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EQUIDISTANTS FOR FAMILIES OF SURFACES

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ABSTRACT. For a smooth surface in $\mathbb{R}^3$ this article investigates certain affine equidistants, that is loci of points at a fixed ratio between points of contact of parallel tangent planes (but excluding ratios 0 and 1 where the equidistant contains one or other point of contact). The situation studied occurs generically in a 1-parameter family, where two parabolic points of the surface have parallel tangent planes at which the unique asymptotic directions are also parallel. The singularities are classified by regarding the equidistants as critical values of a 2-parameter unfolding of maps from $\mathbb{R}^4$ to $\mathbb{R}^3$. In particular, the singularities that occur near the so-called 'supercaustic chord', joining the two special parabolic points, are classified. For a given ratio along this chord either one or three special points are identified at which singularities of the equidistant become more special. Many of the resulting singularities have occurred before in the literature in abstract classifications, so the article also provides a natural geometric setting for these singularities, relating back to the geometry of the surfaces from which they are derived.

1. Introduction

A smooth closed surface in affine 3-space will contain pairs of points at which the affine tangent planes are parallel; indeed the tangent plane at a given point may be parallel to that at several other points if the surface is non-convex. Associated with these pairs of points, and the chords joining them, there are a number of affinely invariant constructions. The *affine equidistants* are the loci of points at a fixed ratio $\lambda : 1-\lambda$ along the chords, and the *centre symmetry set* is the envelope of the chords, which can be locally empty. These constructions have been examined from the point of view of singularity theory in the last few years by several authors; there are many connexions with earlier work such as the 'Wigner caustic' of Berry [2] which, for a curve in the plane, is the equidistant corresponding to a ratio $\lambda = \frac{1}{2}$, that is the midpoints of the parallel tangent chords, and the bifurcations of central symmetry of Janeczko [11]. Notable among recent studies is the work of Domitrz and his co-authors, for example [3].

A generic surface M in affine 3-space will generically have pairs of points at which the tangent planes are parallel and for which both points in the pair are parabolic points of M. For the locus of parabolic points of M is generically a 1-dimensional set, a union of smooth curves, and requiring parallel tangent planes imposes two conditions on a pair of points of this set, so that a finite number of solutions can be expected. In this article we investigate one possible local degeneration of this generic situation by requiring also that the unique asymptotic directions coincide at such a pair of parabolic points with parallel tangent planes. For this to occur the surface M must be contained in a smoothly varying family M_ε of surfaces. Since our investigation is local we shall in fact consider two surface patches M_0 and N_0 which vary in a 1-parameter family $M_\varepsilon, N_\varepsilon$. A similar degeneracy was investigated for plane curves in [6]; we sometimes call it a 'supercaustic' situation. This term is defined in §2.3.

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We find the values $\lambda \neq 0, 1$ for which the ratio $\lambda : 1 - \lambda$ determines an equidistant at which the structure undergoes a qualitative change. There are one or three of these values, depending on the relative orientation of M_0 and N_0. One 'degenerate' value always exists and results in a high codimension singularity; we are able to give a partial analysis of this case. When the other two values exist we call them *special values* (Definition 2.6), and a complete analysis is given.

The article is organized as follows. In §2 we introduce the family of surfaces we shall work with (§2.1), and the maps which we shall classify up to $\mathcal{A}$-equivalence to study the equidistants (§2.2, §2.3). We also show how some of the conditions that arise later can be interpreted geometrically in terms of a scaled reflexion map (§2.4, Definition 2.5). In §3 we find normal forms of maps up to $\mathcal{A}$-equivalence that generate the equidistants: they are the sets of critical values of these maps. We examine in that section general values of the ratio (Generic Case 1.1) and the two 'special' values (Special Case 1.2), leaving the 'degenerate' value (Degenerate Case 2) to §4.

The main results are contained in Proposition 3.2 and the accompanying Figure 1 for Generic Case 1.1; Proposition 3.4 and the accompanying Figure 4 for Special Case 1.2, and Table 1 in §4.6 for Degenerate Case 2.

2. The general setup

2.1. **A generic family of surfaces.** Consider the parabolic set P (assumed to be a nonempty smooth curve) of a generic smooth closed surface M in $\mathbb{R}^3$. We can expect generically to find a finite number of pairs of distinct points on P for which the tangent planes to M are parallel, since the two points give us two degrees of freedom and it is two conditions for the tangent planes to be parallel. However it will not be generically true that the unique asymptotic directions at such a pair of points are parallel. For that we require a 1-parameter family of surfaces and it is this situation which we study here.

Our considerations are local, and also affinely invariant. For this situation we have two surfaces, M_ε and N_ε, varying in a 1-parameter family; using a family of affine transformations of $\mathbb{R}^3$ (coordinates (x, y, z)) we can assume that the origin lies on M_ε, that the origin is a parabolic point of M_ε and that the unique asymptotic direction there is always along the y-axis, for all ε close to 0. Further we can assume that the point $(0, 0, 1)$ lies on N_ε for all small ε and that for $\varepsilon = 0$ this point is parabolic, has horizontal tangent plane parallel to the (x, y)-plane, and has unique asymptotic direction parallel to the y-axis. We realise this setup by the surfaces

$$\begin{aligned} M_\varepsilon : z = f(x,y,\varepsilon) &= f_{20}x^2 + f_{300}x^3 + f_{210}x^2y + f_{120}xy^2 + f_{030}y^3 + \dots \\ &+ \varepsilon\left(f_{301}x^3 + f_{211}x^2y + \dots\right) + \varepsilon^2\left(f_{302}x^3 + \dots\right) + \dots, \end{aligned} \tag{1}$$

$$\begin{aligned} N_\varepsilon : z = 1 + g(x,y,\varepsilon) &= 1 + g_{20}x^2 + g_{300}x^3 + g_{210}x^2y + g_{120}xy^2 + g_{030}y^3 + \dots \\ &+ \varepsilon\left(g_{101}x + g_{011}y + g_{201}x^2 + g_{111}xy + g_{021}y^2 + \dots\right) \\ &+ \varepsilon^2\left(g_{102}x + g_{012}y + \dots\right) + \dots. \end{aligned} \tag{2}$$

For terms other than f_{20}, g_{20}, subscripts ijk indicate that the corresponding monomial is $\varepsilon^k x^i y^j$.

We make the following assumptions about these expansions.

Assumptions 2.1. (i) $f_{20} \neq 0, g_{20} \neq 0$, that is neither M_0 nor N_0 is umbilic at its basepoint $(0, 0, 0)$ or $(0, 0, 1)$.

(ii) $f_{030} \neq 0, g_{030} \neq 0$, that is the parabolic curves of M_0 at the origin and N_0 at $(0, 0, 1)$ are smooth and not tangent to the asymptotic directions there (i.e. these points are not cusps of Gauss). We shall take $f_{030} > 0$ without loss of generality, and we sometimes write

$$f_{030} = f_3^2, \; g_{030} = \pm g_3^2$$

when a definite sign is needed, to avoid square roots appearing in the formulas.

2.2. **Family of maps for the equidistants.** The λ-equidistant for a fixed ε is the locus of points in $\mathbb{R}^3$ of the form $(1-\lambda)\boldsymbol{p}+\lambda\boldsymbol{q}$ where $\boldsymbol{p}\in M_\varepsilon, \boldsymbol{q}\in N_\varepsilon$ and the tangent planes to M_ε at $\boldsymbol{p}$ and N_ε at $\boldsymbol{q}$ are parallel.

We always assume $\lambda\neq 0, \lambda\neq 1$ in what follows.

We use $s=(s_1,s_2)$ as parameters on M_ε and similarly $t=(t_1,t_2)$ for N_ε; we have a 2-parameter family of maps $\mathbb{R}^4\to\mathbb{R}^3$:

$$\mathbb{R}^4\times\mathbb{R}^2\to\mathbb{R}^3,\ (s,t,\varepsilon,\lambda)\mapsto(1-\lambda)(s_1,s_2,f(s_1,s_2,\varepsilon))+\lambda(t_1,t_2,1+g(t_1,t_2,\varepsilon)). \tag{3}$$

Then it is straightforward to check that, for fixed ε and λ, the set of critical values of this map is the λ-equidistant of M_ε and N_ε. We are therefore interested in this family of maps up to $\mathcal{A}$-equivalence. We make the change of variables

$$(1-\lambda)s_1+\lambda t_1=u_1,\ (1-\lambda)s_2+\lambda t_2=u_2,\ \text{and write } \lambda=\lambda_0+\alpha,$$

replacing t_1 and t_2, to rewrite (3) as a map of the form (for any $\lambda_0\neq 0,1$)

$$H:\mathbb{R}^4\times\mathbb{R}^2\to\mathbb{R}^3,\ H(s_1,s_2,u_1,u_2,\varepsilon,\alpha)=(u_1,u_2,h(s_1,s_2,u_1,u_2,\varepsilon,\lambda_0+\alpha)). \tag{4}$$

regarded as a 2-parameter unfolding of the map $H_0(s_1,s_2,u_1,u_2,0,\lambda_0)$. Therefore we have the following.

Proposition 2.2. *The λ-equidistant for fixed ε is the set of points $(u_1,u_2,h)\in\mathbb{R}^3$ for which $\partial h/\partial s_1=\partial h/\partial s_2=0$. For fixed λ the union of all the equidistants, spread out in $\mathbb{R}^4$, the planar sections of which are the $\varepsilon=$ constant equidistants, is the set of points $(u_1,u_2,h,\varepsilon)\in\mathbb{R}^4$ for which the same conditions $\partial h/\partial s_1=\partial h/\partial s_2=0$ hold.*

2.3. **Maps and supercaustics.** Let $\phi:\mathbb{R}^4\to\mathbb{R}^2$ be given, for fixed λ and ε, by

$$\phi(s_1,s_2,u_1,u_2)=(h_{s_1},h_{s_2}),$$

subscripts denoting partial derivatives as usual. Then the corresponding equidistant, given by $\phi^{-1}(0,0)$, is singular when there is a kernel vector of $d\phi$ with image under dH equal to $\mathbf{0}$, these being evaluated at a point of $\phi^{-1}(0,0)$. This requires that

$$\text{rank } J<4 \text{ where } J=\begin{pmatrix} h_{s_1s_1} & h_{s_1s_2} & h_{s_1u_1} & h_{s_1u_2}\\ h_{s_2s_1} & h_{s_2s_2} & h_{s_2u_1} & h_{s_2u_2}\\ 0 & 0 & h_{u_1} & h_{u_2}\\ 0&0&1&0\\ 0&0&0&1\end{pmatrix},$$

that is $h_{s_1s_1}h_{s_2s_2}=h^2_{s_1s_2}$. The singular points of the equidistant for fixed λ and ε are therefore

$$\{(u_1,u_2,h(s_1,s_2,u_1,u_2))\ :\ h_{s_1}=h_{s_2}=h_{s_1s_1}h_{s_2s_2}-h^2_{s_1s_2}=0\}. \tag{5}$$

We note here that, for fixed ε, the 'centre symmetry set' of the pair of surfaces M,N [8], which is the locus of singular points of the equidistants for varying λ, is given by the same formula (5) where h is now a function of s_1,s_2,u_1,u_2,λ but with ε still fixed.

