equals

$$t + t^{4/3} + t^{5/4} + t^{7/6} + 2t^{13/12} + 2t^{11/12} + t^{5/6} + t^{3/4} + t^{2/3} + 2t^{7/12} + 2t^{5/12}.$$

Using `Singular` [13] inside [23], a μ-constant versal deformation of f is given by

$$g(x, y, a, b) := f + ax^3y^2 + bx^4y^2$$

and the Bernstein-Sato polynomial of g for random values of a and b is equal to

$$-17/12, -4/3, -5/4, -7/6, -13/12, -1, -11/12, -5/6, -3/4, -2/3, -7/12, -5/12,$$

so that they do not coincide.

This can be confirmed using `checkRoot` for $s = -17/12$ of [16] in `Singular` [13], where the base field is $\mathbb{C}(a, b)$. Moreover, it can be proved that for general a, b the Tjurina number equals the expected value for Hertling-Stahlke bound, i.e., 14; using [17] the values of Tjurina number are constant in these μ-constant strata.

The previous example shows that the proposed conjecture may not hold when there are vertices with valency greater than 3. Based on the irreducible case we want to study the conjecture for the case where valencies are at most 3.

Modified extended Yano's Conjecture. *Let Σ_μ be the μ-constant stratum of a germ $f:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ of isolated singularity, such that no eigenvalue $\zeta\neq 1$ of the monodromy is mutiple (in particular the valency of the vertices of the resolution graph is at most 3). Then the μ b-exponents $\{\tilde{\beta}_1,\dots,\tilde{\beta}_\mu\}$ of a generic element of Σ_μ are given by the generating series (5)*

Most probably, the hypothesis on the monodromy can be replaced *no repeated non-integral exponent of the monodromy* as the result in [11] for non-degenerate Newton polynomial germs suggests; some examples in the last section go in the same direction. The condition on the valency seems to be more essential, due to Example 1.1.

1.3. Singularities with non-degenerated principal part and commode.

Assume that the power series f has non-degenerated principal part and denote its Newton polygon at 0 by Γ_f , with ℓ facets and commode (Γ_f meets with $x=0$ at $(0,\tau_0)$ and with $y=0$ at $(\sigma_0,0)$). We also assume that the set $\operatorname{Spec}(f)$ of spectral numbers are distinct.

Assume that $f_i(x,y)=1$, with $f_i(x,y)=\frac{c_i x+d_i y}{n_i}$, is the equation of the facet F_i of Γ_f so that $\gcd(c_i,d_i,n_i)=1$, $1\leq i\leq\ell$.

Set

$$\mathcal{N}=\{q\in\mathbb{Q}:\ \sigma_0 q\in\mathbb{N}\, or\, \tau_0 q\in\mathbb{N}\}.$$

Let b_f be the monic polynomial such that its roots are the rational numbers $\sigma_{i,k}:=-\frac{c_i+d_i+k}{n_i}$: with $0\leq k<n_i$ and for all facet F_i such that $\sigma_{i,k}\notin\mathcal{N}$.

Theorem 1.2 ([11, Theorem 1])**.** *For almost all germs of plane curves which have Γ_f as Newton polygon at the origin and all non-integral elements in* $\operatorname{Spec}(f)$ *are distinct then f admits b_f as Bernstein-Sato polynomial.*

Note that Example 1.1 does not satisfy the hypotheses of the above theorem. The minimal embeded resolution graph of germs in Theorem 1.2 has all exceptional divisors of valencies exactly $1,2$ and 3. There are at most 2 divisors with valency 1 and ℓ divisors of valency 3. For all $1\leq i\leq\ell$, let E_i be the corresponding divisor has numerical data $(N_i,\nu_i,\delta_i)=(n_i,c_i+d_i,3)$. So that the roots in this case appear as in the EN-diadram of the germ. So that a generic equisingular deformation of f admits b_f as Bernstein-Sato polynomial.

If two spectral numbers are congruent mod $\mathbb{Z}$, their difference is ±1, and they correspond to a 2-Jordan block of the monodromy, so we can recover the b-exponents from the Bernstein-Sato polynomial.

Proposition 1.3. *If the germ f is Newton non-degenerated with respect to its Newton polygon, commode and all the spectral numbers are distinct then for a generic equisingular deformation of f the b-exponents are given by (5).*

2. Improper integrals

Most of the results in this section come from [1]. We start with 1-variable improper integrals.

