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ON THE TOPOLOGY OF A RESOLUTION OF ISOLATED SINGULARITIES, II

VINCENZO DI GENNARO AND DAVIDE FRANCO

Abstract. Let Y be a complex projective variety of dimension n with isolated singularities, $\pi : X \to Y$ a resolution of singularities, $G := \pi^{-1}(\mathrm{Sing(Y)})$ the exceptional locus. From the Decomposition Theorem one knows that the map $H^{k-1}(G) \to H^k(Y, Y\backslash\mathrm{Sing}(Y))$ vanishes for $k > n$. It is also known that, conversely, assuming this vanishing one can prove the Decomposition Theorem for π in few pages. The purpose of the present paper is to exhibit a direct proof of the vanishing. As a consequence, it follows a complete and short proof of the Decomposition Theorem for π, involving only ordinary cohomology.

1. Introduction

Consider an n-dimensional integral complex projective variety Y with *isolated singularities*. Fix a *resolution of singularities* $\pi : X \to Y$ of Y. This means that X is an irreducible and smooth projective variety, and π is a birational morphism inducing an isomorphism $X\backslash\pi^{-1}(\mathrm{Sing}(Y)) \cong Y\backslash\mathrm{Sing}(Y)$. In this case, the celebrated *Decomposition Theorem* of Deligne, Gabber, Beilinson, and Bernstein [1], assumes the following form:

Theorem 1.1 (The Decomposition Theorem for varieties with isolated singularities)**.** *In $D^b(Y)$, we have a decomposition*

$$R\pi_*\mathbb{Q}_X \cong IC^\bullet_Y[-n] \oplus \mathcal{H}^\bullet,$$

where $\mathcal{H}^\bullet$ is quasi isomorphic to a skyscraper complex on $\mathrm{Sing}(Y)$*. Furthermore, we have*

1. $\mathcal{H}^k(\mathcal{H}^\bullet) \cong H^k(G)$*, for all* $k \geq n$,
2. $\mathcal{H}^k(\mathcal{H}^\bullet) \cong H_{2n-k}(G)$*, for all* $k < n$,

where $G := \pi^{-1}(\mathrm{Sing}(Y))$*, and $H^k(G)$ and $H_{2n-k}(G)$ have $\mathbb{Q}$-coefficients.*

In [6] Goresky and MacPherson remarked that from previous theorem one deduces the following vanishing, concerning ordinary cohomology (compare also with [13, (1.11) Theorem], and with Notations, (*ii*), below):

$$H^{k-1}(G) \to H^k(Y, Y\backslash\mathrm{Sing}(Y)) \ \text{ vanishes for } \ k > n. \tag{1}$$

More recently, in [3] we observed that, conversely, assuming the vanishing (1) one can prove Theorem 1.1 in few pages [3, Theorem 3.1].

Continuing [3], in the present paper we give a direct proof of the vanishing (1), without using Theorem 1.1. Therefore, combining with [3, Theorem 3.1], it follows a complete and short proof of the Decomposition Theorem for π, involving only ordinary cohomology.

Our proof of the vanishing (1) relies on an argument similar to that developed by Navarro in [8, (5.1) Proposition]. First, using certain preliminary results we already stated in [3, Lemma 4.1

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and Lemma 4.2], we reduce to the case the exceptional locus $G = \pi^{-1}(\mathrm{Sing}(Y))$ is a *simple normal crossing divisor* [7, p. 240], and X admits an ample line bundle of the form $\pi^*(\mathcal{L}) \otimes \mathcal{O}_X(D)$, where $\mathcal{L}$ is an ample line bundle on Y, and D a divisor supported on G (see Lemma 3.1 and Lemma 3.2 below). Next, we conclude using again [3, Lemma 4.1], general properties of mixed Hodge Theory, and a slight generalization of a Lemma of Steenbrink appearing in [8, p. 288] (see Lemma 3.4, Lemma 3.5 and Lemma 3.6 below).