It is possible that some singular points of the equidistant arise from singularities of the critical set itself in $\mathbb{R}^4$. In our case this requires, for fixed λ and ε, that the top two rows of the above matrix J are dependent. Indeed, evaluating these rows at $(s_1,s_2,u_1,u_2,\lambda,\varepsilon)=(0,0,0,0,\lambda,0)$ the second row is entirely zero. This means that, for all λ, but $\varepsilon=0$, the critical set itself is singular at the origin of $\mathbb{R}^4$.

Definition 2.3. In the above situation, the λ-axis is called a *supercaustic*; see [6]. The whole of this axis maps to singular points of the equidistants.

Remark 2.4. This depends crucially on the special nature of our surfaces, with not only parallel tangent planes at parabolic points of M_0 and N_0 but also the asymptotic directions at those points being parallel. If instead we assume that the asymptotic directions are distinct (without loss of generality we can take them along the x and y axes) then the top two rows of J become independent for $s_1 = s_2 = u_1 = u_2 = \varepsilon = 0$ and arbitrary λ. In fact, writing g_{020} for the coefficient of y^2 in the parametrization of N_0 and putting $g_{20} = 0$ these rows become

$$\begin{pmatrix} 2(1-\lambda)f_{20} & 0 & 0 & 0 \\ 0 & \frac{2g_{020}(1-\lambda)^2}{\lambda} & 0 & -\frac{2g_{020}(1-\lambda)}{\lambda} \end{pmatrix}.$$

In this case the 'supercaustic' is empty.

2.4. **Scaled reflexion map and contact.** Consider the affine map $\mathcal{S} : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\mathcal{S}(x, y, z) = (\mu x, \mu y, \mu(z-1))$ where $\mu = \frac{\lambda}{\lambda-1} \neq 0$. This leaves the point $(0, 0, \lambda)$ fixed and maps $(0, 0, 1)$ to the origin. We can measure the contact between $\mathcal{S}(N_0)$ and M_0 by composing the parametrization of $\mathcal{S}(N_0)$ given by $(\mu x, \mu y, \mu g(x, y, 0))$ with the equation of M_0, say

$$Z - f(X, Y, 0) = 0.$$

Definition 2.5. The *scaled contact map* is the contact map germ

$$K : \mathbb{R}^2, (0,0) \to \mathbb{R}, 0, \; K(x, y) = \mu g(x, y, 0) - f(\mu x, \mu y, 0), \; \mu = \frac{\lambda}{\lambda - 1} \quad \text{as above.}$$

We shall find this contact map useful in interpreting the conditions which arise from ε-families of equidistants as ε passes through 0.

The 2-jet of K is $K_2(x, y) = \mu(g_{20} - \mu f_{20})x^2$ so that in our situation K is always non-Morse; it has corank 1 and is of type A_k at $(0, 0)$ for some k, provided $f_{20}\lambda + g_{20}(1 - \lambda) \neq 0$ (when this fails we call this the 'Degenerate Case 2'; see §4). The coefficient of y^3 in K is $\mu(g_{030} - \mu^2 f_{030})$ so that K is then of type exactly A_3 provided $f_{030}\lambda^2 - g_{030}(1-\lambda)^2 \neq 0$. If f_{030}, g_{030} are nonzero and have opposite signs then of course this coefficient can never be zero.

Definition 2.6. Assume as above that $f_{20}\lambda + g_{20}(1 - \lambda) \neq 0$. When f_{030}, g_{030} have the same sign (without loss of generality, positive), and the above coefficient $f_{030}\lambda^2 - g_{030}(1-\lambda)^2$ of y^3 is zero, then we refer to the two resulting values of λ as *special values*. Writing $f_{030} = f_3^2, g_{030} = g_3^2$ where we may take $f_3 > 0, g_3 > 0$, these special values of λ are $\dfrac{g_3}{g_3 \pm f_3}$. (We shall usually assume $f_3 \neq g_3$ to avoid one of the special values 'going to infinity'.) These special values of λ give rise to what we shall call Special Case 1.2. This is examined in detail in §3.2.

When λ has a special value, say $\dfrac{g_3}{g_3 + f_3}$, the condition for K to have exactly type A_3 at $(0, 0)$ works out to be

$$(6) \quad (4g_{040}g_{20} - g_{120}^2)f_3^4 + 4g_{040}f_{20}f_3^3 g_3 + 2f_{120}g_{120}f_3^2 g_3^2 + 4f_{040}g_{20}f_3 g_3^3 + (4f_{040}f_{20} - f_{120}^2)g_3^4 \neq 0.$$

This condition will be satisfied by a generic pair of surfaces M_0, N_0. With the other special value the signs in front of the coefficients of $f_3^3 g_3$ and $f_3 g_3^3$ both change to minus.

When the quadratic terms of the contact map K vanish identically, that is when

$$f_{20}\lambda + g_{20}(1 - \lambda) = 0,$$

the cubic terms will in general be nondegenerate and K will generically have type $D_4^{\pm}$, that is $\mathcal{R}$-equivalent to $x^3 \pm xy^2$. The polynomial in the coefficients of f and g which distinguishes the two cases is rather complicated but, remarkably, it has a different interpretation which we give in §4 in the context of self-intersections of the equidistant. See Remark §4.3.

3. The equidistants: normal forms

For a general study of the equidistants we need to expand the function h in (4) using the parametrizations (1) and (2). We begin with $\varepsilon = 0$ and write, for a fixed λ,

$$H_{0\lambda}(s,u) = (u, h_{0\lambda}(s,u)) = H(s,u,0,\lambda).$$

The coefficient of $s_1^i s_2^j u_1^k u_2^\ell$ in $h_{0\lambda}$ will be written $c_{ijk\ell}$. We find:

The 2-jet of $h_{0\lambda}$ at $s = u = 0$ is $(1-\lambda)(\lambda f_{20} + (1-\lambda)g_{20})s_1^2 - 2g_{20}\frac{1-\lambda}{\lambda}s_1u_1$.

Note that the coefficient of s_1u_1 is nonzero.

The main subdivision is between those λ for what $\lambda f_{20} + (1-\lambda)g_{20}$ is nonzero (Generic Case 1) or zero (Degenerate Case 2). We cover the Generic Case here and the Degenerate Case in §4 below.

Case 1 $\lambda f_{20} + (1-\lambda)g_{20} \neq 0$. From §2.4 this is also the condition for the contact function K to have type A_k for some k.

We can now redefine the variable s_1 ('completing the square') to eliminate all terms containing s_1 besides s_1^2 in $h_{0\lambda}$. The coefficient of s_2^3 then becomes

$$c_{0300} = \frac{1-\lambda}{\lambda^2}(f_{030}\lambda^2 - g_{030}(1-\lambda)^2).$$

3.1. **The general values of λ. Generic Case 1.1** $c_{0300} \neq 0$, that is, $Q \neq 0$ where

$$Q = f_{030}\lambda^2 - g_{030}(1-\lambda)^2. \tag{7}$$

From §2.4 this is also the condition for the contact function K to have type A_2 and that λ is not a *special value*.

Consider the 3-jet of $H_{0\lambda}$. There are six degree 3 monomials which do not involve s_1 and which do involve s_2 (any monomial in u_1, u_2 alone can be eliminated by a 'left-change' of coordinates). We still have the freedom to change coordinates in s_2 (involving s_2, u_1, u_2) and in u_1, u_2 (involving u_1, u_2 only). Using only the first of these the terms in $s_2^2u_1$ and $s_2^2u_2$ can be eliminated, leaving

$$\left(u_1, u_2, (1-\lambda)(\lambda f_{20} + (1-\lambda)g_{20})s_1^2 + c_{0300}s_2^3 + s_2\left(c_{0120}u_1^2 + c_{0111}u_1u_2 + c_{0102}u_2^2\right)\right). \tag{8}$$

(The coefficients $c_{ijk\ell}$ need to be updated to take account of the substitutions.) The quadratic form in u_1 and u_2 can be diagonalised, eliminating the term in $s_2u_1u_2$ so that, scaling s_1, the last coordinate in $\mathbb{R}^3$ and s_2, we have 3-jet, say

$$(u_1, u_2, s_1^2 + s_2^3 + as_2u_1^2 + bs_2u_2^2).$$

Suppose that the quadratic form in parentheses in (8) is not a perfect square, that is

$$c_{0111}^2 - 4c_{0120}c_{0102} \neq 0.$$

Then a and b above are nonzero. The condition for this is $R \neq 0$ where

$$R = f_{20}^2 f_{030}\left(g_{120}^2 - 3g_{210}g_{030}\right) - g_{20}^2 g_{030}\left(f_{120}^2 - 3f_{210}f_{030}\right). \tag{9}$$

Since this condition does not involve λ it will be satisfied by a generic pair of surfaces M_0, N_0. Note that the condition separates into a quantity for M_0 unequal to the same quantity for N_0.

Proposition 3.1. *The condition $R \neq 0$ can also be interpreted as saying that the images under the Gauss map of the parabolic curves on M_0 and N_0 have ordinary tangency (that is, 2-point contact) in the Gauss sphere. These images are smooth by Assumptions 2.1.*

Proof The parabolic curves on the two surfaces are given by

$$f_{xx}f_{yy} - f_{xy}^2 = 0 \quad \text{and} \quad g_{xx}g_{yy} - g_{xy}^2 = 0$$

for M_0 and N_0 respectively. The surface M_0 has a parabolic point at the origin and N_0 has a parabolic point at $(0,0,1)$ and since they have parallel asymptotic directions at these points the images of the respective parabolic curves under the Gauss map are tangent. We shall use the modified Gauss maps, that is $(x,y) \mapsto (X,Y) = (f_x, f_y)$ and similarly for g. By a direct calculation, for M_0 the image of the parabolic curve, parametrized by x, under the modified Gauss map has an equation, up to terms in X^2, of the form

$$Y = \frac{3f_{030}f_{210} - f_{120}^2}{12f_{20}^2 f_{030}} X^2$$

with a similar result for N_0. The coefficients of X^2 are unequal, that is the images have ordinary tangency, if and only if the condition R above is nonzero. □

Further scaling allows this case to be reduced to

$$H_{0\lambda}(s,u) = (u_1, u_2, s_1^2 + s_2^3 \pm s_2u_1^2 \pm s_2u_2^2), \tag{10}$$

where the $\pm$ signs are independent, but by interchanging u_1 and u_2 we reduce to three cases, as follows.