Proposition 2.1. *Let $f:[0,1]\times\mathbb{C}\to\mathbb{C}$ be an analytic function. Then the function*

$$s\mapsto\int_0^1 f(t,s)t^s\frac{dt}{t}$$

is holomorphic on $\Re s>0$ and admits a meromorphic continuation to $\mathbb{C}$ with poles contained in $\mathbb{Z}_{\leq 0}$. Moreover, if $f(t,s)$ is algebraic whenever t is algebraic and s rational, then, the residues are algebraic.

If the function f is independent of s, then the above function will be denoted by $G_f(s)$. Let us consider now the 2-variable case.

Proposition 2.2. *Let $f \in \mathbb{R}[x,y]$ such that $f > 0$ in $[0,1]^2$ and let $a_1, b_1, a_2, b_2 \in \mathbb{Z}_{\geq 0}$ (by convention $\frac{b_i}{a_i} = +\infty$ if $a_i = 0$). The function*

$$s \mapsto \int_0^1 \int_0^1 f(x,y)^s x^{a_1 s + b_1} y^{a_2 s + b_2} \frac{dx}{x} \frac{dy}{y}.$$

is holomorphic in $\Re s > \max\left(-\frac{b_1}{a_1}, -\frac{b_2}{a_2}\right)$ and admits a meromorphic continuation on $\mathbb{C}$, where the set of poles is a subset of $S = \left\{-\frac{b_1+\nu_1}{a_1},\ \nu_1 \in \mathbb{Z}_{\geq 0}\right\} \cup \left\{-\frac{b_2+\nu_2}{a_2},\ \nu_2 \in \mathbb{Z}_{\geq 0}\right\}$.

We can be more explicit on those poles.

Proposition 2.3. *With the hypotheses of Proposition 2.2, let $\alpha \in S$.*

(P1) *If $\alpha = -\frac{b_1+\nu_1}{a_1}$ for some $\nu_1 \in \mathbb{Z}_{\geq 0}$ and $\alpha \neq -\frac{b_2+\nu_2}{a_2}$ $\forall \nu_2 \in \mathbb{Z}_{\geq 0}$, then the pole is of order at most one and its residue equals*

$$\frac{1}{\nu_1! a_1} G_{h_{\nu_1,\alpha,x}}(a_2\alpha + b_2), \quad h_{\nu_1,\alpha,x}(y) := \frac{\partial^{\nu_1} f^\alpha}{\partial x^{\nu_1}}(0,y).$$

(P2) *If $\alpha = -\frac{b_2+\nu_2}{a_2}$ for some $\nu_2 \in \mathbb{Z}_{\geq 0}$ and $\alpha \neq -\frac{b_1+\nu_1}{a_1}$ $\forall \nu_1 \in \mathbb{Z}_{\geq 0}$, then the pole is of order at most one and its residue equals*

$$\frac{1}{\nu_2! a_2} G_{h_{\nu_2,\alpha,y}}(a_1\alpha + b_1), \quad h_{\nu_2,\alpha,y}(x) := \frac{\partial^{\nu_2} f^\alpha}{\partial y^{\nu_2}}(x,0).$$

(P3) *If $\alpha = -\frac{b_1+\nu_1}{a_1} = -\frac{b_2+\nu_2}{a_2}$ for some $\nu_1, \nu_2 \in \mathbb{Z}_{\geq} 0$, then the pole is of order at most 2 and the coefficient of $(s-\alpha)^{-2}$ in the Laurent expansion is*

$$\frac{1}{\nu_1!\nu_2! a_1 a_2} \frac{\partial^{\nu_1+\nu_2} f^\alpha}{\partial x^{\nu_1} \partial y^{\nu_2}}(0,0).$$

(P4) *If in the previous situation the pole is of order at most one, then the continuation of the functions $G_{h_{\nu_1,\alpha,x}}$ and $G_{h_{\nu_2,\alpha,y}}$ are holomorphic at $a_2\alpha + b_2$ and $a_1\alpha + b_1$, respectively and its residue equals*

$$\frac{1}{\nu_1! a_1} G_{h_{\nu_1,\alpha,x}}(a_2\alpha + b_2) + \frac{1}{\nu_2! a_2} G_{h_{\nu_2,\alpha,y}}(a_1\alpha + b_1).$$

The last result does not appear in [1] but it can be deduced easily. The following lemma is useful for the residue computations.

