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ON FAMILIES OF LAGRANGIAN SUBMANIFOLDS

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Dedicated to Professor Goo Ishikawa on the occasion of his 60th birthday

Abstract. Lagrangian equivalence among Lagrangian submanifolds and $S.P^+$-Legendrian equivalence among graph-like Legendrian unfoldings are equivalent. We investigate r-parameter families of Lagrangian submanifolds and r-parameter families of graph-like Legendrian unfoldings. Then we show that r-parameter families of Lagrangian equivalence and r-parameter families of $S.P^+$-Legendrian equivalence are equivalent. As an application, we give a generic classification of bifurcations of Lagrangian submanifold germs for lower dimensions.

1. Introduction

The study of singularities of caustics and wave fronts was the starting point of the theory of Lagrangian and Legendrian singularities developed by several mathematicians and physicists (cf. [1], [2, 5, 6, 7, 11, 18, 19, 29, 30]). The caustic is described as the set of critical values of the projection of a Lagrangian submanifold from the phase space onto the configuration space. Lagrangian equivalence among Lagrangian submanifold germs in the phase space was introduced for the study of oscillatory integrals on caustics (cf. [1, 4, 8]). By definition, Lagrangian equivalence implies caustic equivalence (i.e. diffeomorphic caustics). However, it has been known that caustic equivalence does not imply Lagrangian equivalence even generically. This is one of the main differences from the theory of Legendrian singularities. In the theory of Legendrian singularities, wave fronts equivalence (i.e. diffeomorphic wave fronts) implies Legendrian equivalence generically. This is the reason why people considered caustic equivalence instead of Lagrangian equivalence in many situations (cf. [1, 24, 30] etc).

On the other hand, the notion of graph-like Legendrian unfoldings was introduced in [9]. It belongs to a special class of the big Legendrian submanifolds which were introduced in [30]. In §2, we give brief reviews on the theories of Lagrangian singularities (cf. [1, 2, 6]), of big Legendrian submanifolds (cf. [20]) and of graph-like Legendrian unfoldings (cf. [21, 22]), respectively. One of the main results in the theory of graph-like Legendrian unfoldings is that Lagrangian equivalence among Lagrangian submanifolds and $S.P^+$-Legendrian equivalence (which was introduced in [10]) among graph-like Legendrian unfoldings are equivalent, see Theorem 2.8 (cf. [13]). It is known that two graph-like Legendrian unfoldings are $S.P^+$-Legendrian equivalent if and only if the corresponding graph-like wave front set germs are $S.P^+$-diffeomorphic generically [13, 14]. In this sense, $S.P^+$-Legendrian equivalence is geometric equivalence. It follows that the hidden relation between caustics and wave front propagations can be investigated and revealed. In

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fact, we give several applications of Lagrangian singularity theory and graph-like Legendrian unfolding theory (cf. [13, 14, 15, 16, 21, 22, 23]).

On the other hand, if we consider r-parameter families of Lagrangian submanifold germs, the situation is not so simple. In [2, 30], V.I. Arnol'd and V.M. Zakalyukin gave a generic classification of bifurcations of caustics and wave fronts, and hence gave a generic classification of bifurcations of Legendrian submanifold germs by Legendrian equivalence. However, they only gave a generic classification of bifurcations of caustics by caustic equivalence. A generic classification of bifurcations of Lagrangian submanifold germs by Lagrangian equivalence has not been given in any contexts as far as the authors know. In this paper, we consider r-parameter families of Lagrangian submanifolds in §3 and r-parameter families of graph-like Legendrian unfoldings in §4, respectively. As a main result, we show that r-parameter Lagrangian equivalence among Lagrangian submanifolds families and r-parameter $S.P^+$-Legendrian equivalence among graph-like Legendrian unfoldings families are equivalent, see Theorem 5.1 in §5. Since $S.P^+$-Legendrian equivalence is geometric equivalence, it is much easier to investigate than Lagrangian equivalence. Therefore, as an application of Theorem 5.1, we give a generic classification of bifurcations of Lagrangian submanifolds by Lagrangian equivalence for lower dimensions, see Theorem 6.1 in §6. There appear functional moduli in the list of the classification even for lower dimensions.

All maps and manifolds considered here are differentiable of class C^∞.

2. Preliminaries

In order to fix the notations for describing the main results, we give brief reviews on the theories of Lagrangian singularities, of big Legendrian submanifolds and of graph-like Legendrian unfoldings, respectively. We also give a relation between the equivalence relations of Lagrangian submanifolds and graph-like Legendrian unfoldings (cf. [13, 16]).

2.1. **Lagrangian singularities.** We consider the cotangent bundle $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ with the canonical symplectic structure $\omega = \sum_{i=1}^n dp_i \wedge dx_i$, where $(x,p) = (x_1,\dots,x_n,p_1,\dots,p_n)$ is the canonical coordinate on $T^*\mathbb{R}^n$. A submanifold $i : L \subset T^*\mathbb{R}^n$ is said to be a *Lagrangian submanifold* if $\dim L = n$ and $i^*\omega = 0$. The set of the critical values of $\pi \circ i$ is called the *caustic* of $i : L \subset T^*\mathbb{R}^n$, which is denoted by C_L. One of the main results in the theory of Lagrangian singularities is the description of Lagrangian submanifold germs by using families of function germs. For a function germ $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, we say that F is a *Morse family of functions* if the map germ

$$\Delta F = \left(\frac{\partial F}{\partial q_1},\dots,\frac{\partial F}{\partial q_k}\right) : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$$

is non-singular, where $(q,x) = (q_1,\dots,q_k,x_1,\dots,x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, 0)$. In this case, we have a smooth n-dimensional submanifold germ $C(F) = (\Delta F)^{-1}(0) \subset (\mathbb{R}^k \times \mathbb{R}^n, 0)$ and a map germ $L(F) : (C(F), 0) \to T^*\mathbb{R}^n$ defined by

$$L(F)(q,x) = \left(x, \frac{\partial F}{\partial x_1}(q,x),\dots,\frac{\partial F}{\partial x_n}(q,x)\right).$$

We can show that $L(F)(C(F))$ is a Lagrangian submanifold germ. It is known that all Lagrangian submanifold germs in $T^*\mathbb{R}^n$ are constructed by the above method (cf. [2, page 300]).

