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SINGULARITIES AND POLYHEDRA[1]

EGBERT BRIESKORN

I reported about work of my students Thomas Fischer, Alexandra Kaess, Ute Neuschäfer, Frank Rothenhäusler and Stefan Scheidt. This work describes the neighbourhood boundaries of quasi–homogeneous surface singularities in a new way. It is known that these neighbourhood boundaries are quotients G/Γ of a 3–dimensional Lie group G and a discrete subgroup Γ. For example, for the quotient singularities $\mathbf{C}^2/\Gamma$ the group G is $\mathrm{Spin}(3){=}S^3$, the group of unit quaternions, and Γ could for example be one of the three binary polyhedral groups (binary tetrahedral $\mathbb{T}$, binary octahedral $\mathbb{O}$, binary icosahedral $\mathbb{I}$). This gives the three singularities E_6, E_7, E_8. For the next set of examples, the simply–elliptic singularities $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, the group G is the Heisenberg group, and Γ is a congruence subgroup of the lattice of its integral matrices. In most cases however, G is $\mathrm{SU}(1,1)$ or some covering of it, and Γ comes from a Fuchsian group $\overline{\Gamma} \subset \mathrm{PSU}(1,1)$ acting on the hyperbolic plane $\mathbb{H} = \{x \in \mathbb{C} |\, |z| < 1\}$. All of this is well known.

Now I describe a very original construction discovered by Thomas Fischer in his 1992 PhD–thesis:

Let $\overline{\Gamma} \subset \mathrm{PSU}(1,1)$ be discrete with compact quotient $\mathbb{H}/\overline{\Gamma}$. Assume that $\overline{\Gamma}$ has at least one point in $\mathbb{H}$ with nontrivial isotropy subgroup. *Choose* such a point $o \in \mathbb{H}$. Let p be the order of its isotropy group $\{\overline{\gamma} \in \overline{\Gamma} \,|\, \overline{\gamma}(o) = o\}$. Let $\Gamma \subset \mathrm{SU}(1,1)$ be the inverse image of $\overline{\Gamma}$. For many singularities, the neighbourhood boundary is of the form $\mathrm{SU}(1,1)/\Gamma$ with a suitable $\overline{\Gamma}$. For example, for the 14 quasihomogeneous exceptional 1–modular singularities $E_{12}, E_{13}, E_{14}, Z_{11}, Z_{12}, Z_{13}, Q_{10}, Q_{11}, Q_{12}, W_{12}, W_{13}, S_{11}, U_{12}$ the group Γ is the group of orientation–preserving automorphisms of $\mathbb{H}$ in the group $\sum(p,q,r)$ generated by the reflections in the sides of a hyperbolic triangle with angles $\pi/p, \pi/q, \pi/r$. In this case, the choice of $o \in \mathbb{H}$ amounts to choosing one of the integers in the so–called Dolgachev triple (p,q,r). We shall indicate this by underlining this number, e.g. $(2,3,\underline{7})$. Fischer's construction:

$$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \,|a\bar{a} - b\bar{b} = 1 \right\} = \left\{ x \in \mathbb{R}^4 \;\middle|\; x_0^2 + x_1^2 - x_3^2 - x_4^2 = 1 \right\} =: \mathbb{S}$$

is a 3–dimensional pseudosphere with Minkowski–metric with signature $(+,-,-)$. Up to a factor $-1/8$, this agrees with the Killing metric. The construction will be done in $\mathbb{R}^4$ with $\langle x,x\rangle = x_0^2 + x_1^2 - x_3^2 - x_4^2$. Let C^+ be the positive cone $C^+ = \{x \in \mathbb{R}^4 \,|\langle x,x\rangle > 0\}$ and $\pi : C^+ \to \mathbb{S}$ be the retraction by central projection $\pi(x) := x/\sqrt{\langle x,x\rangle}$. For any $g \in \mathbb{S}$, let H_g be the halfspace $H_g := \{x \in \mathbb{R}^4 | \langle x,g\rangle \leq 1\}$. Its boundary ∂H_g is the affine tangent space $\partial H_g = T_g(\mathbb{S})$. For any $z \in \overline{\Gamma}(o)$ in the chosen special orbit $\overline{\Gamma}(o) \subset \mathbb{H}$, let L_z be the coset $L_z = \{\gamma \in \Gamma \mid \gamma(o) = z\}$. It has the cardinality $2p$. Let $Q_z \in \mathbb{R}^4$ be defined by

$$Q_z := \bigcap_{g \in L_z} H_g \; .$$

[1]Tagungsbericht 27/1996, Singularitäten 14.07.-20.07.1996, Mathematisches Forschungsinstitut Oberwolfach (MFO)..