Proposition 3.2. *The normal form* (10) *is as follows, using the notation of* (7) *and* (9). *See Figure 1.*
Subcase 1.1.1 *(positive definite):* $H_{0\lambda}(s,u) = (u_1, u_2, s_1^2 + s_2^3 + s_2u_1^2 + s_2u_2^2)$.
The condition for this is $f_{030}g_{030} < 0$ *and* $QR > 0$. *Bearing in mind the assumptions 2.1 the latter condition is equivalent to* $R > 0$. *This subcase will also be referred to as* A_2^{++}.
Subcase 1.1.2 *(negative definite):* $H_{0\lambda}(s,u) = (u_1, u_2, s_1^2 + s_2^3 - s_2u_1^2 - s_2u_2^2)$.
The condition for this is $f_{030}g_{030} > 0$ *and* $QR > 0$. *This subcase will also be referred to as* A_2^{--}
Subcase 1.1.3 *(indefinite):* $H_{0\lambda}(s,u) = (u_1, u_2, s_1^2 + s_2^3 + s_2u_1^2 - s_2u_2^2)$.
The condition for this is $QR < 0$. *In the case when* $f_{030}g_{030} < 0$ *the condition becomes* $R < 0$. *This subcase will also be referred to as* A_2^{+-}, □

The values of f_{030}, g_{030} and R are fixed by the two surfaces M_0 and N_0. However, assuming $f_{030}g_{030} > 0$, special values of λ exist at which Q as in (7) is zero. Then, as λ passes through such a special value, the normal form changes between negative definite and indefinite, so that the family of equidistants, for ε passing through 0, changes accordingly.

Using standard techniques it can be checked that (10) is 3-$\mathcal{A}$-determined, and that an $\mathcal{A}_e$-versal unfolding is given by adding a multiple of $(0,0,s_2)$ to the above normal form:

$$H_{\varepsilon\lambda}(s,u) = (u_1, u_2, s_1^2 + s_2^3 \pm s_2u_1^2 \pm s_2u_2^2 + \varepsilon s_2). \tag{11}$$

In terms of the original surfaces the coefficient of εs_2 is $-g_{011}(1-\lambda)$, and therefore we require $g_{011} \neq 0$ for a versal unfolding by the parameter ε.

Remark 3.3. It is interesting to relate the above classification to that of the regions on M and N which contribute to the pairs of parallel tangent planes (compare Prop.2.4 and Figure 3 of [5]). A schematic diagram of the common regions for M and N on the Gauss sphere is given in Figure 2 below. The relationship between these and the classification of Proposition 3.2 is as follows.
Subcase 1.1.1 (positive definite, $f_{030}g_{030} < 0$ and $R > 0$): This is (d).
Subcase 1.1.2 (negative definite, $f_{030}g_{030} > 0$ and $QR > 0$): This is (ac).
Subcase 1.1.3 (indefinite): This can arise in two ways, as either (ac) or (b)

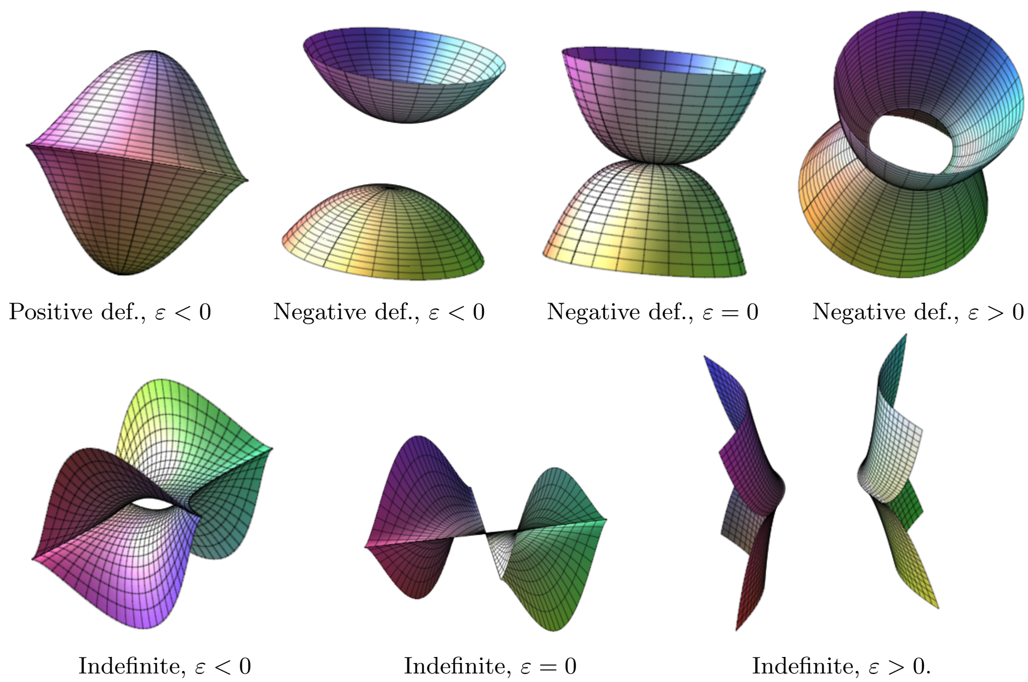

FIGURE 1. The various subcases of Proposition 3.2: Positive definite (for $\varepsilon > 0$ the equidistant is empty and for $\varepsilon < 0$ has a compact cuspidal edge); 1.1.2 Negative definite, where for $\varepsilon > 0$ there is a compact cuspidal edge; 1.1.3 Indefinite, which has two cuspidal edges for $\varepsilon \neq 0$ that form a crossing when $\varepsilon = 0$.

(ac) when $f_{030}g_{030} > 0$ and $QR < 0$,
(b) when $f_{030}g_{030} < 0$ and $R < 0$.

Let us call a pair of points, one from M_ε and the other from N_ε, at which the tangent planes are parallel, 'mates'. Consider for example the top left diagram of Figure 2 and assume that the upper curve is the image of the parabolic curve of N_ε in the Gauss sphere. Each point above this curve is the image of two points of N_ε and two points of M_ε giving altogether four mates. Each point on the upper curve is the image of two points of M_ε and a single parabolic point of N_ε which is a mate for both of them. On the surface M_ε itself there will be a region close to the base-point $(0, 0, 0)$ consisting of those points of M_ε with at least one mate, and usually two mates, on N_ε—a region 'doubly covered by mates on N_ε'. This region will have a local boundary corresponding in the way just described to the parabolic curve on N_ε. Turning to the upper right diagram of Figure 2 the hatched region representing mates now contains a segment of the parabolic curve of M_ε. On the surface N_ε this will result in a closed loop on the boundary of the region of points having mates on M_ε. The situation on the surfaces themselves is illustrated schematically in Figure 3.

3.2. **The 'special values' of λ. Special Case 1.2** $c_{0300} = 0$, that is λ has one of the two special values as in §2.4. Note that this requires f_{030} and g_{030} to have the same sign, which we take as positive, and write $f_{030} = f_3^2, g_{030} = g_3^2$ where $f_3 > 0, g_3 > 0$.

This case will be examined by choosing one of the special values for λ given by $c_{0300} = 0$, namely $\lambda = \dfrac{g_3}{g_3 + f_3}$. We can eliminate the terms in $s_2u_2^2$ and $s_2u_1u_2$ by a substitution of the

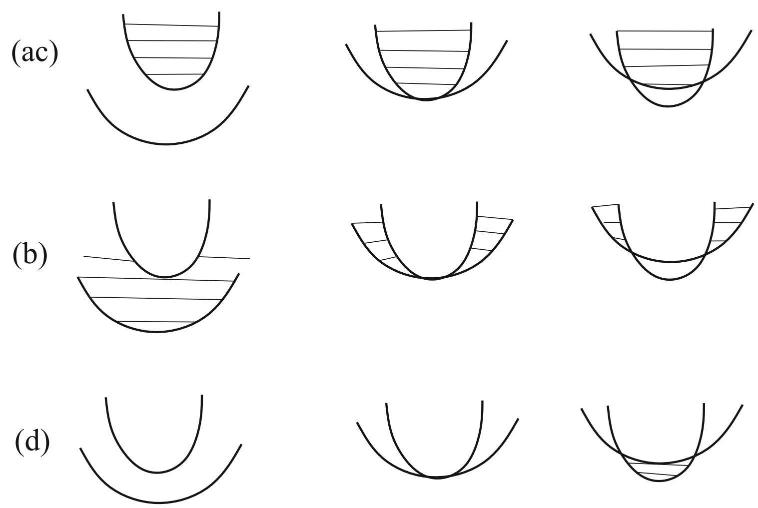

FIGURE 2. Schematic diagrams of the images of the Gauss map for the surfaces M_ε and N_ε. The curves represent the parabolic curves of these surfaces, along which the Gauss map has a fold, and the hatched regions represent the regions where the images of the Gauss maps of M_ε and N_ε intersect, that is the regions of the Gauss sphere representing parallel normals (or parallel tangent planes). Left to right of each row shows varying ε, with the middle diagram $\varepsilon = 0$, and the three possible cases are labelled (ac), (b), (d) as described in the text, to accord with Figure 3 in [5]. Note that the two curves for $\varepsilon = 0$ have ordinary tangency—see Remark 3.1.

form $s_2 = s_2' + au_1 + bu_2$, assuming only the condition $\lambda f_{20} + (1-\lambda)g_{20} \neq 0$ of Generic Case 1. The coefficient of $s_2^2u_2$ then becomes $3f_2^2 \neq 0$ and the remaining degree 3 terms in $h_{0\lambda}$, namely $s_2^2u_1, s_2^2u_2$ and $s_2u_1^2$ can therefore be reduced to the last two by redefining u_2, at the same time making the coefficient of $s_2^2u_2$ equal to 1. The 3-jet of $H_{0\lambda}$ is now of the form (scaling s_1)

$$(u_1, u_2, \pm s_1^2 + s_2^2u_2 + c_{0120}s_2u_1^2),$$

where the updated c_{0120} is nonzero if and only if $R \neq 0$ as in (9), and for generic M_0, N_0 this will be satisfied.