A Morse family of functions $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is called a *generating family* of $L(F)(C(F))$. Let $\pi_n : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be the canonical projection, then we can easily show that the critical value set of $\pi_n|_{C(F)}$ is the *bifurcation set* $\mathcal{B}_F$ of F, where

$$\mathcal{B}_F = \left\{x \in (\mathbb{R}^n, 0) \;\middle|\; \text{there exists } q \in (\mathbb{R}^k, 0) \text{ such that } (q,x) \in C(F),\ \operatorname{rank}\left(\frac{\partial^2 F}{\partial q_i \partial q_j}(q,x)\right) < k\right\},$$

so that we have $C_{L(F)(C(F))} = \mathcal{B}_F$.

We now define an equivalence relation among Lagrangian submanifold germs. Let

$$i : (L, x) \subset (T^*\mathbb{R}^n, p) \quad \text{and} \quad i' : (L', x') \subset (T^*\mathbb{R}^n, p')$$

be Lagrangian submanifold germs. Then we say that i and i' are *Lagrangian equivalent* if there exist a diffeomorphism germ $\sigma : (L, x) \to (L', x')$, a symplectic diffeomorphism germ $\hat{\tau} : (T^*\mathbb{R}^n, p) \to (T^*\mathbb{R}^n, p')$ and a diffeomorphism germ $\tau : (\mathbb{R}^n, \pi(p)) \to (\mathbb{R}^n, \pi(p'))$ such that $\hat{\tau} \circ i = i' \circ \sigma$ and $\pi \circ \hat{\tau} = \tau \circ \pi$. Then the caustic C_L is diffeomorphic to the caustic $C_{L'}$ by the diffeomorphism germ τ. However, it has been known that caustic equivalence does not imply Lagrangian equivalence even generically (cf. [2, 12, 16]).

A Lagrangian submanifold germ in $T^*\mathbb{R}^n$ at a point is said to be *Lagrange stable* if for every map with the given germ there is a neighbourhood in the space of Lagrangian submanifolds (in the Whitney C^∞-topology) and a neighbourhood of the original point such that each Lagrangian submanifold belonging to the first neighbourhood has in the second neighbourhood a point at which its germ is Lagrangian equivalent to the original germ.

We can interpret the Lagrangian equivalence by using the notion of generating families. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be function germs. We say that F and G are *P-$\mathcal{R}^+$-equivalent* if there exist a diffeomorphism germ

$$\Phi : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}^k \times \mathbb{R}^n, 0)$$

of the form $\Phi(q, x) = (\phi_1(q, x), \phi_2(x))$ and a function germ $\alpha : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that $G(q, x) = F(\Phi(q, x)) + \alpha(x)$. For any $F_1 : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $F_2 : (\mathbb{R}^{k'} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, F_1 and F_2 are said to be *stably P-$\mathcal{R}^+$-equivalent* if they become P-$\mathcal{R}^+$-equivalent after the addition to the arguments q_i of new arguments q'_i and to the functions F_i of non-degenerate quadratic forms Q_i in the new arguments, that is, $F_1 + Q_1$ and $F_2 + Q_2$ are P-$\mathcal{R}^+$-equivalent. Then we have the following theorem (cf. [2, pages 304 and 325]):

Theorem 2.1. *Let $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $G : (\mathbb{R}^{k'} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be Morse families of functions. Then $L(F)(C(F))$ and $L(G)(C(G))$ are Lagrangian equivalent if and only if F and G are stably P-$\mathcal{R}^+$-equivalent.*

2.2. **The theory of wave front propagations.** We consider one-parameter families of wave fronts and their bifurcations. The principal idea is that a one-parameter family of wave fronts is considered to be a wave front whose dimension is one dimension higher than each member of the family. This is called a *big wave front*. We distinguish space and time coordinates, so that we denote $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and coordinates are denoted by $(x, t) = (x_1, \ldots, x_n, t) \in \mathbb{R}^n \times \mathbb{R}$. Then we consider the projective cotangent bundle $\overline{\pi} : PT^*(\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ over $\mathbb{R}^n \times \mathbb{R}$. Let $\overline{\Pi} : TPT^*(\mathbb{R}^n \times \mathbb{R}) \to PT^*(\mathbb{R}^n \times \mathbb{R})$ be the tangent bundle over $PT^*(\mathbb{R}^n \times \mathbb{R})$ and

$$d\overline{\pi} : TPT^*(\mathbb{R}^n \times \mathbb{R}) \to T(\mathbb{R}^n \times \mathbb{R})$$