Our strategy has some points of contact with that of de Cataldo and Migliorini [2, pp. 572-575], but also some important differences. Indeed, in both approaches the main point consists in proving that the map $\epsilon_i : H_{n+i}(G) \to H^{n+i}(G)$ is an isomorphism for all $i \geq 0$ (see (3) and Remark 3.3 below). However, in order to accomplish this, de Cataldo and Migliorini reduce to the semismall case, where they crucially prove that the Hodge-Riemann bilinear relations hold true for a divisor on X which is the pull-back of an ample divisor on Y. Instead, we reduce to the divisorial case, and lean on the classical Hodge-Riemann bilinear relations for an ample line bundle on X (see the proof of Lemma 3.6 below).

It is known that the Decomposition Theorem generalizes to the case Y is the germ of an isolated singularity [2, p. 551], [10], [11], [12]. We do not know whether our approach can be used also in this case. In fact, it does not seem to us completely local in nature, in view of the crucial role played by the Hodge-Riemann bilinear relations for X. We have in mind to return on this question in a future paper.

2. Notations

(i) For a function $f : A \to B$ we denote by $\Im(f)$ the image of f, i.e., $\Im(f) = f(A)$.

(ii) We recall how the map $H^{k-1}(G) \to H^k(Y, Y \backslash \mathrm{Sing}(Y))$ appearing in (1) is defined (compare with [3, p. 198, (iv)]). Embed Y in some projective space $\mathbb{P}^N$. For all $y \in \mathrm{Sing}(Y)$ choose a small closed ball $S_y \subset \mathbb{P}^N$ around y, and set $B_y := S_y \cap Y$, $D_y := \pi^{-1}(B_y)$, $B := \bigcup_{y \in \mathrm{Sing}(Y)} B_y$, and $D := \pi^{-1}(B)$. B_y is homeomorphic to the cone over the link ∂B_y of the singularity $y \in Y$, with vertex at y. B_y is contractible, by excision we have

$$H^k(Y, Y \backslash \mathrm{Sing}(Y)) \cong H^k(B, B \backslash \mathrm{Sing}(Y)) \cong H^k(B, \partial B)$$

for all k, and from the cohomology long exact sequence of the pair $(B, \partial B)$, we get

$$H^k(Y, Y \backslash \mathrm{Sing}(Y)) \cong H^{k-1}(\partial B)$$

for all $k \geq 2$. Since $\partial D \cong \partial B$ via π, we deduce

$$H^k(Y, Y \backslash \mathrm{Sing}(Y)) \cong H^{k-1}(\partial D)$$

for all $k \geq 2$. On the other hand, G is homotopy equivalent to D. Therefore, we also have

$$H^{k-1}(G) \cong H^{k-1}(D).$$

It follows that we may identify the restriction map $H^{k-1}(D) \to H^{k-1}(\partial D)$ with a map

$$H^{k-1}(G) \to H^k(Y, Y \backslash \mathrm{Sing}(Y)).$$

3. The proof of the vanishing

We need the following two lemmas. The first one is certainly well known, but we prove it for lack of a suitable reference.

Lemma 3.1. *Let $\pi : X \to Y$ be a resolution of singularities of Y. Then there exists a resolution of singularities $p : Z \to Y$ of Y satisfying the following conditions:*

(1) there is a morphism $\pi_X : Z \to X$ such that $p = \pi \circ \pi_X$;

(2) $\Gamma := p^{-1}(\mathrm{Sing}(Y))$ is a simple normal crossing divisor (s.n.c.) on Z;

(3) for every ample line bundle $\mathcal{L}$ on Y, there are integers $a, a_1, \ldots, a_r$ $(a > 0)$ such that $p^*(\mathcal{L}^{\otimes a}) \otimes \mathcal{O}_Z\left(\sum_{i=1}^r a_i \Gamma_i\right)$ is an ample line bundle on Z, the sum being taken over the components of Γ.