Lemma 2.4. *Let $p \in \mathbb{N}$ and $c \in \mathbb{R}_{>0}$. Given $s_1, s_2 \in \mathbb{C}$ such that $-\alpha = s_1 + s_2 > 0$ then*

$$G_{(y^p+c)^\alpha}(ps_1) + G_{(1+cx^p)^\alpha}(ps_2) = \frac{c^{-s_2}}{p} \boldsymbol{B}(s_1, s_2) \tag{6}$$

where $\boldsymbol{B}$ is the beta function.

In [1], we proceeded as follows. For a fixed equisingularity type, we consider *generic* polynomial representatives f with real algebraic coefficients, in some field $\mathbb{K}$, and such that for a suitable semi-algebraic compact domain $\mathcal{D}$, we had $f > 0$ in $\mathcal{D} \setminus \{(0,0)\}$ (the origin is in the boundary of $\mathcal{D}$). For a special choice of coordinates and a *weight* function g we consider the following integrals

$$\mathcal{I}(f, g, \beta_1, \beta_2, \beta_3)(s) := \int_{\mathcal{D}} f(x,y)^s x^{\beta_1} y^{\beta_2} g(x,y)^{\beta_3} \frac{dx}{x} \frac{dy}{y} \tag{7}$$

where $\beta_1, \beta_2, \beta_3 + 1 \in \mathbb{Z}_{>0}$. These integrals are holomorphic in a semiplane of $\mathbb{C}$ and admitted a meromorphic continuation (see Example 4.3 for an idea of the proof). The knowledge of the residues allowed us to prove the following theorem.

Theorem 2.5. *Let $f \in \mathbb{K}[x,y]$ be as above. Let α be a pole of $\mathcal{I}(f,\beta_1,\beta_2,\beta_3)(s)$ with transcendental residue, and such that $\alpha+1$ is not a pole of $\mathcal{I}(f,\beta_1',\beta_2',\beta_3')(s)$ for any $(\beta_1',\beta_2',\beta_3')$. Then α is a root of the Bernstein-Sato polynomial $b_f(s)$ of f.*

3. Partial proof of the conjecture

We are going to prove the modified extended conjecture when the number of rupture vertices is small.

Theorem 3.1. *The extended Yano's conjecture holds for germs of plane curve singularities with no multiple eigenvalues of the monodromy (except maybe 1), and such that there are at most two rupture vertices and their valency is at most 3.*

Sketch of the proof. As we have seen in Example 1.1, the valency condition and the non-existence of multiple values distinct from 1 seem to be essential. The condition of 1 or 2 branching vertices is only technical.

There are three types of such singularities.

(S1) The resolution graph is linear.
(S2) The germ is the product of two irreducible germs with one-Puiseux pair (m,n) and intersection number $> mn$, and eventually two smooth branches with intersection numbers m,n with the singular branches.
(S3) The resolution graph coincides with the one of a two-Puiseux pair irreducible (which is part of the germ).

The case (S1) is a consequence of [11, Theorem 1]. The case (S2) is represented by the μ-constant versal deformation of $f = x^\epsilon y^\eta((y^m - x^n)^2 - x^u y^v)$, where $\epsilon,\eta \in \{0,1\}$ and u,v depend on the intersection number of the two singular branches. We omit the cases where there are multiple eigenvalues distinct from 1. We follow the strategy in [1]. The presence of x,y does not affect this strategy as we explain later for (S3). If there are more than 2 branches, 1 is a multiple eigenvalue of the monodromy. Nevertheless, the only point where this condition is needed is for Varchenko's lower semicontinuity [24] and only eigenvalues distinct from 1 cannot be multiple for this result.

Let us finish with (S3). Let us consider the improper integral $\mathcal{I}(f,g,\beta_1,\beta_2,\beta_3)$ of (7), studied in [1], where $\beta_1,\beta_2,\beta_3+1 \in \mathbb{Z}_{>0}$, f,g are real polynomials positive on $[0,1]^2 \setminus \{(0,0)\}$, f is a 2-Puiseux-pair germ singularity for which the Newton polygone is of type $(y^m \pm x^n)^p$, g is a 1-Puiseux pair singularity with Newton polygone $y^m \pm x^n$ and maximal contact with f. For (S3) we replace f by $x^\epsilon y^\eta f g^\gamma$, $\epsilon,\eta,\gamma \in \{0,1\}$. We repeat the process as in [1]. ☐

4. Computations on examples with multiple eigenvalues

Example 4.1. Let us consider $f(x,y) = y^5 + x^2y^2 + x^5$; its μ-constant miniversal deformation is a singleton, so its Bernstein-Sato polynomial coincides with the generic one. This singularity does not satisfy [11, Theorem 1] since the exponents $\pm\frac{1}{10}, \pm\frac{3}{10}$ appear twice ($\pm\frac{1}{2}$ appear only once). Using `Singular`, the Bernstein polynomial is

$$\left(s+\frac{1}{2}\right)^2\left(s+\frac{7}{10}\right)\left(s+\frac{9}{10}\right)(s+1)\left(s+\frac{11}{10}\right)\left(s+\frac{13}{10}\right).$$

The extended conjecture is satisfied even though we are not in the hypotheses of the modified one.