the differential map of $\overline{\pi}$. For any $X \in TPT^*(\mathbb{R}^n \times \mathbb{R})$, there exists an element $\alpha \in T^*_{(x,t)}(\mathbb{R}^n \times \mathbb{R})$ such that $\overline{\Pi}(X) = [\alpha]$. For an element $V \in T_{(x,t)}(\mathbb{R}^n \times \mathbb{R})$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define *the canonical contact structure* on $PT^*(\mathbb{R}^n \times \mathbb{R})$ by $K = \{X \in TPT^*(\mathbb{R}^n \times \mathbb{R}) \mid \overline{\Pi}(X)(d\overline{\pi}(X)) = 0\}$. Because of the trivialization $PT^*(\mathbb{R}^n \times \mathbb{R}) \cong (\mathbb{R}^n \times \mathbb{R}) \times P(\mathbb{R}^n \times \mathbb{R})^*$, we call

$$((x_1, \ldots, x_n, t), [\xi_1 : \cdots : \xi_n : \tau])$$

homogeneous coordinates, where $[\xi_1 : \cdots : \xi_n : \tau]$ are the homogeneous coordinates of the dual projective space $P(\mathbb{R}^n \times \mathbb{R})^*$. It is easy to show that $X \in K_{((x,t),[\xi:\tau])}$ if and only if

$$\sum_{i=1}^{n} \mu_i \xi_i + \lambda\tau = 0,$$

where $d\overline{\pi}(X) = \sum_{i=1}^n \mu_i(\partial/\partial x_i) + \lambda(\partial/\partial t)$. We remark that $PT^*(\mathbb{R}^n \times \mathbb{R})$ is a fiberwise compactification of the 1-jet space $J^1(\mathbb{R}^n, \mathbb{R})$ as follows: We consider an affine open subset

$$U_\tau = \{((x,t),[\xi:\tau]) | \tau \neq 0\}$$

of $PT^*(\mathbb{R}^n \times \mathbb{R})$. For any $((x,t),[\xi:\tau]) \in U_\tau$, we have

$$((x_1,\dots,x_n,t),[\xi_1 : \cdots : \xi_n : \tau]) = ((x_1,\dots,x_n,t),[-(\xi_1/\tau) : \cdots : -(\xi_n/\tau) : -1]),$$

so that we may adapt the corresponding *affine coordinates* $((x_1,\dots,x_n,t),(p_1,\dots,p_n))$, where $p_i = -\xi_i/\tau$. On U_τ we can easily show that $\theta^{-1}(0) = K|U_\tau$, where $\theta = dt - \sum_{i=1}^n p_i dx_i$. This means that U_τ may be identified with the 1-jet space $J^1(\mathbb{R}^n, \mathbb{R})$. We set

$$U_\tau = J^1_{GA}(\mathbb{R}^n, \mathbb{R}) \subset PT^*(\mathbb{R}^n \times \mathbb{R}).$$

We call the above coordinate system *a system of graph-like affine coordinates.* Throughout this paper, we use this identification.

A submanifold $i : \mathcal{L} \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ is *a Legendrian submanifold* if $\dim \mathcal{L} = n$ and $di_p(T_p\mathcal{L}) \subset K_{i(p)}$ for any $p \in \mathcal{L}$. We say that a point $p \in \mathcal{L}$ is *a Legendrian singular point* if $\operatorname{rank} d(\overline{\pi} \circ i)_p < n$. For a Legendrian submanifold $i : \mathcal{L} \subset PT^*(\mathbb{R}^n \times \mathbb{R})$, $\overline{\pi} \circ i(\mathcal{L}) = W(\mathcal{L})$ is called *a big wave front.* We have a family of *small fronts*:

$$W_t(\mathcal{L}) = \pi_1(\pi_2^{-1}(t) \cap W(\mathcal{L})) \quad (t \in \mathbb{R}),$$

where $\pi_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $\pi_2 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are the canonical projections defined by $\pi_1(x,t) = x$ and $\pi_2(x,t) = t$ respectively. In this sense, we call $\mathcal{L}$ a *big Legendrian submanifold.*

The *discriminant of the family* $\{W_t(\mathcal{L})\}_{t\in\mathbb{R}}$ is defined as the image of singular points of $\pi_1|_{W(\mathcal{L})}$. In the general case, the discriminant consists of three components: *the caustic* $C_L = \pi_1(\Sigma(W(\mathcal{L})))$, where $\Sigma(W(\mathcal{L}))$ is the set of singular points of $W(\mathcal{L})$ (i.e. the critical value set of the Legendrian mappings $\overline{\pi}|_{\mathcal{L}} = \overline{\pi} \circ i$); *the Maxwell stratified set* $M_{\mathcal{L}}$, the projection of the closure of the self intersection set of $W(\mathcal{L})$; and the critical value set $\Delta_{\mathcal{L}}$ of $\pi_1|_{W(\mathcal{L})\setminus\Sigma(W(\mathcal{L}))}$. In [20, 21, 31], it has been stated that $\Delta_{\mathcal{L}}$ is the *envelope of the family of momentary fronts.* However, we remark that $\Delta_{\mathcal{L}}$ is not necessarily the envelope of the family of the projection of smooth momentary fronts $\overline{\pi}(W_t(\mathcal{L}))$. It may happen that $\pi_2^{-1}(t) \cap W(\mathcal{L})$ is non-singular while $\pi_1|_{\pi_2^{-1}(t)\cap W(\mathcal{L})}$ has singularities, so that $\Delta_{\mathcal{L}}$ is the set of critical values of the family of mappings $\pi_1|_{\pi_2^{-1}(t)\cap W(\mathcal{L})}$ for smooth $\pi_2^{-1}(t) \cap W(\mathcal{L})$ (cf. [12]).