Passing to the 4-jet of $H_{0\lambda}$, we can first remove all monomials divisible by s_1 besides $\pm s_1^2$ by completing the square, and then eliminate all degree 4 monomials besides s_2^4 and $s_2^3u_1$, without adding any new monomials of degree 3. This can be done, for example, by substitutions of the form $s_2 = s_2' +$ quadratic terms in s_2', u_1', u_2', $u_1 = u_1' +$ quadratic terms in u_1', u_2', and similarly for u_2. A left change of coordinates will then restore the first two components of $H_{0\lambda}$ to (u_1, u_2).

The 4-jet is now reduced to

$$\left(u_1, u_2, \pm s_1^2 + s_2^2u_2 + c_{0120}s_2u_1^2 + c_{0400}s_2^4 + c_{0310}s_2^3u_1\right).$$

This is 4-$\mathcal{A}$-determined provided all the coefficients are nonzero. The coefficient c_{0400} is nonzero if and only if the 'exactly A_3 contact condition' (6) holds. Unfortunately we do not know a geometrical criterion for the coefficient of $s_2^2u_2$ to be nonzero; it involves only the coefficients in the functions f, g which define the surfaces M_0, N_0.

Scaling reduces all but the coefficient of s_1^2 to 1 and we summarize this discussion as follows.

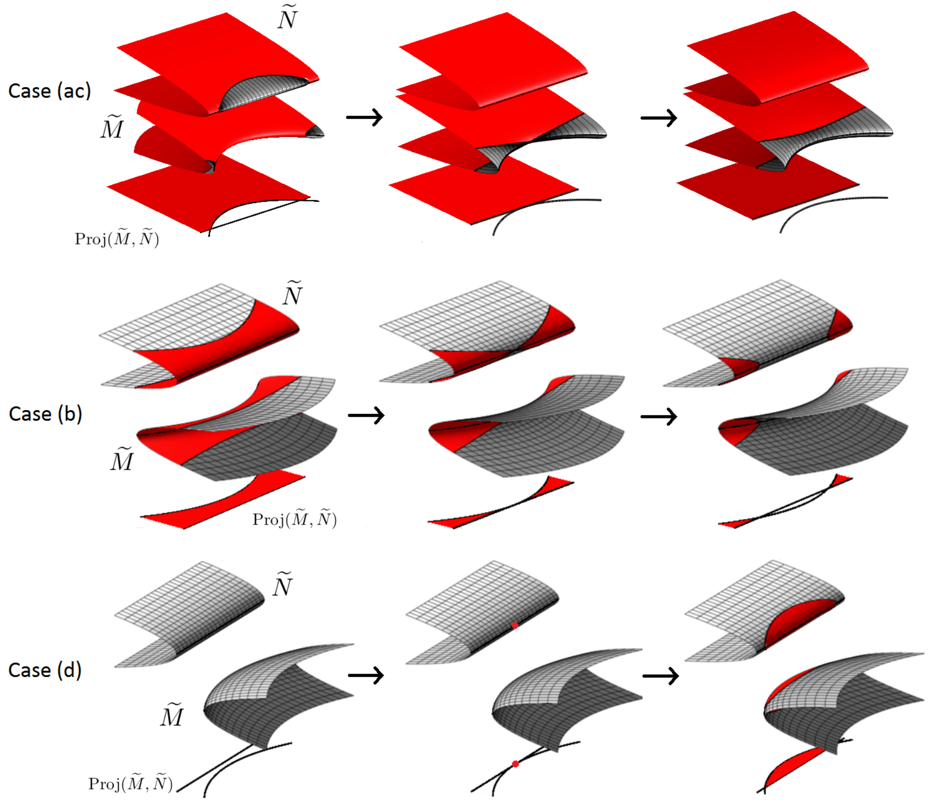

FIGURE 3. In this diagram, the Gauss map of the surfaces M_ε and N_ε is represented by vertical projection and the surfaces in this schematic representation are labelled $\widetilde{M}, \widetilde{N}$. The rows and columns are arranged as in Figure 2. See the above text for further explanation.

Proposition 3.4. *For Special Case 1.2, that is $f_{030} = f_3^2, g_{030} = g_3^2$, a special value of $\lambda = g_3/(g_3 \pm f_3)$ (Definition 2.6 or $Q = 0$ as in (7)) but $\lambda f_{20} + (1-\lambda)g_{20} \neq 0$, the function $H_{0\lambda}$ reduces under $\mathcal{A}$-equivalence to the normal form*

$$H_{0\lambda}(s_1, s_2, u_1, u_2) = \left(u_1, u_2, \pm s_1^2 + s_2^2 u_2 + s_2 u_1^2 + s_2^4 + s_2^3 u_1 + (ps_2 + qs_2^3)\right), \tag{12}$$

provided the geometrical conditions $R \neq 0$ (9), and 'exactly A_3-contact' (6) hold, together with a third condition on M_0, N_0 which will be generically satisfied. The terms $ps_2 + qs_2^3$ in brackets represent an $\mathcal{A}_e$-versal unfolding provided the geometrical condition $g_{011} \neq 0$ in (1) holds. See Figure 4 for a 'clock diagram' of the equidistants in the (p, q)-plane. □

A similar normal form, without the fourth variable s_1, but with an additional ambiguity of sign, occurs as 4_2^2 in [12]; see also [9]. The sign in front of s_1^2 will not affect our results since the critical set of $H_{0\lambda}$ has $s_1 = 0$. The versal unfolding condition means that as ε changes through 0 the normal to N tilts in a direction with a nonzero component along the y-axis, which is the asymptotic direction at $\varepsilon = 0$.

When λ moves away from a special value then, in (12), p remains at 0 while q becomes small and nonzero. We can then reduce (12) as in Generic Case 1.1, as follows. The 3-jet of (12) becomes $(u_1, u_2, s_1^2+s_2^2u_2+s_2u_1^2+qs_2^3)$ with $q \neq 0$. Replacing s_2 by ms_2+nu_2 where $3qn+1=0$ and $qm^3=1$, and then removing terms in the third component involving only u_1, u_2, reduces this to

$$\left(u_1, u_2, s_1^2 + \frac{1}{q^{1/3}}s_2u_1^2 - \frac{1}{3q^{4/3}}s_2u_2^2 + s_2^3\right).$$

The product of terms in front of $s_2u_1^2$ and $s_2u_2^2$ therefore has the sign of $-q$ and hence changes as q passes through 0. Furthermore it is not possible for both signs to be positive. We deduce the following.

Corollary 3.5. *Moving λ through a special value $\lambda = g_3/(g_3 \pm f_3)$ but keeping $\varepsilon = 0$ the type of equidistant always changes between Subcase 1.1.2 (negative definite) and Subcase 1.1.3 (indefinite) as in Proposition 3.2. It is not possible to realize the positive definite Subcase 1.1.1.*

Figure 4 shows a typical way in which equidistants near to a special value evolve as λ and ε change.

3.3. **Some further details of Special Case 1.2.** We take $\lambda_0 = \frac{g_3}{g_3+f_3}$ as a special value, assuming $f_{20} \neq 0, g_{20} \neq 0, f_3 > 0, g_3 > 0, \lambda_0 f_{20} + (1-\lambda_0)g_{20} \neq 0$, i.e. $f_{20}g_3 + g_{20}f_3 \neq 0$, and also $R \neq 0$ (9) hold. We write $\lambda = \lambda_0 + \alpha$ for nearby values, and examine the full versal unfolding $\widetilde{H}$ of H, as follows.

Thus the family of equidistants can be reduced to

(13) $$\widetilde{H}(s_1, s_2, u_1, u_2, p, q) = \left(u_1, u_2, \pm s_1^2 + s_2^2u_2 + s_2u_1^2 + s_2^4 + s_2^3u_1 + ps_2 + qs_2^3\right) = (u_1, u_2, \tilde{h}),$$

say, where p, q are unfolding parameters that are closely related to ε, α respectively.

As an aid to understanding the equidistants for (ε, α) close to $(0,0)$ we can calculate the loci in the (p,q)-plane at which the structure of the singular set or the self-intersection set on the equidistant changes.

(1) **Singular set** For fixed p, q the singular set is the image under $\widetilde{H}$ of the set of points (using suffices for partial derivatives)

$$(0, s_2, u_1, u_2) \text{ such that } \tilde{h}_{s_2} = \tilde{h}_{s_2s_2} = 0.$$

Eliminating u_2, the equations reduce to

$$u_1^2 - 3s_2^2u_1 + (p - 3s_2^2q - 8s_2^3) = 0,$$

and the condition for this to have real solutions for u_1 is

$$9s_2^4 + 32s_2^3 + 12qs_2^2 - 4p \geq 0.$$

We are therefore interested in finding the pairs (p,q) for which there is a change in the number of real intervals in the set of s_2 satisfying this inequality. This will occur when the discriminant with respect to s_2 vanishes, and that gives a locus of the form

(14) $$p = 0 \text{ or } p = \tfrac{1}{16}q^3 + \tfrac{9}{1024}q^4 + \dots.$$

See Figure 5.