Proof. Let $\pi_1 : X_1 \to Y$ be a resolution of singularities verifying condition (2), i.e. such that $\pi_1^{-1}(\mathrm{Sing}(Y))$ is a s.n.c. divisor [7, Theorem 4.1.3]. One can construct $\pi_1 : X_1 \to Y$ via a sequence $X_1 = B_h \to B_{h-1} \to \cdots \to B_1 = Y$ of blowings-up along smooth centers supported in the singular locus of Y [7, loc. cit.]. Fix an ample line bundle $\mathcal{L}$ on Y, and an integer $h > 0$ such that $\mathcal{L}^{\otimes h}$ is very ample, corresponding to a closed immersion $i : Y \subset \mathbb{P}$ of Y in some projective space $\mathbb{P}$ ($\mathcal{L}^{\otimes h} = i^*(\mathcal{O}_{\mathbb{P}}(1))$). Each element B_j of this sequence is contained in an element C_j of a sequence $C_h \to C_{h-1} \to \cdots \to C_1 = \mathbb{P}$ of blowings-up along the *same* smooth centers, starting from $\mathbb{P}$. By [5, Proposition 6.7, (e)], we know that the Picard group of each C_j is generated by the pull-back of the hyperplane class of $\mathbb{P}$, and certain divisor classes supported in the singular locus of Y. Therefore, an ample line bundle $\mathcal{M}$ on C_h is *necessarily* the pull-back of a positive power of $\mathcal{O}_{\mathbb{P}}(1)$, tensored with a line bundle like $\mathcal{O}_{C_h}(E)$, with E divisor supported in the singular locus of Y. Restricting such ample bundle $\mathcal{M}$ on X_1, we get a line bundle as in (3). This proves that there exists a resolution $\pi_1 : X_1 \to Y$ satisfying conditions (2) and (3).

Now consider the fibred product $X \times_Y X_1$. It contains $U := Y \backslash \mathrm{Sing}(Y)$. Let $\overline{U}$ be the closure of U in $X \times_Y X_1$. Applying [7, loc. cit.] to the blowing-up of $\overline{U}$ along $\overline{U} \backslash U$, we may construct a resolution of singularities $\varphi : Z \to \overline{U}$ of $\overline{U}$, inducing an isomorphism $\varphi^{-1}(U) \cong U$, and such that $\Gamma := \varphi^{-1}(\overline{U} \backslash U)$ is a s.n.c. divisor on Z. Composing $\varphi : Z \to \overline{U}$ with the inclusion $\overline{U} \subset X \times_Y X_1$, and the projections $X \times_Y X_1 \to X$ and $X \times_Y X_1 \to X_1$, we get maps $\pi_{X_1} : Z \to X_1$, $\pi_X : Z \to X$ and $p : Z \to Y$, with $p = \pi_1 \circ \pi_{X_1} = \pi \circ \pi_X$:

$$\begin{tikzcd} & Z \arrow[dl, "\pi_{X_1}"'] \arrow[dr, "\pi_X"] \arrow[dd, "p"] & \\ X_1 \arrow[dr, "\pi_1"'] & & X \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

The morphism $p : Z \to Y$ is the map we are looking for. In fact, $p : Z \to Y$ is a resolution of singularities of Y. Moreover, it satisfies conditions (1) and (2) because $p = \pi \circ \pi_X$ and $\Gamma = \varphi^{-1}(\overline{U} \backslash U) = p^{-1}(\mathrm{Sing}(Y))$ is a s.n.c. divisor. It remains to check condition (3) for p.

To this aim, fix an ample line bundle $\mathcal{L}$ on Y. Since $\pi_1 : X_1 \to Y$ satisfies condition (3), there exists an integer $a > 0$ and a divisor E_1 on X_1, supported in the singular locus of Y, such that $\mathcal{L}_1 := \pi_1^*(\mathcal{L}^{\otimes a}) \otimes \mathcal{O}_{X_1}(E_1)$ is an ample line bundle on X_1. Since $\pi_{X_1} : Z \to X_1$ is a birational morphism between *smooth* projective varieties, by [7, Corollary 4.1.4 and Remark 4.1.5] we know that there exists an integer $b > 0$ and a divisor E_2 on Z, supported in the singular locus of Y, such that $\pi_{X_1}^*(\mathcal{L}_1^{\otimes b}) \otimes \mathcal{O}_Z(E_2)$ is an ample line bundle on Z. Since $p = \pi_1 \circ \pi_{X_1}$, it follows that also $p : Z \to Y$ satisfies condition (3). $\square$