For any Legendrian submanifold germ $i : (\mathcal{L}, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$, there exists a generating family of i by the theory of Legendrian singularities [2]. Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be a function germ such that $(\mathcal{F}, d_2\mathcal{F}) : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R} \times \mathbb{R}^k, 0)$ is non-singular, where

$$d_2\mathcal{F}(q,x,t) = \left(\frac{\partial \mathcal{F}}{\partial q_1}(q,x,t), \dots, \frac{\partial \mathcal{F}}{\partial q_k}(q,x,t)\right).$$

In this case, we call $\mathcal{F}$ a *big Morse family of hypersurfaces.* Then $\Sigma_*(\mathcal{F}) = (\mathcal{F}, d_2\mathcal{F})^{-1}(0)$ is a smooth n-dimensional submanifold germ. Define $\mathcal{L}_{\mathcal{F}} : (\Sigma_*(\mathcal{F}), 0) \to PT^*(\mathbb{R}^n \times \mathbb{R})$ by

$$\mathcal{L}_{\mathcal{F}}(q,x,t) = \left(x, t, \left[\frac{\partial \mathcal{F}}{\partial x}(q,x,t) : \frac{\partial \mathcal{F}}{\partial t}(q,x,t)\right]\right),$$

where

$$\left[\frac{\partial\mathcal{F}}{\partial x}(q,x,t):\frac{\partial\mathcal{F}}{\partial t}(q,x,t)\right]=\left[\frac{\partial\mathcal{F}}{\partial x_1}(q,x,t):\cdots:\frac{\partial\mathcal{F}}{\partial x_n}(q,x,t):\frac{\partial\mathcal{F}}{\partial t}(q,x,t)\right].$$

It is easy to show that $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ is a Legendrian submanifold germ. It is known that all big Legendrian submanifold germs are constructed by the above method (cf. [1,30]). We call $\mathcal{F}$ a *generating family of* $\mathcal{L}_{\mathcal{F}}$. The big wave front coincides with the *discriminant set* $D(\mathcal{F})$ of $\mathcal{F}$, where

$$D(\mathcal{F})=\left\{(x,t)\in(\mathbb{R}^n\times\mathbb{R},0)\;\middle|\;\text{there exists } q\in(\mathbb{R}^k,0)\text{ such that }(q,x,t)\in\Sigma_*(\mathcal{F})\right\},$$

so that we have $W(\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})))=D(\mathcal{F})$.

We now consider an equivalence relation among big Legendrian submanifolds which preserves both the qualitative pictures of bifurcations and the discriminant of families of small fronts. Let $i:(\mathcal{L},p_0)\subset(PT^*(\mathbb{R}^n\times\mathbb{R}),p_0)$ and $i':(\mathcal{L}',p_0')\subset(PT^*(\mathbb{R}^n\times\mathbb{R}),p_0')$ be big Legendrian submanifold germs. We say that i and i' are *strictly parametrized*$^+$ *Legendrian equivalent* (or, briefly, *$S.P^+$-Legendrian equivalent*) if there exist diffeomorphism germs

$$\Phi:(\mathbb{R}^n\times\mathbb{R},\overline{\pi}(p_0))\to(\mathbb{R}^n\times\mathbb{R},\overline{\pi}(p_0'))$$

of the form $\Phi(x,t)=(\phi_1(x),t+\alpha(x))$ and $\Psi:(\mathcal{L},p_0)\to(\mathcal{L}',p_0')$ such that $\widehat{\Phi}\circ i=i\circ\Psi$, where $\widehat{\Phi}:(PT^*(\mathbb{R}^n\times\mathbb{R}),p_0)\to(PT^*(\mathbb{R}^n\times\mathbb{R}),p_0')$ is the unique contact lift of Φ. This equivalence relation was independently introduced in [10, 31] for the different purposes, respectively. We can define the notion of stability of big Legendrian submanifold germs with respect to $S.P^+$-Legendrian equivalence similar to the definition of Lagrangian stability in §2.1 (cf. [2, Part III]). However, we omit to give the definition here.

We study $S.P^+$-Legendrian equivalence by using the notion of generating families of Legendrian submanifold germs. Let $\mathcal{E}_{(q,x,t)}$ be the $\mathbb{R}$-algebra of function germs of (q,x,t)-variables. For function germs $\mathcal{F},\mathcal{G}:(\mathbb{R}^k\times(\mathbb{R}^n\times\mathbb{R}),0)\to(\mathbb{R},0)$, we say that $\mathcal{F}$ and $\mathcal{G}$ are *space-$S.P^+$-$\mathcal{K}$-equivalent* (or, briefly, *s-$S.P^+$-$\mathcal{K}$-equivalent*) if there exists a diffeomorphism germ

$$\Phi:(\mathbb{R}^k\times(\mathbb{R}^n\times\mathbb{R}),0)\to(\mathbb{R}^k\times(\mathbb{R}^n\times\mathbb{R}),0)$$

of the form $\Phi(q,x,t)=(\phi(q,x,t),\phi_1(x),t+\alpha(x))$ such that $\langle\mathcal{F}\circ\Phi\rangle_{\mathcal{E}_{(q,x,t)}}=\langle\mathcal{G}\rangle_{\mathcal{E}_{(q,x,t)}}$. The notion of $S.P^+$-$\mathcal{K}$-versal deformation plays an important role for our purpose. We define the extended tangent space of $\overline{f}:(\mathbb{R}^k\times\mathbb{R},0)\to(\mathbb{R},0)$ relative to $S.P^+$-$\mathcal{K}$ by

$$T_e(S.P^+\text{-}\mathcal{K})(\overline{f})=\left\langle\frac{\partial\overline{f}}{\partial q_1},\ldots,\frac{\partial\overline{f}}{\partial q_k},\overline{f}\right\rangle_{\mathcal{E}_{(q,t)}}+\left\langle\frac{\partial\overline{f}}{\partial t}\right\rangle_{\mathbb{R}}.$$