(2) **Self-intersection locus** Suppose $(0, s_{21}, u_1, u_2)$ and $(0, s_{22}, u_1, u_2)$ are both in the critical set of $\widetilde{H}$ ($h_{s_1} = 0$ gives $s_1 = 0$) and have the same image under $\widetilde{H}$. Then with a little more trouble we can eliminate the u variables and obtain a condition in s_{21}, s_{22} alone. It is slightly more convenient to write $s_{21} = v_1 + v_2,\ s_{22} = v_1 - v_2$; then in fact we require $v_1(4v_1^3 + 16v_1^2 + 8qv_1 + p + q^2) \geq 0$. The number of v_1-intervals on which this holds will change when the discriminant with respect to v_1 vanishes. One case here gives

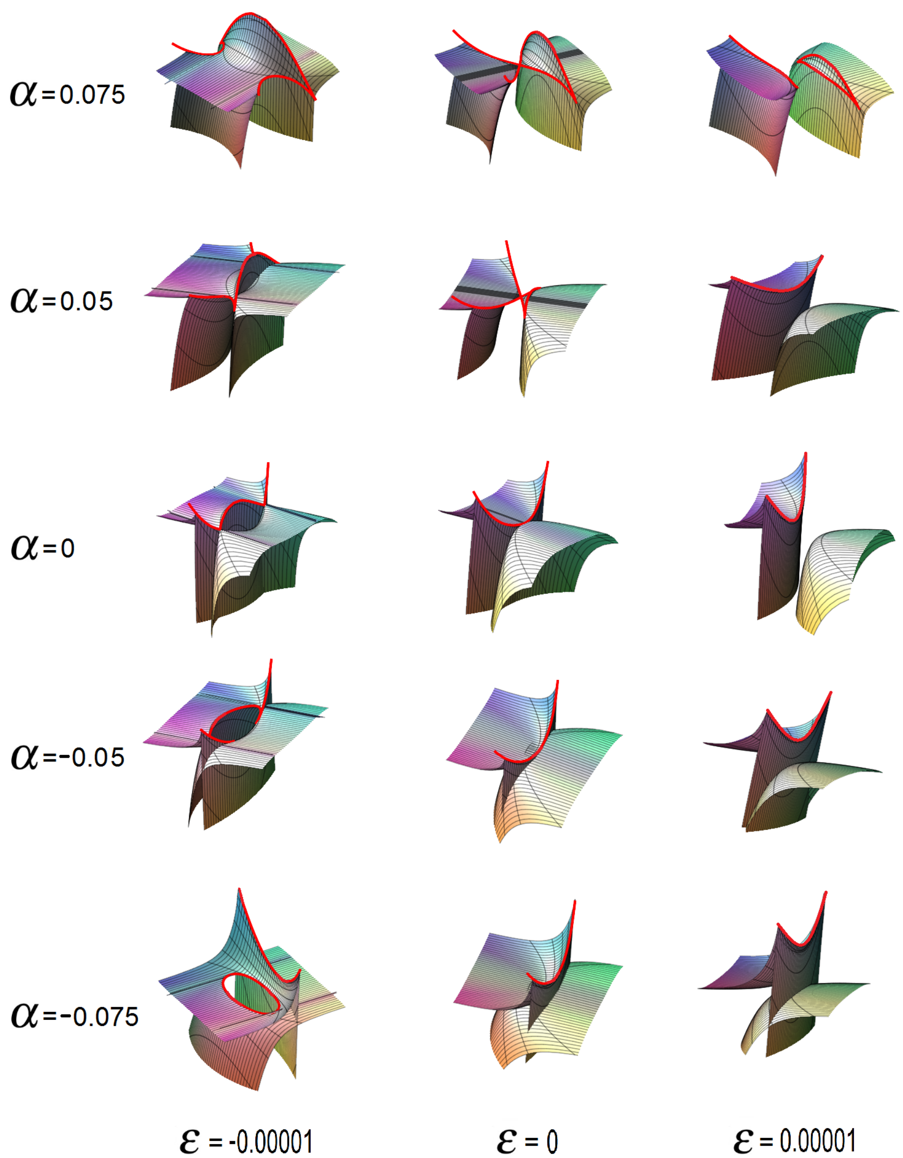

FIGURE 4. Special Case 1.2. A typical 'clock diagram' of equidistants close to a special value of $\lambda_0 = g_3/(g_3 \pm f_3)$. The vertical axis represents $\lambda = \lambda_0 + \alpha$ and the horizontal axis the parameter ε in the family of surfaces.

the same condition as (i) above, but we are concerned with the remaining possibility: taking into account that v_1, v_2 must both have real solutions the locus in the (p, q)-plane is

(15) $$p = -q^2, \ q \geq 0,$$

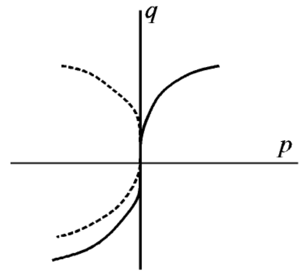

FIGURE 5. Special Case 1.2. A schematic drawing of two curves in the p, q-plane at which the structure of the equidistant in the family (13) changes, either because the cuspidal edge set changes (solid curve, together with the q-axis) or the self-intersection set changes (dashed curve).

where of course the double root is $v_1 = 0$, that is $s_{22} = -s_{21}$. (The other potential double root when $p = -q^2$ leads to $q = 2$ and is therefore not relevant to a neighbourhood of the origin in the (p, q)-plane.) See Figure 5.

4. Degenerate Case 2

In this section we give some details of Degenerate Case 2, that is $\lambda f_{20} + (1-\lambda)g_{20} = 0$. This gives a unique value of λ, namely $\lambda = \dfrac{g_{20}}{g_{20} - f_{20}}$. (If $f_{20} = g_{20}$ then, using $\lambda f_{20} + (1-\lambda)g_{20} = 0$, it follows that $f_{20} = g_{20} = 0$, contrary to our assumptions.) Thus whatever surfaces M_0, N_0 we start with there will be an equidistant which falls into this case. It turns out to be a rich area for investigation; here we shall give some invariants which help to separate out the many subcases. One of these invariants classifies the effect of changing λ slightly from the degenerate value, while preserving the geometrical situation of two surfaces with parallel tangent planes at parabolic points where the asymptotic directions are parallel, that is $\varepsilon = 0$ in (1), (2). See Proposition 4.1.

4.1. **A normal form for Degenerate Case 2.** The 2-jet of $H_{0\lambda}$ is now $(u_1, u_2, 2f_{20}s_1u_1)$. Writing the third component as $u_1(s_1 + \text{h.o.t.})+$ terms independent of u_1 and then using the bracketed expression to redefine s_1 we can eliminate u_1 from the higher terms. Then replacing s_2 by an expression of the form $s_2 + au_2$ we can remove the degree 3 terms $s_1u_2^2$ and $s_1s_2u_2$. When this is done, the coefficient of $s_2^2u_2$ becomes $3g_{030}f_{20}^2/g_{20}^2 \neq 0$ and the coefficient of $s_2u_2^2$ becomes $3f_{20}g_{030}(g_{20} - f_{20})/g_{20}^2 \neq 0$. We shall also assume that the coefficient of s_1^3 is nonzero to avoid further degeneration. We can now use scaling to reduce the 3-jet of $H_{0\lambda}$ to

$$\left(u_1,\ u_2,\ s_1u_1 + s_1^3 + s_2^2u_2 + s_2u_2^2 + bs_1^2s_2 + cs_1^2u_2 + ds_1s_2^2 + es_2^3\right),$$

for coefficients b, c, d, e. The 4-jet can then be reduced by similar arguments, including scaling, to

$$(u_1, u_2, h) = \left(u_1, u_2, s_1u_1 + s_1^3 + s_2^2u_2 + s_2u_2^2 + bs_1^2s_2 + cs_1^2u_2 + ds_1s_2^2 + es_2^3 + s_1^4 + (ps_2 + qs_1^2)\right),$$

provided the coefficient of s_1^4 is nonzero: this and the 4-$\mathcal{A}$-determinacy of this 4-jet hold generically, by standard calculations. The terms in brackets, $ps_1 + qs_1^2$, represent an $\mathcal{A}$-versal unfolding of this germ. We have not been able to reduce the number of coefficients b, c, d, e. We shall work with (4.1) as a 'normal form' and when appropriate interpret the coefficients in terms of the surfaces M_0, N_0.

The equidistant for M_0, N_0 and $\lambda = g_{20}/(g_{20} - f_{20})$ is then locally diffeomorphic to the image under (4.1) of the set $\{(s_1, s_2, u_1, u_2) : h_{s_1} = h_{s_2} = 0\}$. Here, $h_{s_1} = 0$ defines u_1 as a smooth function of the other three variables, while $h_{s_2} = 0$ can be written

$$\frac{\partial h}{\partial s_2} = (s_2 + u_2)^2 + bs_1^2 + 2ds_1s_2 + (3e-1)s_2^2 = (s_2 + u_2)^2 - T(s_1, s_2) = 0, \tag{16}$$

say where T is a quadratic form in s_1, s_2 which we shall assume to be nondegenerate, that is $d^2 - b(3e - 1) \neq 0$.

4.2. **Plotting the equidistants.** It is also useful to rewrite the equation of the quadric cone C, given by $h_{s_2} = 0$, where $p = q = 0$ in (4.1), and provided $b \neq 0$, as

$$C: \quad (s_2 + u_2)^2 + b\left(s_1 + \frac{d}{b}s_2\right)^2 + \left(\frac{3be - b - d^2}{b}\right)s_2^2 = 0. \tag{17}$$

Note that this is a single point at the origin if and only if all coefficients are > 0 (since the first one is > 0), that is

$$b > 0, \ d^2 + b - 3be < 0;$$

compare Proposition 4.1.

The equidistant (for $p = q = 0$) is the image of C under the map $\mathbb{R}^3 \to \mathbb{R}^3$ given by

$$(s_1, s_2, u_2) \mapsto (u_1, u_2, \overline{h}(s_1, s_2, u_2))$$

where on the right-hand side u_1 is expressed in terms of s_1, s_2, u_2 using $h_{s_1} = 0$ and this is substituted into h, giving the function $\overline{h}$.

We can find a 'good' parametrization of the equidistant by using coordinates (x_1, x_2, s_2) and writing (17) as

$$x_1^2 + bx_2^2 + ks_2^2, \text{ where } k = \tfrac{3be-b-d^2}{b}, \ x_1 = s_2 + u_2, \ x_2 = s_1 + \tfrac{d}{b}s_2.$$

Thus the substitution to use in $\overline{h}$ is $u_2 = x_1 - s_2, \ s_1 = x_2 - \frac{d}{b}s_2$. The equidistant is then plotted as follows.

(1) If $b > 0$ and C is not a single point then $k < 0$ (i.e. $d^2 + b - 3be > 0$) and we write
$$x_1^2 + bx_2^2 = (-k)s_2^2,$$
so that for any $(x_1, x_2) \neq (0, 0)$ we have two distinct values for s_2: there is no restriction on the values of x_1, x_2. We use x_1, x_2 as parameters and the two 'halves' of C are given by the two values of s_2.

(2) If $b < 0, k > 0$ (i.e. $d^2 + b - 3be > 0$) then we similarly write $x_1^2 + ks_2^2 = (-b)x_2^2$, so that for any $(x_1, s_2) \neq (0, 0)$ we have two distinct values for x_2. Here x_1, s_2 are used as parameters.