Q_z is a 4–dimensional prism, the product of $\mathbb{R}^2$ with a plane $2p$–gon. Consider

$$P := \bigcup_{z \in \overline{\Gamma}(o)} Q_z$$

and $\partial_+ P := \partial P \cap C^+$.

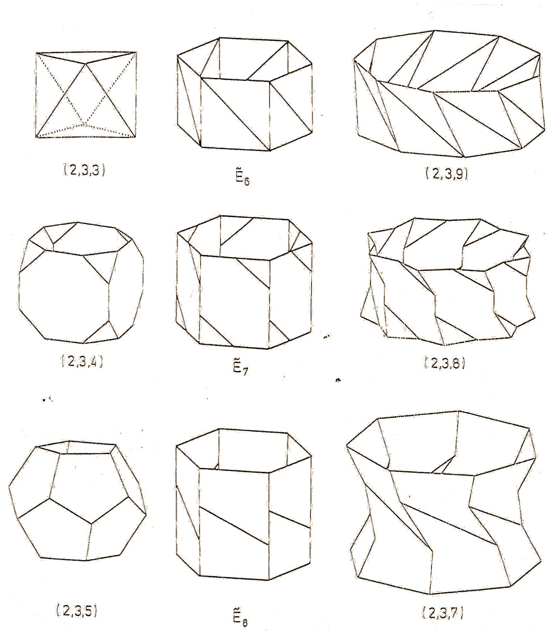
$\partial_+ P$ is the support of a 3–dimensional polyhedral complex and $\pi : \partial_+ P \to \mathbb{S}$ is a homeomorphism, which transfers the polyhedral structure to $\mathbb{S}$. The following definition and theorem of Fischer analyzes this structure:

Definition: $F_g = C^+ \cap \partial H_g \cap (Q_{g(o)} \smallsetminus \bigcup_{\substack{z \in \Gamma(o) \\ z \neq g(o)}} Q_z)$.

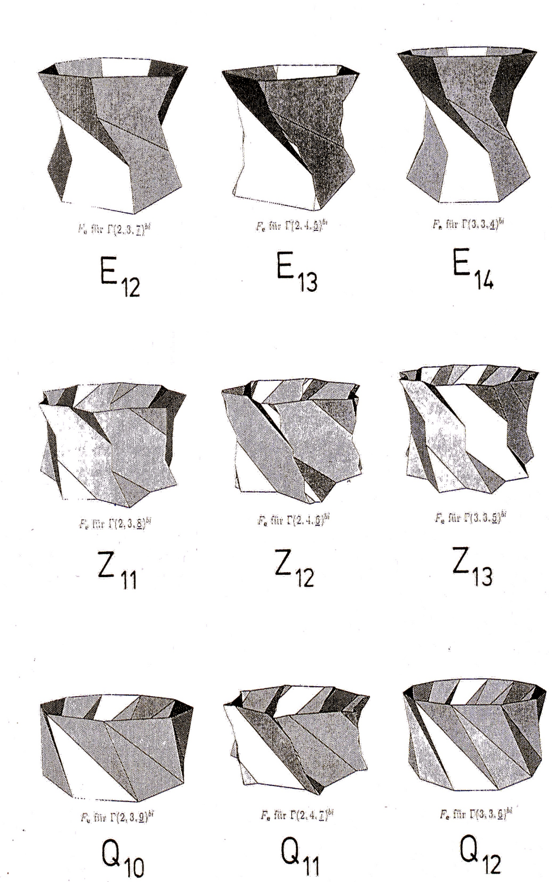
Theorem:

(1) F_g is a compact polyhedron in the Minkowski–3–space ∂H_g
(2) $\{F_g\}_{g \in \Gamma}$ is the set of 3–dimensional faces of a 3–dimensional polyhedral complex with support $\partial_+ P$.
(3) Γ operates simply transitively on $\{F_g | g \in \Gamma\}$.
(4) $\{\pi(F_g)\}$ is a tesselation of $\mathbb{S}$ by totally geodesic polyhedra in this Minkowski–pseudosphere. Γ acts simply transitively on the set of these $\pi(F_g)$, so each of them can serve as a fundamental domain.
(5) Hence $\mathbb{S}/\Gamma$ is obtained from F_G by pairing faces and identifying them in a specified way given by Γ and the construction.

Fischer calculated the examples $(2, 3, \underline{7}), (2, 3, \underline{8}), (2, 3, \underline{9})$. These fit in very well with the classical cases $E_6 = (2, 3, \underline{3}), E_7 = (2, 3, \underline{4})$ and $E_8 = (2, 3, \underline{5})$. I myself added an analysis of the cases $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$. The following pictures show the resulting 9 fundamental domains:



The other four students worked out all 14 exceptional $(p, q, , \underline{r})$ with the exception of $r = 2$. As a result, a pattern seems to emerge. The following shows a sample of their pictures:



I presented some conjectures on the series–patterns. Work in progress by Ludwig Balke may lead to a new and original way of looking at symmetry–breaking.

The following pages show the handwritten notes of Brieskorn from the "Vortragsbuch" of the singularities workshop 1996 in Oberwolfach.

Singularities and Polyhedra.

Egbert Brieskorn, Bonn

I reported about work of my students Thomas Fischer, Alexandra Kaess, Ute Neuschäfer, Frank Rothenhäusler and Stefan Scheidt. This work describes the neighbourhood boundaries of quasi-homogeneous surface singularities in a new way. It is known that these neighbourhood boundaries are quotients G/Γ of a 3-dimensional Liegroup and a discrete subgroup Γ. For example, for the quotient singularities $\mathbb{C}^2/\Gamma$ the group G is $\mathrm{Spin}(3) = S^3$, the group of unit quaternions, and Γ could for example be one of the three binary polyhedral groups (binary tetrahedral $\mathbb{T}$, binary octahedral $\mathbb{O}$, binary icosahedral $\mathbb{I}$. ~~As an~~ This gives the three singularities E_6, E_7, E_8. For the next set of examples, the simply-elliptic singularities $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ the group G is the Heisenberg group, and Γ is a congruence subgroup of the lattice of its integral matrices. In most cases however, G is $SU(1,1)$ or some covering of it, and Γ comes from a Fuchsian group $\bar{\Gamma} \subset PSU(1,1)$ acting on the hyperbolic plane $\mathbb{H} = \{z \in \mathbb{C} \mid |z| < 1\}$. All this is well known.

Now I describe a very original construction discovered by Thomas Fischer in his 1992 Ph.D. thesis.

Let $\bar{\Gamma} \subset PSU(1,1)$ be discrete with compact quotient $\mathbb{H}/\bar{\Gamma}$.
Assume, that $\bar{\Gamma}$ has at least one point in $\mathbb{H}$ with nontrivial isotropy subgroup. Choose such a point $o \in \mathbb{H}$.
Let p be the order of its isotropy group $\{\bar{\gamma} \in \bar{\Gamma} \mid \bar{\gamma}(p) = o\}$.
Let $\Gamma \subset SU(1,1)$ be the inverse image of $\bar{\Gamma}$.
For many singularities, the neighbourhood boundary is of the form $SU(1,1)/\Gamma$ with a suitable $\bar{\Gamma}$.
For example, for the 14 quasihomogeneous exceptional 1-modular singularities $E_{12}, E_{13}, E_{14}, Z_{11}, Z_{12}, Z_{13}, Q_{10}, Q_{11}, Q_{12},$

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288

Definition: $F_g = C^+ \cap \partial H_g \cap \big(Q_{g(0)} \setminus \bigcup_{\substack{z \in \Gamma(0) \\ z \neq g(0)}} \mathring{Q}_z \big)$

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Egbert Brieskorn