Example 4.2. Let us consider $f(x,y) = y^5 + x^2y^2 + x^7$; its μ-constant versal deformation is also a singleton, so its Bernstein polynomial coincides with the generic one. This singularity *does* satisfy

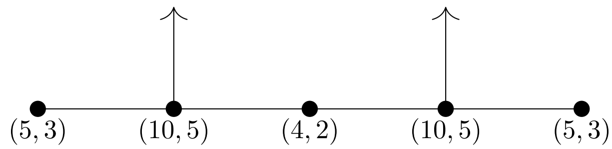
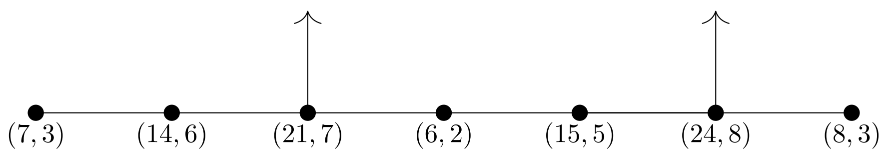


FIGURE 2. Resolution graph of $y^5 + x^2y^2 + x^5$ with (N, ν)-data.

[11, Theorem 1] since $\pm\frac{1}{2}$ appear as exponents of the monodromy, even though $\exp\left(2i\pi\frac{\pm 1}{2}\right) = -1$ is a double eigenvalue. Using `Singular`, we can confirm the expected Bernstein-Sato polynomial.

Example 4.3. Let us consider $f(x, y) = x^3y^3 + x^7 + y^8$; a μ-constant versal deformation is given by $f_{t,s}(x, y) := x^3y^3 + x^7 + tx^6y + sxy^7 + y^8$. As in the previous example the hypotheses of [11, Theorem 1] are satisfied and hence the extended conjecture holds; note that there are multiple eigenvalues for the monodromy but the exponents of the monodromy are distinct.



Yano's candidates start at $\frac{1}{3} = \frac{7}{21} = \frac{8}{24}$. The particular Bernstein-Sato polynomials may depend on s, t; let us study some jumps using improper integrals. Choose $t, s \in \mathbb{R}_{\geq 0}$; note that $f_{t,s} > 0$ in $[0, 1]^2 \setminus \{(0, 0)\}$. Let us denote, for $\beta_1, \beta_2 \in \mathbb{Z}_{\geq 1}$:

$$\mathcal{I}_{\beta_1,\beta_2} = \int_{[0,1]^2} f_{t,s}(x, y)^s x^{\beta_1} y^{\beta_2} \frac{dx}{x}\frac{dy}{y}$$

Let us decompose this square in two domains:

$$\{(x, y) \in [0, 1]^2 \mid x^{\frac{4}{3}} \leq y \leq 1\}, \quad \{(x, y) \in [0, 1]^2 \mid 0 \leq y \leq x^{\frac{4}{3}}\}.$$

Integrating on each subdomain we decompose $\mathcal{I}_{\beta_1,\beta_2} = \mathcal{I}_{1,\beta_1,\beta_2} + \mathcal{I}_{2,\beta_1,\beta_2}$.

Let us consider the change of variables $x \mapsto xy^3$, $y \mapsto y^4$:

$$x \mapsto xy^3, \quad y \mapsto y^4 \Longrightarrow \mathcal{I}_{1,\beta_1,\beta_2} = 4\int_{[0,1]^2} \tilde{f}_{t,s}(x, y)^s x^{\beta_1} y^{3\beta_1+4\beta_2+21s} \frac{dx}{x}\frac{dy}{y}$$

where

$$\tilde{f}_{t,s}(x, y) := tx^6y + sxy^{10} + x^7 + x^3 + y^{11}.$$

In the same way is $x \mapsto x^3$, $y \mapsto x^4y$;