Lemma 3.2. *Let $\pi : X \to Y$ and $p : Z \to Y$ be resolutions of singularities of Y. Set $U := Y \backslash \mathrm{Sing}(Y)$, $G := \pi^{-1}(\mathrm{Sing}(Y))$ and $\Gamma := p^{-1}(\mathrm{Sing}(Y))$. Assume there is a morphism*

$\pi_X : Z \to X$ such that $p = \pi \circ \pi_X$. Fix an integer $k > n$. If the map $H^{k-1}(\Gamma) \to H^k(Y, U)$ vanishes, then also the map $H^{k-1}(G) \to H^k(Y, U)$ vanishes.

Proof. Consider the following commutative diagram given by the pull-back:

$$\begin{tikzcd} H^{k-1}(Y) \arrow[r, "\pi^*"] \arrow[dr, "\alpha^*_{k-1}"'] & H^{k-1}(X) \arrow[r, "\pi_X^*"] \arrow[d, "\beta^*_{k-1}"] & H^{k-1}(Z) \arrow[dl, "\gamma^*_{k-1}"] \\ & H^{k-1}(U). & \end{tikzcd}$$

If the map $H^{k-1}(\Gamma) \to H^k(Y, U)$ vanishes, from [3, Lemma 4.1 and 4.2], it follows that $\Im(\alpha^*_{k-1}) = \Im(\gamma^*_{k-1})$. Since previous diagram commutes, we also have

$$\Im(\alpha^*_{k-1}) \subseteq \Im(\beta^*_{k-1}) \subseteq \Im(\gamma^*_{k-1}).$$

Therefore we get $\Im(\alpha^*_{k-1}) = \Im(\beta^*_{k-1})$. Again from [3, loc. cit.], it follows that the map $H^{k-1}(G) \to H^k(Y, U)$ vanishes. □

By previous Lemma 3.1 and Lemma 3.2, in order to prove the vanishing (1) we may assume that G is a s.n.c. divisor, and that there exists a very ample line bundle $\mathcal{L}$ on Y, and integers $a_1, \dots, a_r$, such that

$$\mathcal{M} := \pi^*(\mathcal{L}) \otimes \mathcal{O}_X \left(\sum_{j=1}^{r} a_j G_j \right) \tag{2}$$

is a very ample line bundle on X, the sum being taken over the components of G. Denote by $[\mathcal{M}] \in H^2(X)$ its cohomology class, and by $\mu \in H^2(G)$ its restriction to G. Denote by

$$\eta_i : H_{n-i}(G) \to H^{n+i}(G),$$

the map obtained composing the pull-back $H^{n+i}(X) \to H^{n+i}(G)$, with the isomorphism induced by Poincaré duality $H_{n-i}(X) \to H^{n+i}(X)$, and the push-forward $H_{n-i}(G) \to H_{n-i}(X)$. By [3, (3) p. 198] and [14, Lemma 14, p. 351] we see that, for a fixed $i \geq 0$, to prove the vanishing of the map $H^{n+i}(G) \to H^{n+i+1}(Y, U)$ ($U := Y \backslash \mathrm{Sing}(Y)$) is equivalent to prove that the map η_i is onto. Now consider the map

$$\epsilon_i : H_{n+i}(G) \to H^{n+i}(G) \tag{3}$$

obtained composing $\eta_i : H_{n-i}(G) \to H^{n+i}(G)$ with the cap-product $H_{n+i}(G) \xrightarrow{\mu^i \cap \cdot} H_{n-i}(G)$:

$$\begin{tikzcd} H_{n+i}(G) \arrow[rr, "\epsilon_i"] \arrow[dr, "\mu^i \cap \cdot"'] & & H^{n+i}(G) \\ & H_{n-i}(G). \arrow[ur, "\eta_i"'] & \end{tikzcd}$$