We say that $\mathcal{F}$ is an *$S.P^+$-$\mathcal{K}$-versal deformation* of $\overline{f}=\mathcal{F}|_{\mathbb{R}^k\times\{0\}\times\mathbb{R}}$ if it satisfies

$$\mathcal{E}_{(q,t)}=T_e(S.P^+\text{-}\mathcal{K})(\overline{f})+\left\langle\frac{\partial\mathcal{F}}{\partial x_1}|_{\mathbb{R}^k\times\{0\}\times\mathbb{R}},\ldots,\frac{\partial\mathcal{F}}{\partial x_n}|_{\mathbb{R}^k\times\{0\}\times\mathbb{R}}\right\rangle_{\mathbb{R}}.$$

Then we also have the following theorem.

Theorem 2.2. *Let $\mathcal{F}:(\mathbb{R}^k\times(\mathbb{R}^n\times\mathbb{R}),0)\to(\mathbb{R},0)$ and $\mathcal{G}:(\mathbb{R}^{k'}\times(\mathbb{R}^n\times\mathbb{R}),0)\to(\mathbb{R},0)$ be big Morse families of hypersurfaces.*
(1) *$\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ and $\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are $S.P^+$-Legendrian equivalent if and only if $\mathcal{F}$ and $\mathcal{G}$ are stably s-$S.P^+$-$\mathcal{K}$-equivalent.*
(2) *$\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ is $S.P^+$-Legendre stable if and only if $\mathcal{F}$ is an $S.P^+$-$\mathcal{K}$-versal deformation of $\overline{f}=\mathcal{F}|_{\mathbb{R}^k\times\{0\}\times\mathbb{R}}$.*

Since the big Legendrian submanifold germ $i : (\mathcal{L}, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ is uniquely determined on the regular part of the big wave front $W(\mathcal{L})$, we have the following simple but significant property of Legendrian submanifold germs:

Proposition 2.3. *Let $i : (\mathcal{L}, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (\mathcal{L}', p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ be big Legendrian submanifold germs such that $\overline{\pi} \circ i, \overline{\pi} \circ i'$ are proper map germs and the regular sets of these map germs are dense respectively. Then $(\mathcal{L}, p_0) = (\mathcal{L}', p_0)$ if and only if*

$$(W(\mathcal{L}), \overline{\pi}(p_0)) = (W(\mathcal{L}'), \overline{\pi}(p_0)).$$

This result has been firstly pointed out by Zakalyukin [30]. Also see [25]. The assumption in the above proposition is a generic condition for i, i'. In particular, if i and i' are $S.P^+$-Legendre stable, then these satisfy the assumption.

Concerning the discriminant and the bifurcation of momentary fronts, we define the following equivalence relation among big wave front germs. Let $i : (\mathcal{L}, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (\mathcal{L}', p'_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p'_0)$ be big Legendrian submanifold germs. We say that $W(\mathcal{L})$ and $W(\mathcal{L}')$ are *$S.P^+$-diffeomorphic* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p_0)) \to (\mathbb{R}^n \times \mathbb{R}, \overline{\pi}(p'_0))$$

of the form $\Phi(x, t) = (\phi_1(x), t + \alpha(x))$ such that $\Phi(W(\mathcal{L})) = W(\mathcal{L}')$. Remark that the $S.P^+$-diffeomorphism among big wave front germs preserves the diffeomorphism types of discriminants [31]. By Proposition 2.3, we have the following proposition.

Proposition 2.4. *Let $i : (\mathcal{L}, p_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p_0)$ and $i' : (\mathcal{L}', p'_0) \subset (PT^*(\mathbb{R}^n \times \mathbb{R}), p'_0)$ be big Legendrian submanifold germs such that $\overline{\pi} \circ i, \overline{\pi} \circ i'$ are proper map germs and the regular sets of those map germs are dense respectively. Then i and i' are $S.P^+$-Legendrian equivalent if and only if $(W(\mathcal{L}), \overline{\pi}(p_0))$ and $(W(\mathcal{L}'), \overline{\pi}(p'_0))$ are $S.P^+$-diffeomorphic.*

2.3. **Graph-like Legendrian unfoldings.** In this subsection we explain the theory of graph-like Legendrian unfoldings. Graph-like Legendrian unfoldings belong to a special class of big Legendrian submanifolds. A big Legendrian submanifold $i : \mathcal{L} \subset PT^*(\mathbb{R}^n \times \mathbb{R})$ is said to be a *graph-like Legendrian unfolding* if $\mathcal{L} \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$.
We call $W(\mathcal{L}) = \overline{\pi}(\mathcal{L})$ a *graph-like wave front* of $\mathcal{L}$, where $\overline{\pi} : J^1_{GA}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ is the canonical projection. We define the mapping $\Pi : J^1_{GA}(\mathbb{R}^n, \mathbb{R}) \to T^*\mathbb{R}^n$ by $\Pi(x, t, p) = (x, p)$, where $(x, t, p) = (x_1, \dots, x_n, t, p_1, \dots, p_n)$ and the canonical contact form on $J^1_{GA}(\mathbb{R}^n, \mathbb{R})$ is given by $\theta = dt - \Sigma_{i=1}^n p_i dx_i$. Then we have the following proposition.