$$x \mapsto x^3, \quad y \mapsto x^4y \Longrightarrow \mathcal{I}_{2,\beta_1,\beta_2} = 3\int_{]0,1]^2} f^*_{t,s}(x, y)^s x^{3\beta_1+4\beta_2+21s} y^{\beta_2} \frac{dx}{x}\frac{dy}{y}.$$

where

$$f^*_{t,s}(x, y) := txy + sx^{10}y^7 + x^{11}y^8 + y^3 + 1.$$

Note that I_{2,β_1,β_2} satisfies the hypotheses of Proposition 2.2, which was the goal of these changes of variables. Since it is not the case for I_{1,β_1,β_2}, let us perform a decomposition of the square as

$$\{(x, y) \in [0, 1]^2 \mid 0 \leq y \leq x^{\frac{3}{11}}\}, \quad \{(x, y) \in [0, 1]^2 \mid x^{\frac{3}{11}} \leq y \leq 1\},$$

and denote the corresponding integral decomposition as $I_{1,\beta_1,\beta_2} = I_{1,1,\beta_1,\beta_2} + I_{1,2,\beta_1,\beta_2}$. Suitable changes of variables yield:

$$x \mapsto x^{11},\ y \mapsto x^3 y \Longrightarrow \mathcal{I}_{1,1,\beta_1,\beta_2} = 44\int_{[0,1]^2} \hat{f}_{t,s}(x,y)^s x^{4(5\beta_1+3\beta_2+24s)} y^{3\beta_1+4\beta_2+21s} \frac{dx}{x}\frac{dy}{y},$$

where

$$\hat{f}_{t,s}(x,y) := tx^{36}y + sx^8y^{10} + x^{44} + y^{11} + 1,$$

and

$$x \mapsto xy^{11}, \quad y \mapsto y^3 \Longrightarrow \mathcal{I}_{1,2,\beta_1,\beta_2} = 12\int_{[0,1]^2} \check{f}_{t,s}(x,y)^s x^{\beta_1} y^{4(5\beta_1+3\beta_2+24s)} \frac{dx}{x}\frac{dy}{y},$$

where

$$\check{f}_{t,s}(x,y) := tx^6y^{36} + sxy^8 + x^7y^{44} + x^3 + 1.$$

The candidate pole $-\frac{8}{21}$ can be pole only for $\beta_1 = \beta_2 = 1$, and in this case the residue is

$$\begin{gathered}\frac{44}{21}\int_0^1 \frac{\partial \hat{f}^{-\frac{8}{21}}}{\partial y}(x,0)x^{-\frac{32}{7}}\frac{dx}{x} + \frac{3}{21}\int_0^1 \frac{\partial f^{*-\frac{8}{21}}}{\partial x}(0,y)y\frac{dy}{y} = \\ -\frac{8\cdot 44t}{21^2}\int_0^1 (1+x^{44})^{-\frac{29}{21}} x^{\frac{220}{7}}\frac{dx}{x} - \frac{3\cdot 8t}{21^2}\int_0^1 (1+y^3)^{-\frac{29}{21}} y^2\frac{dy}{y} = \\ -\frac{8t}{21^2}\int_0^1 (1+u)^{-\frac{29}{21}} u^{\frac{5}{7}}\frac{du}{u} - \frac{8t}{21^2}\int_0^1 (1+u)^{-\frac{29}{21}} u^{\frac{2}{3}}\frac{du}{u} = -\frac{8t}{21^2}\boldsymbol{B}\left(\frac{5}{7},\frac{2}{3}\right).\end{gathered}$$

Hence, for $t \neq 0$ (and algebraic), we have that $-\frac{8}{21}$ is a root of the Bernstein-Sato polynomial. Note that we can prove that $-\frac{29}{21}$ is a pole of $\mathcal{I}_{7,2}$ with transcendental residue for any (algebraic) value of t,s. In particular, $-\frac{29}{21}$ is a root of the Bernstein polynomial if $t=0$ and s is algebraic after Theorem 2.5. Note that $-\frac{8}{21}$ and $-\frac{29}{21}$ cannot be simultaneously roots of the Bernstein-Sato polynomial, since $\exp\left(-2i\pi\frac{8}{21}\right) = \exp\left(-2i\pi\frac{29}{21}\right)$ is a simple eigenvalue of the monodromy. These results are confirmed by `Singular` and `checkRoot`. We have then proved that there is a function f_0 in the μ-constant stratum such that $-\frac{8}{21}$ is not a root of Bernstein-Sato polynomial for f_0, compare with [2]