(3) Finally if $b < 0, k < 0$ (i.e. $d^2 + b - 3be < 0$) then we write $x_1^2 = (-b)x_2^2 + (-k)s_2^2$ and for any $(x_2, s_2) \neq (0, 0)$ we have two distinct values for x_1. Here x_2, s_2 are used as parameters.

For values of (p, q) other than $(0, 0)$ the equation of C acquires an extra term $-p$ on the right-hand side, thus creating a hyperboloid of one or two sheets (or an ellipsoid when C is a single point). In fact the hyperboloid has one sheet when $bkp > 0$, that is $(d^2 + b - 3be)p < 0)$, and two sheets when $bkp < 0$, that is $(d^2 + b - 3be)p > 0)$. In the two-sheet situation the same method as above plots the equidistant, without restrictions on the values of the parameters. In the one-sheet situation the points in the parameter plane lie outside an ellipse, the 'waist' of the hyperboloid. This ellipse is given in the three situations above by $x_1^2 + bx_2^2 = -p, \ x_1^2 + ks_2^2 = -p$ and $(-b)x_2^2 + (-k)s_2^2 = p$ respectively. In the situation where C is a single point, and $p < 0$, the points in the parameter plane lie inside an ellipse. In all situations, q does not affect the hyperboloid or ellipsoid, but of course its value affects the function $\overline{h}$.

4.3. **Nearby non-special values of λ.** Here, we examine the effect of adding in the term qs_1^2 in (4.1). This represents changing λ from the value $g_{20}/(g_{20}-f_{20})$ to a nearby value, which will be of the type considered in Generic Case 1.1, provided the coefficient e of s_2^3 in (4.1) is nonzero, and to avoid further degeneracy we shall assume this to be true. We determine here, in terms of b, c, d, e, which subcase of Proposition 3.2 is obtained, and then refer this back to the surfaces M_0, N_0. (The subcase does not depend on the sign of q in the added term qs_1^2.) To do this we reduce (4.1), with $p = 0$ but with qs_1^2 present, to the normal form found above for Generic Case 1.1, by making the 'left' and 'right' changes of coordinates as sketched above. We can restrict attention for this to the terms of (4.1) of degree ≤ 3 since the Generic Case 1.1 germ is 3-$\mathcal{A}$-determined. Thus we start by redefining s_1 ('completing the square') to change the degree 2 terms to s_1^2, remove the terms in u_1, u_2 only, remove the remaining terms besides s_1^2 that are divisible by s_1 and then redefine s_2 by adding suitable multiples of u_1 and u_2. The result of this is to reduce the 3-jet of (4.1) by $\mathcal{A}$-equivalence to the form

$$\left(u_1,\ u_1,\ qs_1^2 + es_2^3 + \frac{s_2}{12eq^2}\left((3be-d^2)u_1^2 + 4qdu_1u_2 + 4q^2(3e-1)u_2^2\right)\right).$$

The discriminant of the quadratic form in u_1, u_2 is $(d^2+b-3be)/3eq^2$, so this form is definite if and only if $e(b+d^2-3be) < 0$. Scaling so that the terms in s_1^2, s_2^3 have coefficients equal to 1 multiplies the quadratic form in u_1, u_2 by $(q^2e)^{-1/3}$, and from this we deduce the following, where (i) and (ii) are derived by direct calculations from the parametrizations of M_0 and N_0.

Proposition 4.1. *The normal form (4.1) for Degenerate Case 2, with $p = 0$ but q nonzero and small, corresponding to a small change in λ, gives the following subcases of Generic Case 1.1 (general λ):*
Subcase 1.1.1 (positive definite, $++$): $e > \frac{1}{3}$ and $d^2+b-3be < 0$,
Subcase 1.1.2 (negative definite, $--$): $e < \frac{1}{3}$ and $e(d^2+b-3be) < 0$,
Subcase 1.1.3 (indefinite, $+-$): $e(d^2+b-3be > 0$.

In terms of the surfaces M_0, N_0,
(i) *When $f_{030}g_{030} > 0$, so $f_{030} = f_3^2$, $g_{030} = g_3^2$, $e < \frac{1}{3}$ and has the sign of $f_{20}g_3^2 - g_{20}f_3^2$ while $d^2+b-3be$ has the sign of $-R$ as in (9).*
(ii) *When $f_{030}g_{030} < 0$, so $f_{030} = f_3^2$, $g_{030} = -g_3^2$, $e > \frac{1}{3}$ and $d^2+b-3be$ has the sign of R.*

4.4. **Invariants distinguishing subcases of Degenerate Case 2.** We shall use the following:

(1) The number of cuspidal edges on the equidistant for $p = q = 0$, which can be 0, 2 or 4 (see below);
(2) The number of self-intersection curves on the equidistant for $p = q = 0$, which can be 0, 1, 2 or 3 (see §4.5);
(3) The subcase of Generic Case 1.1 given in Proposition 4.1 which is obtained by changing λ slightly.

This might give $3 \times 4 \times 3 = 36$ subcases but fortunately many of these combinations cannot be realized. We shall give values of b, c, d, e realizing of all possible subcases in §4.6, Table 1 below.

For given values of these invariants, the interval in which e lies, either $e < 0$ or $0 < e < \frac{1}{3}$ or $e > \frac{1}{3}$ could in principle affect the equidistant but so far as we are aware the basic geometrical structure—the qualitative nature of the equidistant—is not affected.

The number of cuspidal edges, that is 1-dimensionial singular sets, on the equidistant, can be calcuated as follows. We can regard $h_{s_2} = 0$, as in §4.2 above, as the equation of a quadric cone C in $\mathbb{R}^3$ with coordinates (s_1, s_2, u_2). The quadric cone C is nondegenerate since T in (16) is a nondegenerate quadratic form, and consists of the origin alone if and only if T is negative definite (that is, $d^2 < b(3e-1)$ and $b > 0$), otherwise it is a real cone, or equivalently a real nonsingular conic in $\mathbb{R}P^2$.

When T is not negative definite, the equidistant therefore has two 'branches', which are the images of the two halves of the cone; these branches may intersect (apart from at the origin) and will generally themselves be singular. Writing the equation of C more briefly as $\gamma(s_1, s_2, u_2) = 0$, the singular set of the equidistant is the image of certain curves on C, given by the additional equation

$$\overline{h}_{s_1}\gamma_{s_2} - \overline{h}_{s_2}\gamma_{s_1} = 0.$$

(This can be written in terms of h itself as $h_{s_1s_1}h_{s_2s_2} - h^2_{s_1s_2} = 0$.) The lowest terms of the left hand side are of degree 2 in s_1, s_2, u_2 and therefore give another conic C_2 in $\mathbb{R}P^2$. The equation of C_2 is in fact

$$(b^2 - 3d)s_1^2 + (bd - 9e)s_1s_2 - (cd + 3)s_1u_2 + (d^2 - 3be)s_2^2 - (3ce + b)s_2u_2 - cu_2^2 = 0.$$

This meets the nonsingular conic $\gamma = 0$ in 0, 2 or 4 real points. (The conic C_2 cannot in fact be a single point: examination of the matrix of the above quadratic form in variables s_1, s_2, u_2 defining C_2 shows that its determinant is always ≤ 0 so the quadratic form cannot be positive definite, and negative definiteness is also ruled out by examining the signs of the other leading minors. The leading 1×1 minor cannot be < 0 at the same time as the leading 2×2 minor is > 0.) There are therefore 0, 2 or 4 curves through the origin on C whose images are the singular points, the cuspidal edges, of the equidistant. These cuspidal edges pass through the origin, lying on both 'sheets' of the equidistant.

The number of cuspidal edges can be calculated for example by substituting

$$(s_1, s_2, u_2) = (mt, nt, t)$$

in the equations of C and C_2, taking out the factor t^2 and finding the common solutions of the two resulting quadratic equations in m, n. Eliminating one of m, n gives a degree 4 equation in the other and there are standard algebraic techniques for computing the number of real solutions of a quartic equation—or for given (b, c, d, e) we can solve numerically. The results for the Classes I-X are given in Table 1 below.

4.5. **Self-intersections of the equidistant in Degenerate Case 2.** We start with the normal form (4.1) in §4, namely

$$(u_1, u_2, h) = \left(u_1,\ u_2,\ s_1u_1 + s_1^3 + s_2^2u_2 + s_2u_2^2 + bs_1^2s_2 + cs_1^2u_2 + ds_1s_2^2 + es_2^3 + s_1^4 + ps_2 + qs_1^2\right),$$

subject to the critical set conditions $h_{s_1} = h_{s_2} = 0$. We include the unfolding terms $ps_2 + qs_1^2$ though we are particularly interested in the self-intersections for $p = q = 0$. We can immediately solve $h_{s_1} = 0$ for u_1:

$$u_1 = -2bs_1s_2 - 2cs_1u_2 - ds_2^2 - 3s_1^2 - 4s_1^3 - 2qs_1,$$

so that the equations which state that two domain points (s_1, s_2, u_1, u_2) and say (t_1, t_2, u_1, u_2) have the same image take the following form.
(SI1): the above formula for u_1 gives the same answer for both domain points;
(SI2): the formula for h above gives the same answer for both domain points;
(SI3): $h_{s_2}(s_1, s_2, u_1, u_2) = 0$; and
(SI4): $h_{t_2}(t_1, t_2, u_1, u_2) = 0$.