Proposition 2.5 ([12])**.** *For a graph-like Legendrian unfolding $\mathcal{L} \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, $z \in \mathcal{L}$ is a singular point of $\overline{\pi}|_{\mathcal{L}} : \mathcal{L} \to \mathbb{R}^n \times \mathbb{R}$ if and only if it is a singular point of $\pi_1 \circ \overline{\pi}|_{\mathcal{L}} : \mathcal{L} \to \mathbb{R}^n$. Moreover, $\Pi|_{\mathcal{L}} : \mathcal{L} \to T^*\mathbb{R}^n$ is immersive, so that $\Pi(\mathcal{L})$ is a Lagrangian submanifold in $T^*\mathbb{R}^n$.*

We have the following corollary of Proposition 2.5.

Corollary 2.6 ([12])**.** *For a graph-like Legendrian unfolding $\mathcal{L} \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, $\Delta_{\mathcal{L}}$ is the empty set so that the discriminant of the family of momentary fronts is $C_{\mathcal{L}} \cup M_{\mathcal{L}}$.*

Since $\mathcal{L}$ is a big Legendrian submanifold in $PT^*(\mathbb{R}^n \times \mathbb{R})$, it has a generating family

$$\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$$

at least locally. Since $\mathcal{L} \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R}) = U_\tau \subset PT^*(\mathbb{R}^n \times \mathbb{R})$, it satisfies the condition $(\partial \mathcal{F}/\partial t)(0) \neq 0$. Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be a big Morse family of hypersurfaces. We say that $\mathcal{F}$ is a *graph-like Morse family of hypersurfaces* if $(\partial \mathcal{F}/\partial t)(0) \neq 0$. It is easy to show that the corresponding big Legendrian submanifold germ is a graph-like Legendrian unfolding. Of course, all graph-like Legendrian unfolding germs can be constructed by the above way. We also say that $\mathcal{F}$ is a *graph-like generating family* of $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$. We remark that the notion of graph-like Legendrian unfoldings and corresponding generating families have been introduced by the first named author in [9] to describe the perestroikas of wave fronts given as the solutions for general eikonal equations.

We can consider the following more restrictive class of graph-like generating families: Let $\mathcal{F}$ be a graph-like Morse family of hypersurfaces. By the implicit function theorem, there exists a function $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that $\langle \mathcal{F}(q,x,t)\rangle_{\mathcal{E}_{(q,x,t)}} = \langle F(q,x) - t\rangle_{\mathcal{E}_{(q,x,t)}}$. Then we have the following proposition.

Proposition 2.7 ([22]). *Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ and $F : (\mathbb{R}^k \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be function germs such that $\langle \mathcal{F}(q,x,t)\rangle_{\mathcal{E}_{(q,x,t)}} = \langle F(q,x) - t\rangle_{\mathcal{E}_{(q,x,t)}}$. Then $\mathcal{F}$ is a graph-like Morse family of hypersurfaces if and only if F is a Morse family of functions.*

We now consider the case $\mathcal{F}(q,x,t) = \lambda(q,x,t)(F(q,x) - t)$, for $\lambda(0) \neq 0$. In this case,

$$\Sigma_*(\mathcal{F}) = \{(q,x,F(q,x)) \in (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \mid (q,x) \in C(F)\},$$

where $C(F) = \Delta F^{-1}(0)$. Moreover, we have the Lagrangian submanifold germ

$$L(F)(C(F)) \subset T^*\mathbb{R}^n,$$

where $L(F)$ is defined by

$$L(F)(q,x) = \left(x, \frac{\partial F}{\partial x_1}(q,x), \dots, \frac{\partial F}{\partial x_n}(q,x)\right).$$

Since $\mathcal{F}$ is a graph-like Morse family of hypersurfaces, we have a big Legendrian submanifold germ $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})) \subset J^1_{GA}(\mathbb{R}^n, \mathbb{R})$, where $\mathcal{L}_{\mathcal{F}} : (\Sigma_*(\mathcal{F}), 0) \to J^1_{GA}(\mathbb{R}^n, \mathbb{R}) = T^*\mathbb{R}^n \times \mathbb{R}$ is defined by

$$\mathcal{L}_{\mathcal{F}}(q,x,t) = \left(x, t, -\frac{\frac{\partial \mathcal{F}}{\partial x_1}(q,x,t)}{\frac{\partial \mathcal{F}}{\partial t}(q,x,t)}, \dots, -\frac{\frac{\partial \mathcal{F}}{\partial x_n}(q,x,t)}{\frac{\partial \mathcal{F}}{\partial t}(q,x,t)}\right).$$

We also define $\mathfrak{L}_F : (C(F), 0) \to J^1_{GA}(\mathbb{R}^n, \mathbb{R})$ by

$$\mathfrak{L}_F(q,x) = \left(x, F(q,x), \frac{\partial F}{\partial x_1}(q,x), \dots, \frac{\partial F}{\partial x_n}(q,x)\right).$$

Since $\partial\mathcal{F}/\partial x_i = (\partial\lambda/\partial x_i)(F - t) + \lambda \partial F/\partial x_i$ and $\partial\mathcal{F}/\partial t = (\partial\lambda/\partial t)(F - t) - \lambda$, we have

$$(\partial\mathcal{F}/\partial x_i)(q,x,t) = \lambda(q,x,t)(\partial F/\partial x_i)(q,x,t)$$

and

$$(\partial\mathcal{F}/\partial t)(q,x,t) = -\lambda(q,x,t)$$

for $(q,x,t) \in \Sigma_*(\mathcal{F})$. It follows that $\mathfrak{L}_F(C(F)) = \mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$. By definition, we have

$$\Pi(\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))) = \Pi(\mathfrak{L}_F(C(F))) = L(F)(C(F)).$$

The graph-like wave front of $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})) = \mathfrak{L}_F(C(F))$ is the graph of $F|_{C(F)}$. This is the reason why we call it a graph-like Legendrian unfolding.