Example 4.4. Let us consider $f_\pm(x,y) := (x^4-y^3)^2 + x^6y^2$ which corresponds to the case (S3). A μ-constant versal deformation is given by $f_{\mathbf{t}}(x,y) = f_\pm(x,y) + t_1x^8y + t_2x^9$. Let $\mathcal{D} := \{(x,y) \in [0,1]^2 \mid 0 \le y \le x^{\frac{4}{3}}\}$ and for $t_1,t_2 \in \mathbb{R}_{\ge 0}$, consider

$$\mathcal{I}_{\beta_1,\beta_2,\beta_3} := \int_{\mathcal{D}} f_{\mathbf{t}}(x,y)^s x^{\beta_1} y^{\beta_2} (x^4-y^3)^{\beta_3} \frac{dx}{x}\frac{dy}{y}$$

for $\beta_1,\beta_2,\beta_3+1 \in \mathbb{Z}_{>0}$. In order to check that it is holomorphic with meromorphic continuation, we perform a first change of variable:

$$x \mapsto x^3, y \mapsto x^4(1-y) \Longrightarrow \mathcal{I}_{\beta_1,\beta_2,\beta_3} = 3\int_{[0,1]^2} \tilde{f}_{\mathbf{t}}(x,y)^s x^{3\beta_1+4\beta_2+12\beta_3+24s} y^{\beta_3+1} q(y)\frac{dx}{x}\frac{dy}{y}$$

where $q(y) := (1-y)^{\beta_2-1}(3-3y+y^2)^{\beta_3}$ and

$$\tilde{f}_{\mathbf{t}}(x,y) = y^2(3-3y+y^2)^2 + x^2(1-y)^2 + t_1x^4(1-y) + t_2x^3.$$

Decomposing the square in two triangles with the diagonal line, we can decompose

$$\mathcal{I}_{\beta_1,\beta_2,\beta_3} = \mathcal{I}_{1,\beta_1,\beta_2,\beta_3} + \mathcal{I}_{2,\beta_1,\beta_2,\beta_3};$$

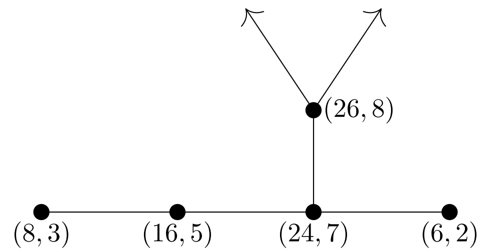


FIGURE 3.

with the following changes of variables we obtain

$$x \mapsto x, y \mapsto xy \Longrightarrow \mathcal{I}_{1,\beta_1,\beta_2,\beta_3}{=}3\int_{[0,1]^2} \hat{f}_{\mathbf{t}}(x,y)^s x^{3\beta_1+4\beta_2+13\beta_3+1+26s} y^{\beta_3+1} q(xy) \frac{dx}{x}\frac{dy}{y}$$

and $x \mapsto xy,\ y \mapsto y \Longrightarrow$:

$$\mathcal{I}_{2,\beta_1,\beta_2,\beta_3}{=}3\int_{[0,1]^2} \check{f}_{\mathbf{t}}(x,y)^s x^{3\beta_1+4\beta_2+12\beta_3+24s} y^{3\beta_1+4\beta_2+13\beta_3+1+26s} q(y) \frac{dx}{x}\frac{dy}{y},$$

where

$$\hat{f}_{\mathbf{t}}(x,y) = y^2(3-3xy+x^2y^2)^2 + (1-xy)^2 + t_1x^2(1-xy) + t_2x,$$
$$\tilde{f}_{\mathbf{t}}(x,y) = (3-3y+y^2)^2 + x^2(1-y)^2 + t_1x^4y^2(1-y) + t_2x^3y.$$

Example 4.5. A μ-constant miniversal deformation for $f(x,y) = (y^2-x^3)^2 + x^{12}$ is constant. It does not satisfy the hypotheses of the modified extended conjecture, since there are multiple eigenvalues (and multiple exponents of the monodromy) but, nevertheless, the extended conjecture holds.

Example 4.6. Let $f(x,y) := x(y^3-x^2)(y^2-x^{10})$, with μ-constant miniversal deformation $f_t(x,y) := f(x,y) + ty^7$. This example has multiple eigenvalues (besides 1) and it is a counterexample for the extended conjecture. It is not hard to prove that $\frac{19}{13}$ is not a Yano's candidate while $-\frac{19}{13}$ is a root of the Bernstein polynomial as it can be checked with `checkRoot` in `Singular` (working over $\mathbb{C}(t)$ instead of randomly evaluating t).