In order to prove that the map $\eta_i : H_{n-i}(G) \to H^{n+i}(G)$ is onto, it suffices to prove that the map ϵ_i is an isomorphism. Summing up, *in order to prove the vanishing (1), it suffices to prove that the map $\epsilon_i : H_{n+i}(G) \to H^{n+i}(G)$ is an isomorphism for all $i \geq 0$.*

Remark 3.3. For all $y \in \mathrm{Sing}(Y)$, set $G_y := \pi^{-1}(y)$. By [8, (1.4) Corollaire] we see that, for $1 \leq i < n$,

$${}^p\mathcal{H}^{-i}(R\pi_* \mathbb{Q}_X[n])_y \cong H_{n+i}(G_y) \quad \text{and} \quad {}^p\mathcal{H}^{i}(R\pi_* \mathbb{Q}_X[n])_y \cong H^{n+i}(G_y).$$

Therefore, to prove that the map $\epsilon_i : H_{n+i}(G) \to H^{n+i}(G)$ is an isomorphism for $i \geq 1$, is equivalent to prove perverse Hard-Lefschetz Theorem for π and $\mathcal{M}$ [8, (6.1) Proposition], [2, Theorem 3.3.1., p. 573], [4, Theorem 5.4.8, p. 160].

Now we are going to prove that the map ϵ_i is an isomorphism for all $i \geq 0$.

To this purpose, let $\widetilde{G} \to G$ be the normalization of G. And consider the following commutative diagram:

$$\begin{tikzcd}
H_{n+i}(\widetilde{G}) \arrow[rr, "a"] \arrow[dr, "\gamma"] \arrow[dd, "\beta"'] & & H_{n+i}(G) \arrow[dl, "b"'] \arrow[dd, "\epsilon_i"] \\
& H_{n+i}(X) \arrow[dl, "\rho"'] \arrow[dr, "c"] & \\
H^{n+i}(\widetilde{G}) & & H^{n+i}(G), \arrow[ll, "d"]
\end{tikzcd}$$

where:

- the maps a, b and γ are the push-forward, and d is the pull-back;
- the map c is the composition of the pull-back $H^{n+i}(X) \to H^{n+i}(G)$, with the cup-product $H^{n-i}(X) \stackrel{[\mathcal{M}]^i \cup \cdot}{\longrightarrow} H^{n+i}(X)$, and the isomorphism $PD^{-1} : H_{n+i}(X) \to H^{n-i}(X)$ induced by Poincaré Duality;
- the map ρ is the composition of the pull-back $H^{n+i}(X) \to H^{n+i}(\widetilde{G})$, with the cup-product $H^{n-i}(X) \stackrel{[\mathcal{M}]^i \cup \cdot}{\longrightarrow} H^{n+i}(X)$, and the isomorphism $PD^{-1} : H_{n+i}(X) \to H^{n-i}(X)$ induced by Poincaré Duality;
- $\beta := \rho \circ \gamma$.

We need the following lemmas. We prove them in a moment.

Lemma 3.4. *The push-forward map $b : H_{n+i}(G) \to H_{n+i}(X)$ is injective.*

Lemma 3.5. *The push-forward map $a : H_{n+i}(\widetilde{G}) \to H_{n+i}(G)$ is onto.*

Lemma 3.6. $\ker \beta \subseteq \ker \gamma$.

Taking into account previous lemmas, a simple diagram chase proves that ϵ_i is injective, hence an isomorphism because $H_{n+i}(G)$ and $H^{n+i}(G)$ have the same dimension. In fact, assume $\epsilon_i(x) = 0$. Let $y \in H_{n+i}(\widetilde{G})$ such that $a(y) = x$. Then $d(\epsilon_i(a(y))) = 0$, i.e. $\beta(y) = 0$. Then $\gamma(y) = 0$, and therefore $b(a(y)) = 0$. It follows that $a(y) = 0$, i.e. $x = 0$.