It is convenient to make the substitution $s_1 = x_1 + y_1, t_1 = x_1 - y_1, s_2 = x_2 + y_2, t_2 = x_2 - y_2$, so that the 'trivial solution' $s_1 = t_1, s_2 = t_2$ becomes $y_1 = y_2 = 0$. Furthermore replacing y_1 by $-y_1$ and y_2 by $-y_2$ interchanges (s_1, s_2) and (t_1, t_2), that is interchanges the two domain points (s_1, s_2, u_1, u_2) and (t_1, t_2, u_1, u_2) with the same image in $\mathbb{R}^3$ under the normal form map (4.1). With this substitution the equations become say (SI1′), etc., and we use (SI3′)-(SI4′) to solve for u_2:

$$u_2 = -\frac{bx_1y_1 + dx_1y_2 + dx_2y_1 + 3ex_2y_2}{y_2},$$

where the denominator y_2 is harmless since it is easy to check that if $y_2 = 0$ then the other equations imply that $y_1 = 0$ too. *Note that this expression does not involve p, q.*

We can solve (SI1′) for x_2:

$$x_2 = \frac{bcx_1y_1^2 + cdx_1y_1y_2 - bx_1y_2^2 - 6x_1^2y_1y_2 - 2y_1^3y_2 - 3x_1y_1y_2 - qy_1y_2}{-cdy_1^2 - 3cey_1y_2 + by_1y_2 + dy_2^2}.$$

This time we may need to investigate the vanishing of the denominator, but assuming the denominator is nonzero and substituting for x_2 we find that the equation (SI2′)-y_2((SI3′)+(SI4′)) reduces to

$$\text{SI5} : by_1^2y_2 + dy_1y_2^2 + ey_2^3 + 4x_1y_1^3 + y_1^3 = 0. \tag{18}$$

This is to be treated as the equation of a surface in 3-space (x_1, y_1, y_2) which contains the x_1-axis, since $(x_1, 0, 0)$ is always a solution. The surface will have a certain number of 'sheets' passing through the origin, equal to the number of values of k which make the first coordinate zero in the following parametriztion of SI5 by k and y_1.

$$\left(-\frac{ek^3 + dk^2 + bk + 1}{4},\ y_1,\ ky_1\right). \tag{19}$$

If $y_1 = 0$ in (18), then $y_2 = 0$ and x_1 is arbitrary; and indeed, being cubic in k, (19) gives all points $(x_1, 0, 0)$, possibly for more than one (real) k. If $y_1 \neq 0$ then we solve (18) for x_1 and writing $y_2 = ky_1$ produces the given value $-\frac{1}{4}(ek^3 + dk^2 + bk + 1)$ for x_1. Conversely, every point (19) satisfies (18) by substitution. Hence (19) parametrizes the complete surface (18). Two examples are shown in Figure 6.

Note that the surface (18) and the parametrization (19) are independent of the unfolding parameters p, q.

Proposition 4.2. *The number of smooth real sheets of the surface (18) through the origin in (x_1, y_1, y_2)-space is 1 or 3 according as*

$$27e^2 + 2b(2b^2 - 9d)e + d^2(4d - b^2) > 0 \text{ or } < 0 \text{ respectively.}$$

This number is therefore the maximum number of self-intersection branches of the equidistant, for any p, q. If $b^2 < 3d$ then the displayed expression is > 0 for all values of e.

Proof This is a matter of calculating the discriminant of the cubic polynomial ek^3+dk^2+bk+1 in k, and the discriminant $16(b^2 - 3d)^3$ of the displayed quadratic polynomial in e. The sheets will be smooth provided the cubic in k has no repeated root, that is provided the discriminant is nonzero. □

Remark 4.3. In §2.4 we noted that, in the current Degenerate Case 2, the sign of a certain polynomial in the coefficients of the two surfaces M_0, N_0 determines whether the 'scaled contact map' has type D_4^+ or D_4^-. By reducing to normal form as in §3 we can re-express this polynomial in terms of the coefficients b, c, d, e of the normal form. When this is done, we find that the condition for one (resp. three) sheets as in the above proposition coincides with the condition for D_4^+ (resp. D_4^-) in the scaled contact map. We do not know the full significance of this fact.

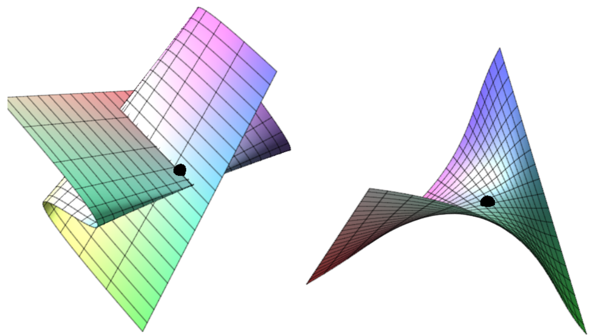

FIGURE 6. The surface given by (18) or (19), for (left) $b = 8, c = -4, d = -3, e = -1$, with three smooth sheets through the origin, which is marked by a black dot; (right) $b = -8, c = 4, d = -3, e = -1$, with one smooth sheet. (See Proposition 4.2.) These are respectively Class III and Class IX in Table 1 below. Note that in the first of these there are nevertheless only two self-intersection curves of the equidistant for $p = q = 0$, using the criterion of Proposition 4.6. In fact the picture for Class II is very similar to the left-hand figure, but there is only one self-intersection curve of the equidistant for $p = q = 0$.

Substituting $x_1 = -\frac{1}{4}(ek^3 + dk^2 + bk + 1)$ and $y_2 = ky_1$ in one of the conditions on x_1, y_1, y_2 not fully used yet (for example, SI2′) we obtain a single equation in y_1, k (involving now p and q) which determines the branches of the self-intersection set of the equidistant. We are interested in values of k close to a zero k_0 of the polynomial $ek^3 + dk^2 + bk + 1$, so we now write $k = k_0 + z$ say where z, as well as y_1, p, q, will be small. Since k_0 satisfies a cubic equation we can express k_0^3 in terms of k_0 and k_0^2, namely as $k_0^3 = (-dk_0^2 - bk_0 - 1)/e$, and therefore all higher powers of k_0 can be expressed in terms of k_0, k_0^2 as well.

Definition 4.4. For a chosen value of k_0, the polynomial in y_1, z, p, q just formed, the zero set of which determines the solutions to (SI1)-(SI4) or their equivalents (SI1′)-(SI4′), and hence determines the points corresponding to self-intersections of the equidistant, will be called $L(k_0)$. In the special case $p = q = 0$, we shall write $L_0(k_0)$ for the polynomial in y_1 and z.

We deduce the following; the statements 2-5 are easily checked by direct calculation.

Proposition 4.5. (1) *For each real root k_0 of $ek^3 + dk^2 + bk + 1 = 0$ one smooth sheet of the surface (18) is parametrized by (y_1, z) and the points which correspond to self-intersections on the equidistant for any p, q are given by the additional equation $L(k_0) = 0$.*

(2) *The polynomials $L(k_0)$ and $L_0(k_0)$ contain only the powers y_1^2 and y_1^4 of y_1. For any p, q the zero-set of $L(k_0)$ is symmetric about the y_1-axis in the (y_1, z)-plane.*

(3) *The other variable z occurs to powers ≤ 14 in $L(k_0)$. The coefficient of z^{14} is in fact $27e^5(3e - 1)$ which will not be zero since $e = 0, \frac{1}{3}$ are excluded values.*

(4) *The linear part of $L(k_0)$ has the form constant $\times p$. The nonzero quadratic terms are in y_1^2, z^2, zp, zq and q^2.*

(5) *The 2-jet of $L_0(k_0)$ has the form $c_0y_1^2 + c_2z^2$.*

The last statement above implies that, for $p = q = 0$, a given sheet of the surface (18), that is a given value of k_0, will correspond to a branch of the self-intersection set of the equidistant if and only of c_0, c_2 have opposite signs. When $c_0c_2 > 0$ there is only an isolated point at $y_1 = z = 0$. When $c_0c_2 < 0$ the two real branches of the set $L_0(k_0) = 0$ (forming a crossing at the origin $y_1 = z = 0$) will give only one branch of the self-intersection set because, as noted above, replacing y_1 by $-y_1$, and hence $y_2 = ky_1$ by $-y_2 = k(-y_1)$, merely interchanges the domain points contributing to the self-intersection.

Each of c_0, c_2 is quadratic in k_0; multiplying them gives an expression of degree 4 which can be reduced to degree 2 again using the equation $ek^3 + dk^2 + bk + 1 = 0$. Writing the resulting quadratic expression as $N = N_0(b,c,d,e) + N_1(b,c,d,e)k_0 + N_2(b,c,d,e)k_0^2$ we have the following, which is used to determine the number of self-intersection branches of the equidistant in the ten classes of Table 1.

Proposition 4.6. *The number of real branches of the self-intersection set of the equidistant for $p = q = 0$ is the number of solutions $k = k_0$ of $ek^3 + dk^2 + bk + 1 = 0$ at which the quadratic N is < 0.*

As (p,q) moves away from $(0,0)$ we can still trace the zero set of $L(y_0)$ in the (y_1, z)-plane. An isolated point may disappear or open into a symmetric loop, which represents a self-intersection of the equidistant having two endpoints, if the loop crosses the y_1-axis, and a closed self-intersection curve if it does not. A crossing will become a 'hyperbola'; if it crosses the y_1-axis then the corresponding self-intersection curve will have two endpoints and if not then it will be an unbroken arc. This is illustrated in the next section.

4.6. **Examples.** Considering different realizable values of the three invariants in §4.4, we have the ten classes of equidistant given in Table 1. It is also possible in some of these classes to allow values of e in different ranges $e < 0,\ 0 < e < \frac{1}{3},\ e > \frac{1}{3}$ but this does not appear to affect the equidistant in any qualitative way. We can compute the curves in the (p,q)-plane alomg which the cusp edges or the self-intersection curves on the equidistant underfgo a qualitative change. (The ten cases of the table in fact have ten distinct configurations of these curves.)

Class	Cusp edges	self-int	Subcase (Prop. 4.1)	b	c	d	e
I	0	0	$++$	8	4	-3	1
II	0	1	$+-$	8	-4	-3	$\frac{1}{6}$
III	0	2	$--$	8	-4	-3	-1
IV	2	0	$+-$	-13	6	-3	-5
V	2	1	$--$	1	2	3	-1
VI	2	2	$+-$	8	4	-3	$\frac{1}{6}$
VII	2	3	$--$	-13	-6	1	$\frac{1}{6}$
VIII	2	3	$+-$	-8	4	1	$\frac{1}{6}$
IX	4	1	$+-$	-8	4	-3	-1
X	4	3	$+-$	-8	6	-3	10

TABLE 1. Ten distinct classes of Case 2, giving all possible realizations of the three invariants of §4.4, and examples of values of b, c, d, e which realize these invariants. The fourth column refers to the 'non-special' type which results from changing λ slightly from the degenerate value.