For a graph-like Morse family of hypersurfaces $\mathcal{F}(q,x,t) = \lambda(q,x,t)(F(q,x) - t)$, $\mathcal{F}(q,x,t)$ and $\overline{F}(q,x,t) = F(q,x) - t$ are s-$S.P^+$-$\mathcal{K}$-equivalent, so that we consider $\overline{F}(q,x,t) = F(q,x) - t$ as a graph-like Morse family of hypersurfaces. Since $\overline{F}(q,x,t)$ is a big Morse family, we can use all the definitions of equivalence relations in §2.2. Moreover, we can translate the propositions and theorems into corresponding assertions in terms of graph-like Legendrian unfoldings. We can also consider the stability of graph-like Legendrian unfolding with respect to $S.P^+$-Legendrian equivalence which is analogous to the stability of Lagrangian submanifold germs with respect to Lagrangian equivalence in §2.1.

2.4. **Equivalence relations.** We consider a relation between the equivalence relations of Lagrangian submanifold germs and of graph-like Legendrian unfoldings (cf. [9, 10, 16, 20, 21, 31]).

Theorem 2.8 ([13]). *Let*

$$\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0) \quad \text{and} \quad \mathcal{G} : (\mathbb{R}^{k'} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$$

be graph-like Morse families of hypersurfaces of the forms $\mathcal{F}(q, x, t) = \lambda(q, x, t)(F(q, x) - t)$ *and* $\mathcal{G}(q', x, t) = \mu(q', x, t)(G(q', x) - t)$. *Then Lagrangian submanifold germs* $L(F)(C(F))$ *and* $L(G)(C(G))$ *are Lagrangian equivalent if and only if the graph-like Legendrian unfoldings* $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ *and* $\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ *are* $S.P^+$*-Legendrian equivalent.*

By definition, the set of Legendrian singular points of the graph-like Legendrian unfolding $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ coincides with the set of singular points of $\pi \circ L(F)$. Therefore the singularities of graph-like wave front of $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ lie on the caustic of $L(F)$. It follows that we can apply Proposition 2.4 to $S.P^+$-Legendrian equivalence. We have the following direct corollaries of Theorem 2.8.

Corollary 2.9. *With the same notations as those in Theorem 2.8, suppose that* $\overline{\pi} \circ \mathcal{L}_{\mathcal{F}}, \overline{\pi} \circ \mathcal{L}_{\mathcal{G}}$ *are proper map germs and the regular sets of these map germs are dense respectively. Then Lagrangian submanifold germs* $L(F)(C(F))$ *and* $L(G)(C(G))$ *are Lagrangian equivalent if and only if* $W(\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})))$ *and* $W(\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G})))$ *are* $S.P^+$*-diffeomorphic.*

Corollary 2.10. *Suppose that* $\mathcal{F}(q, x, t) = \lambda(q, x, t)(F(q, x) - t)$ *is a graph-like Morse family of hypersurfaces. Then* $L(F)(C(F))$ *is Lagrange stable if and only if* $\mathcal{L}(\Sigma_*(\mathcal{F}))$ *is* $S.P^+$*-Legendre stable.*

3. Families of Lagrangian submanifolds

We say that $i_r : L \times \mathbb{R}^r \subset T^*\mathbb{R}^n$ is an *r-parameter family of Lagrangian submanifolds* if $i|_{L \times \{s\}} : L \times \{s\} \subset T^*\mathbb{R}^n$ is a Lagrangian submanifold for each $s = (s_1, \ldots, s_r) \in \mathbb{R}^r$. By the theory of Lagrangian singularity in §2.1, we have a Morse family of functions. Let

$$F : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0), (q, x, s) \to F(q, x, s)$$

be an r-parameter family of Morse families of functions, that is, for each fixed $s \in (\mathbb{R}^r, 0)$, $F_s(q, x) = F(q, x, s)$ is a Morse family of functions and it depends smoothly on s.

We consider the cotangent bundle $\pi_r : T^*(\mathbb{R}^n \times \mathbb{R}^r) \to \mathbb{R}^n \times \mathbb{R}^r$ over $\mathbb{R}^n \times \mathbb{R}^r$. Let

$$(x, s, p, u) = (x_i, s_j, p_i, u_j), \ i = 1, \ldots, n, j = 1, \ldots, r$$

be the canonical coordinates on $T^*(\mathbb{R}^n \times \mathbb{R}^r)$. Then the canonical symplectic structure on $T^*(\mathbb{R}^n \times \mathbb{R}^r)$ is given by the canonical 2-form $\omega_r = \sum_{i=1}^n dp_i \wedge dx_i + \sum_{j=1}^r du_j \wedge ds_j$. We denote the canonical projection by $\widetilde{\pi}_r : T^*(\mathbb{R}^n \times \mathbb{R}^r) \to T^*\mathbb{R}^n$.