To conclude, we are going to prove previous lemmas. We only prove Lemma 3.5 and Lemma 3.6 because Lemma 3.4 follows from [3, Lemma 4.1]. We may assume all cohomology and homology groups are with $\mathbb{C}$-coefficients.

Proof of Lemma 3.5. Since X is smooth, by [9, Proposition 4.20, p. 102], we know that $H^{n+i}(X)$ has no weights of order $n+i-1$, i.e. $W_{n+i-1}H^{n+i}(X) = 0$. Since the pull-back

$$H^{n+i}(X) \to H^{n+i}(G)$$

is onto [3, Lemma 4.1], and is a morphism of mixed Hodge structure, it follows that also $H^{n+i}(G)$ has no weights of order $n+i-1$, i.e. $W_{n+i-1}H^{n+i}(G) = 0$ [9, Corollary 3.6, p. 65, and Theorem

5.33, (iii), p. 126]. On the other hand, since $\widetilde{G} \to G$ is a resolution (and $\widetilde{G}$ and G are projective), we have

$$W_{n+i-1}H^{n+i}(G) = \ker(H^{n+i}(G) \to H^{n+i}(\widetilde{G}))$$

[9, Corollary 5.42, p. 133, and Remark 5.15, 1), p. 119]. Therefore, the pull-back

$$H^{n+i}(G) \to H^{n+i}(\widetilde{G})$$

is injective. This is equivalent to saying that the push-forward map $H_{n+i}(\widetilde{G}) \to H_{n+i}(G)$ is onto. □

Proof of Lemma 3.6. Since G is a s.n.c. divisor, its irreducible components $G_1, \dots, G_r$ are smooth, and $\widetilde{G}$ is simply the disjoint union of them. Via Poincaré Duality on each components of G, and on X, we may identify $H_{n+i}(\widetilde{G})$ with $H^{n-i-2}(\widetilde{G})$, $H_{n+i}(X)$ with $H^{n-i}(X)$, and the push-forward $\gamma : H_{n+i}(\widetilde{G}) \to H_{n+i}(X)$ with a Gysin map $\gamma' : H^{n-i-2}(\widetilde{G}) \to H^{n-i}(X)$. Hence, to prove that $\ker\beta \subseteq \ker\gamma$ is equivalent to prove that

$$\ker\beta' \subseteq \ker\gamma',$$

where $\beta' := \rho' \circ \gamma'$, and ρ' denotes the composition of the pull-back $H^{n+i}(X) \to H^{n+i}(\widetilde{G})$, with the cup-product $H^{n-i}(X) \stackrel{[\mathcal{M}]^i\cup\cdot}{\longrightarrow} H^{n+i}(X)$:

$$\begin{tikzcd} H^{n-i-2}(\widetilde{G}) \arrow[dd, "\beta'"'] \arrow[dr, "\gamma'"] & \\ & H^{n-i}(X) \arrow[dl, "\rho'"] \\ H^{n+i}(\widetilde{G}). & \end{tikzcd}$$

Let $v \in \ker\beta'$.

Since the Gysin map and the pull-back are morphisms of Hodge structures [9, Corollary 1.13, p. 17, and Lemma 1.19, p.19], we may assume that $v \in H^{p,q}(\widetilde{G})$, with $p+q = n-i-2$. Set:

$$\lambda : H^{n-i}(X) \to H^{n+i+2}(X), \quad \lambda(x) := x \cup [\mathcal{M}]^{i+1},$$

$$\epsilon : H^{n-i}(X) \to H^{n+i+2}(X), \quad \epsilon(x) := x \cup \left([\mathcal{M}]^{i+1} - \left[\mathcal{O}_X\left(\sum_{j=1}^{r} a_j G_j\right)\right]^{i+1}\right),$$

$$\delta : H^{n-i}(X) \to H^{n+i+2}(X), \quad \delta(x) := x \cup \left[\mathcal{O}_X\left(\sum_{j=1}^{r} a_j G_j\right)\right]^{i+1},$$

$$\gamma_j : H^{n+i}(G_j) \to H^{n+i+2}(X), \quad \gamma_j := \text{the Gysin map}.$$