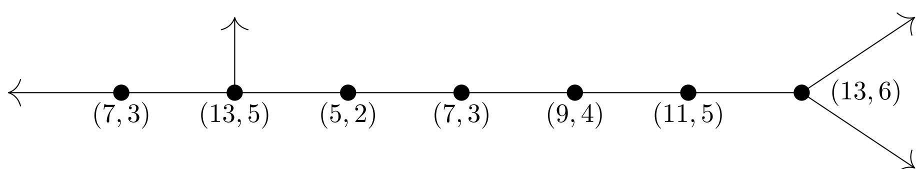


FIGURE 4. Resolution graph for Example 4.6

Example 4.7. Let $f(x,y) := y^{10} - x^3y^5 - x^{12}$. A μ-constant versal deformation is given by

$$f_{\mathbf{t}}(x,y) := f(x,y) + t_1x^7y^3 + t_2xy^9 + t_3x^9y^2 + t_4x^8y^3 + t_5x^{11}y$$
$$+t_6x^{10}y^2 + t_7x^9y^3 + t_8x^{11}y^2 + t_9x^{10}y^3 + t_{10}x^{11}y^3.$$

Using random values we can prove that $-\frac{19}{15}$ and $-\frac{4}{15}$ are both roots of the Bernstein polynomial, but only $\frac{4}{15}$ is a Yano's candidate.

References

[1] E. Artal Bartolo, Pi. Cassou-Noguès, I. Luengo, and A. Melle-Hernández, *Yano's conjecture for 2-Puiseux pairs irreducible plane curve singularities*, Publ. RIMS Kyoto Univ. **53** (2017), no. 1, 211–239. DOI: 10.4171/PRIMS/53-1-7

[2] E. Artal Bartolo, Pi. Cassou-Noguès, I. Luengo, and A. Melle-Hernández, *Bernstein polynomial of 2-Puiseux pairs irreducible plane curve singularities*, Methods Appl. Anal. **24** (2017), no. 2, 185–214. DOI: 10.4310/MAA.2017.v24.n2.a2

[3] I.N. Bernšteĭn, *Analytic continuation of generalized functions with respect to a parameter*, Funkcional. Anal. i Priložen. **6** (1972), no. 4, 26–40. DOI: 10.1007/BF01077645

[4] J.-E. Björk, *The Bernstein class of modules on algebraic manifolds*, Paul Dubreil and Marie-Paule Malliavin Algebra Seminar, 33rd Year (Paris, 1980), Lecture Notes in Math., vol. 867, Springer, Berlin-New York, 1981, pp. 148–156. DOI: 10.1007/BFb0090385

[5] J. Briançon, F. Geandier, and Ph. Maisonobe, *Déformation d'une singularité isolée d'hypersurface et polynômes de Bernstein*, Bull. Soc. Math. France **120** (1992), no. 1, 15–49. DOI: 10.24033/bsmf.2178

[6] J. Briançon, M. Granger, Ph. Maisonobe, and M. Miniconi, *Algorithme de calcul du polynôme de Bernstein: cas non dégénéré*, Ann. Inst. Fourier (Grenoble) **39** (1989), no. 3, 553–610. DOI: 10.5802/aif.1177

[7] J. Briançon, P. Maisonobe, and T. Torrelli, *Matrice magique associée à un germe de courbe plane et division par l'idéal jacobien*, Ann. Inst. Fourier (Grenoble) **57** (2007), no. 3, 919–953. DOI: 10.5802/aif.2281

[8] E. Brieskorn, *Die Monodromie der isolierten Singularitäten von Hyperflächen*, Manuscripta Math. **2** (1970), 103–161. DOI: 10.1007/BF01155695

[9] Pi. Cassou-Noguès, *Racines de polyômes de Bernstein*, Ann. Inst. Fourier (Grenoble) **36** (1986), no. 4, 1–30. DOI: 10.5802/aif.1067

[10] Pi. Cassou-Noguès, *Séries de Dirichlet et intégrales associées à un polynôme à deux indéterminées*, J. Number Theory **23** (1986), no. 1, 1–54. DOI: 10.1016/0022-314X(86)90002-8

[11] Pi. Cassou-Noguès, *Polynôme de Bernstein générique*, Abh. Math. Sem. Univ. Hamburg **58** (1988), 103–123. DOI: 10.1007/BF02941372

[12] S. C. Coutinho, *A primer of algebraic D-modules*, London Mathematical Society Student Texts, vol. 33, Cambridge University Press, Cambridge, 1995. DOI: 10.1017/CBO9780511623653

[13] Wolfram Decker, Gert-Martin Greuel, Gerhard Pfister, and Hans Schönemann, SINGULAR *4-1-1 — A computer algebra system for polynomial computations*, `http://www.singular.uni-kl.de`, 2018.