We shall now give more detail on Case II of the table, showing how the cuspidal edges and self-intersections of the equidistant evolve as (p,q) in (4.1) makes a circuit of the origin. Figure 8 shows the transformations in the cuspidal edge as (p,q) moves in such a circuit and Figure 9 gives schematic diagrams of the corresponding equidistants, indicating their self-intersections and cusp

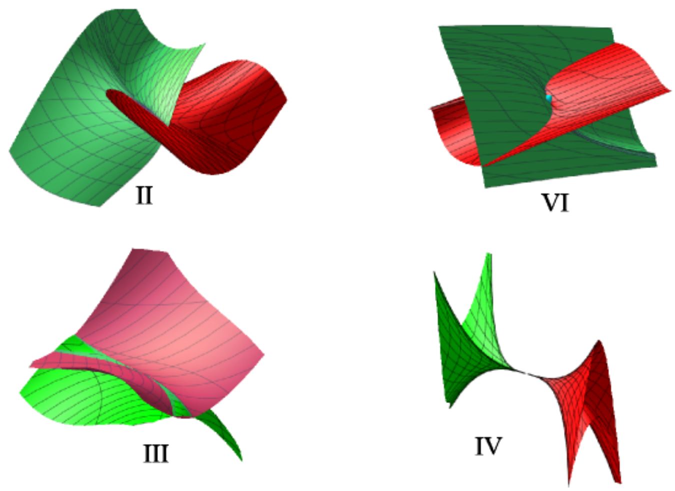

FIGURE 7. Cases II, III, IV and VI from Table 1, for $p = q = 0$. The origin is marked for Case VI, where there are two very narrow swallowtails passing through the origin, contributing two cusp edges and one self-intersection, and the other self-intersection is visible where the sheets pass through one-another.

edges. We use the following labelling on these figures to indicate transitions (perestroikas) in the structure of the equidistant.

Notation 4.7. $A_2^{++}, A_2^{--}, A_2^{+-}$ refer to Subcases 1.1.1, 1.1.2 and 1.1.3, as in Proposition 3.2. The corresponding transitions have also been described as 'Zeldovich's pancakes' or 'flying saucers', 'the death of a compact component of an edge', and 'the hyperbolic transformation of an edge', respectively. See also [9, 10].

A_3^+, A_3^- refer to the 'swallowtail-lips' and 'swallowtail-beaks' singularity respectively.

D_4^- refers to the 'pyramid' singularity (and D_4^+ would similarly be the 'purse' singularity).

$TA_1^{3,1}$, called such in [10, 1] (see also [9]) refers to the situation where three smooth sheets of the equidistant are pairwise transversal to each other, but the curve of intersection of any two of them is tangent to the third sheet at the moment of bifurcation.

5. Conclusion and further work

There have been many recent studies of singularities of (affine) equidistants of surfaces. For a single equidistant of a fixed surface, the generic singularities are A_1, A_2, A_3 (see for example [8, 4]); for a fixed surface, but allowing the ratio λ defining the equidistant to vary, the generic singularities are now A_1 (smooth surface), A_2 (cusp edge), A_3 (swallowtail), $A_3^{\pm}$ (swallowtail beaks/lips transition), A_4 (butterfly) and also $D_4^{\pm}$ (purse/pyramid) (compare [7]). The context of the present paper is to extend this to 1-parameter families of surfaces, the parameter in the family being ε in our notation, so that there are now two parameters to consider, λ and ε. The particular degeneracy in the ε family studied here comes from a 'supercaustic chord',

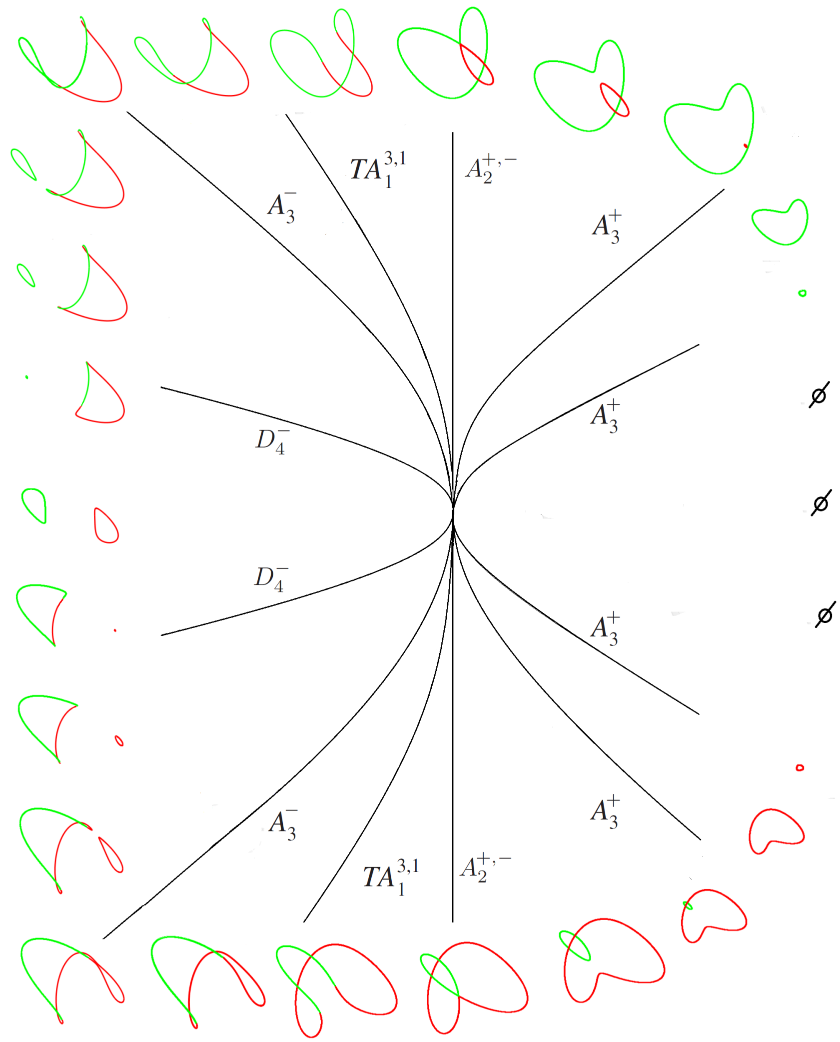

FIGURE 8. Pre-images of the cuspidal edges on the equidistants in Class II of Table 1 for unfolding parameters (p, q) making a circuit of the origin. The colours correspond to either the two parts of a hyperboloid of two sheets as in §4.2 or to the two parts into which a hyperboloid of one sheet is cut by the plane through the 'waist'. For the labelling of transitions, see Notation 4.7.

that is a chord joining two parabolic points with parallel tangent planes and parallel asymptotic directions. This occurs generically only in a 1-parameter family of surfaces. Along such a chord there may be special values of λ where singularities become more degenerate, depending on the relative local geometry of the surface patches at the ends of the chord. When two such special values exist (our Case 1.2) this corresponds to the intersection of an A_3 stratum with the supercaustic. In addition, there always exists a value of λ, which we call the degenerate Case 2. This corresponds to the intersection of a D_4 stratum with the supercaustic, and we

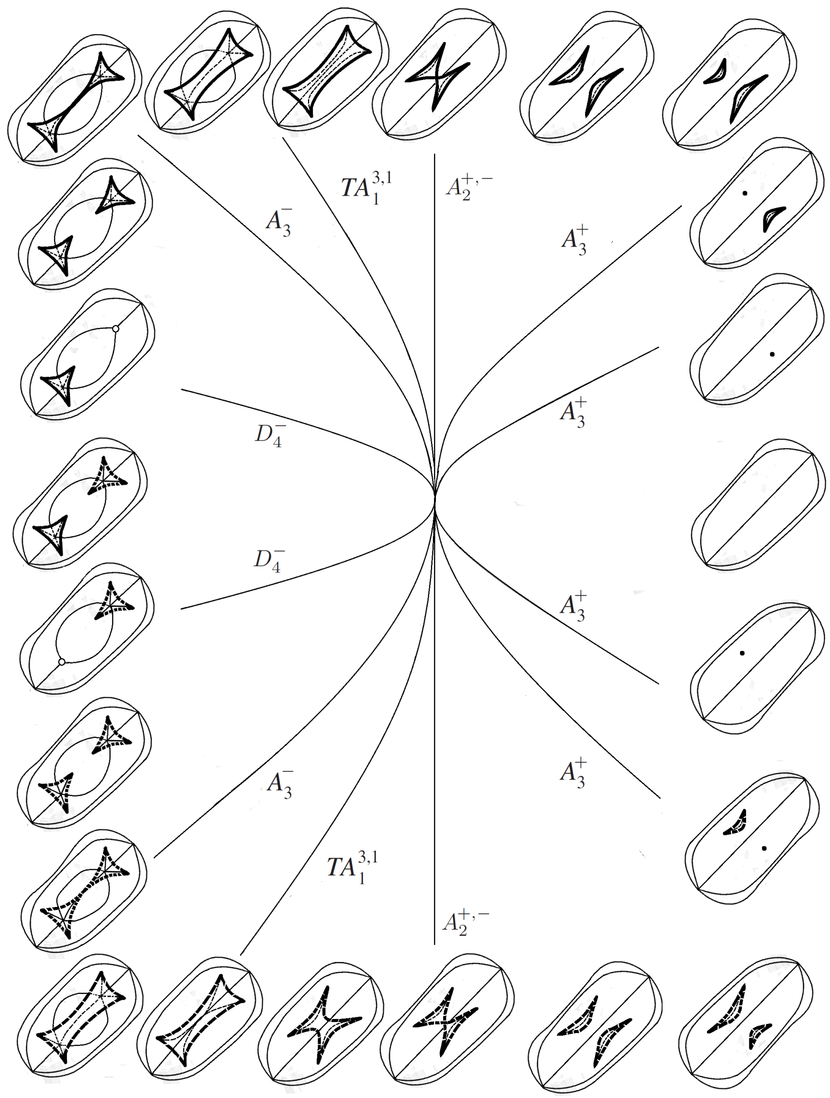

FIGURE 9. Schematic diagram of the equidistants for Class II of Table 1, with the unfolding parameters (p, q) making a circuit of the origin. The figure shows cuspidal edges (thick lines) and self-intersections (thin lines) with solid and dashed curves indicating visibility from one direction. For the labelling, see Notation 4.7.

elucidate ten geometrically distinct cases. Our paper also gives a natural geometric setting for many singularity types which belong to the list of corank 1 maps from $\mathbb{R}^3$ to $\mathbb{R}^3$ ([12, 9]), with the addition of a quadratic term in the extra variable which does not affect the critical set. The cases where equidistants are defined by $\lambda = 0$ or 1 remain to be studied.

A second natural 1-parameter family of surfaces is derived from the 'tangential' case in which two surface pieces share a common tangent plane (see for example [8]); here boundary singularities occur in the generic case, so that making one contact point parabolic in a 1-parameter family will introduce additional boundary singularities. The full adjacency diagram for singularities of equidistants of 1-parameter families of surfaces, not restricted to the supercaustic case, also remains to be found.

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