We have:

$$\lambda(\gamma'(v)) = (\epsilon+\delta)(\gamma'(v)) = \epsilon(\gamma'(v)) + \delta(\gamma'(v)). \tag{4}$$

Notice that, since the singular locus of Y is finite, we have:

$$[\pi^*(\mathcal{L})] \cup [\mathcal{O}_X(G_j)] = 0 \in H^4(X) \tag{5}$$

for every component G_j of G. Hence (compare with (2)):

$$[\mathcal{M}]^{i+1} = [\pi^*(\mathcal{L})]^{i+1} + \left[\mathcal{O}_X\left(\sum_{j=1}^{r} a_j G_j\right)\right]^{i+1}. \tag{6}$$

Taking into account (5), (6), and the projection formula [9, p. 424], it follows that:

$$\epsilon(\gamma'(v)) = \gamma'(v) \cup [\pi^*(\mathcal{L})]^{i+1} = 0.$$

Continuing previous computation (4), and using again (5) and (6) as before, we get:

$$\lambda(\gamma'(v)) = \delta(\gamma'(v)) = \left[\left(\sum_{j=1}^{r} a_j\gamma_j\right)\circ\rho'\circ\gamma'\right](v) = \left[\left(\sum_{j=1}^{r} a_j\gamma_j\right)\circ\beta'\right](v) = 0.$$

This proves that, if $v \in \ker\beta'$, then $\gamma'(v)$ is a primitive cohomology class [9, p. 25 and 26].

Now denote by $f_j : G_j \to X$ the inclusion, that we may see as the composition of the natural map $\widetilde{G} \to X$, with the inclusion $l_j : G_j \to \widetilde{G}$. Denote by $v_1, \dots, v_r$ the components of $v \in H^{n-i-2}(\widetilde{G}) = \oplus_{j=1}^r H^{n-i-2}(G_j)$. We have:

$$\overline{\gamma'(v)} \cup \gamma'(v) \cup [\mathcal{M}]^i = \gamma'(\overline{v}) \cup \gamma'(v) \cup [\mathcal{M}]^i = \sum_{j=1}^{r} \gamma'(\overline{v_j}) \cup \gamma'(v) \cup [\mathcal{M}]^i.$$

By the projection formula we may write:

$$\gamma'(\overline{v_j}) \cup \left(\gamma'(v) \cup [\mathcal{M}]^i\right) = \overline{v_j} \cup \left(f_j^*(\gamma'(v) \cup [\mathcal{M}]^i)\right).$$

Since

$$f_j^*(\gamma'(v) \cup [\mathcal{M}]^i) = (l_j^* \circ \rho' \circ \gamma')(v) = (l_j^* \circ \beta')(v) = 0 \in H^{n+i}(G_j),$$

we deduce

$$\overline{v_j} \cup \left(f_j^*(\gamma'(v) \cup [\mathcal{M}]^i)\right) = 0 \in H^{2n-2}(G_j) \cong \mathbb{C},$$

and so

$$\overline{\gamma'(v)} \cup \gamma'(v) \cup [\mathcal{M}]^i = 0 \in H^{2n}(X) \cong \mathbb{C}.$$

Summing up, $\gamma'(v)$ lies in $H^{p+1,q+1}(X)$, is primitive, and $\overline{\gamma'(v)} \cup \gamma'(v) \cup [\mathcal{M}]^i = 0$. By the Hodge-Riemann bilinear relations it follows that $\gamma'(v) = 0$. □

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Vincenzo Di Gennaro , Università di Roma "Tor Vergata", Dipartimento di Matematica, Via della Ricerca Scientifica, 00133 Roma, Italy.

Email address: digennar@axp.mat.uniroma2.it

Davide Franco, Università di Napoli "Federico II", Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Via Cintia, 80126 Napoli, Italy.

Email address: davide.franco@unina.it