[14] D. Eisenbud and W.D. Neumann, *Three-dimensional link theory and invariants of plane curve singularities*, Annals of Mathematics Studies, vol. 110, Princeton University Press, Princeton, NJ, 1985.

[15] M. Kashiwara, *B-functions and holonomic systems. Rationality of roots of B-functions*, Invent. Math. **38** (1976/77), no. 1, 33–53. DOI: 10.1007/BF01390168

[16] V. Levandovskyy and J. Martín-Morales, *Algorithms for checking rational roots of b-functions and their applications*, J. Algebra **352** (2012), 408–429. DOI: 10.1016/j.jalgebra.2011.12.002

[17] I. Luengo and G. Pfister, *Normal forms and moduli spaces of curve singularities with semigroup* $\langle 2p, 2q, 2pq+d\rangle$, Compositio Math. **76** (1990), no. 1-2, 247–264, Algebraic geometry (Berlin, 1988). DOI: 10.1007/978-94-009-0685-3_12

[18] B. Malgrange, *Le polynôme de Bernstein d'une singularité isolée*, Fourier integral operators and partial differential equations (Colloq. Internat., Univ. Nice, Nice, 1974), Springer, Berlin, 1975, pp. 98–119. Lecture Notes in Math., Vol. 459. DOI: 10.1007/BFb0074194

[19] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, No. 61, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968.

[20] M. Saito, *On the structure of Brieskorn lattice*, Ann. Inst. Fourier (Grenoble) **39** (1989), no. 1, 27–72. DOI: 10.5802/aif.1157

[21] M. Saito, *On microlocal b-function*, Bull. Soc. Math. France **122** (1994), no. 2, 163–184. DOI: 10.24033/bsmf.2227

[22] J.H.M. Steenbrink, *Mixed Hodge structure on the vanishing cohomology*, Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 525–563.

[23] The Sage Developers, *Sagemath, the Sage Mathematics Software System (Version 8.1)*, 2017, `http://www.sagemath.org`.

[24] A.N. Varčenko, *Gauss-Manin connection of isolated singular point and Bernstein polynomial*, Bull. Sci. Math. (2) **104** (1980), no. 2, 205–223.

[25] A.N. Varčenko, *The complex singularity index does not change along the stratum* $\mu = \text{const}$, Funktsional. Anal. i Prilozhen. **16** (1982), no. 1, 1–12, 96.

[26] C.T.C. Wall, *Singular points of plane curves*, London Mathematical Society Student Texts, vol. 63, Cambridge University Press, Cambridge, 2004. DOI: 10.1017/CBO9780511617560

[27] T. Yano, *Exponents of singularities of plane irreducible curves*, Sci. Rep. Saitama Univ. Ser. A **10** (1982), no. 2, 21–28.

E. Artal Bartolo, Departamento de Matemáticas-IUMA, Universidad de Zaragoza, c/ Pedro Cerbuna 12, 50009 Zaragoza, SPAIN

Email address: artal@unizar.es

Pi. Cassou-Noguès, Institut de Mathématiques de Bordeaux, Université de Bordeaux, 350, Cours de la Libération, 33405, Talence Cedex 05, FRANCE

Email address: Pierrette.Cassou-nogues@math.u-bordeaux.fr

I. Luengo, ICMAT (CSIC-UAM-UC3M-UCM), Dpto. de Álgebra, Geometría y Topología, Universidad Complutense de Madrid, Plaza de las Ciencias s/n, Ciudad Universitaria, 28040 Madrid, SPAIN

Email address: iluengo@ucm.es

A. Melle-Hernández, Instituto de Matemática Interdisciplinar (IMI), Dpto. de Álgebra, Geometría y Topología, Universidad Complutense de Madrid, Plaza de las Ciencias s/n, Ciudad Universitaria, 28040 Madrid, SPAIN

Email address: amelle@ucm.es