Let $F : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0), (q, x, s) \mapsto F(q, x, s)$ be an r-parameter family of Morse families of functions. Then it is also a Morse family of functions as an $(n+r)$-parameter family of function germs. Therefore we have a Lagrangian submanifold germ $L(F)(C(F)) \subset T^*(\mathbb{R}^n \times \mathbb{R}^r)$, where $L(F) : (C(F), 0) \to T^*(\mathbb{R}^n \times \mathbb{R}^r)$ is defined in §2.1. Moreover, $\widetilde{\pi}_r \circ L(F)(C(F)) \subset T^*\mathbb{R}^n$ is an r-parameter family of Lagrangian submanifold germs. We call $L(F)(C(F))$ a *big Lagrangian submanifold germ.*

Let $i_r : (L \times \mathbb{R}^r, (x, 0)) \subset (T^*(\mathbb{R}^n \times \mathbb{R}^r), p)$ and $i'_r : (L' \times \mathbb{R}^r, (x', 0)) \subset (T^*(\mathbb{R}^n \times \mathbb{R}^r), p')$ be big Lagrangian submanifold germs. We say that i_r and i'_r are *r-parameter Lagrangian equivalent* (or, briefly, *r-Lagrangian equivalent*) if there exist a diffeomorphism germ

$$\sigma : (L \times \mathbb{R}^r, (x, 0)) \to (L' \times \mathbb{R}^r, (x', 0))$$

of the form $\sigma(u,s)=(\sigma_1(u,s),\varphi(s))$, a symplectic diffeomorphism germ

$$\hat{\tau}:(T^*(\mathbb{R}^n\times\mathbb{R}^r),p)\to(T^*(\mathbb{R}^n\times\mathbb{R}^r),p')$$

and a diffeomorphism germ $\tau:(\mathbb{R}^n\times\mathbb{R}^r,\pi(p))\to(\mathbb{R}^n\times\mathbb{R}^r,\pi(p'))$ of the form

$$\tau(x,s)=(\tau_1(x,s),\varphi(s))$$

such that $\hat{\tau}\circ i_r=i'_r\circ\sigma$ and $\pi_r\circ\hat{\tau}=\tau\circ\pi_r$.

Let $F,G:(\mathbb{R}^k\times\mathbb{R}^n\times\mathbb{R}^r,0)\to(\mathbb{R},0)$ be function germs. We say that F and G are *P-$\mathcal{R}^+$-equivalent as r-parameter families* (or, briefly, *r-P-$\mathcal{R}^+$-equivalent*) if there exist a diffeomorphism germ $\Phi:(\mathbb{R}^k\times\mathbb{R}^n\times\mathbb{R}^r,0)\to(\mathbb{R}^k\times\mathbb{R}^n\times\mathbb{R}^r,0)$ of the form $\Phi(q,x,s)=(\phi_1(q,x,s),\phi_2(x,s),\varphi(s))$ and a function germ $\alpha:(\mathbb{R}^n\times\mathbb{R}^r,0)\to(\mathbb{R},0)$ such that $G(q,x,s)=F(\Phi(q,x,s))+\alpha(x,s)$. Then we also have the following theorem.

Theorem 3.1. *Let $F:(\mathbb{R}^k\times\mathbb{R}^n\times\mathbb{R}^r,0)\to(\mathbb{R},0)$ and $G:(\mathbb{R}^{k'}\times\mathbb{R}^n\times\mathbb{R}^r,0)\to(\mathbb{R},0)$ be r-parameter families of Morse families of functions. Then $L(F)(C(F))$ and $L(G)(C(G))$ are r-Lagrangian equivalent if and only if F and G are stably r-P-$\mathcal{R}^+$-equivalent.*

We also consider the stability of r-parameter families of Lagrangian submanifolds with respect to r-Lagrangian equivalence.

4. Families of graph-like Legendrian unfoldings

A big Legendrian submanifold

$$i:\mathcal{L}\times\mathbb{R}^r\subset PT^*(\mathbb{R}^n\times\mathbb{R}^r\times\mathbb{R})$$

is said to be an *r-parameter family of graph-like Legendrian unfoldings* if

$$\mathcal{L}\times\mathbb{R}^r\subset J^1_{GA}(\mathbb{R}^n\times\mathbb{R}^r,\mathbb{R}).$$

We call $W(\mathcal{L}\times\mathbb{R}^r)=\overline{\pi}_r(\mathcal{L}\times\mathbb{R}^r)$ an *r-parameter family of graph-like wave fronts* of $\mathcal{L}\times\mathbb{R}^r$, where $\overline{\pi}_r:J^1_{GA}(\mathbb{R}^n\times\mathbb{R}^r,\mathbb{R})\to\mathbb{R}^n\times\mathbb{R}^r\times\mathbb{R}$ is the canonical projection. By the theory of Legendrian singularity in §2.3, we have a graph-like Legendrian unfolding corresponding to the family of graph-like Legendrian unfoldings. Let

$$\mathcal{F}:(\mathbb{R}^k\times(\mathbb{R}^n\times\mathbb{R}^r\times\mathbb{R}),0)\to(\mathbb{R},0),(q,x,s,t)\to\mathcal{F}(q,x,s,t)$$

be an r-parameter family of graph-like Morse families of hypersurfaces, that is, for each fixed $s\in(\mathbb{R}^r,0)$, $\mathcal{F}_s(q,x,t)=\mathcal{F}(q,x,s,t)$ is a graph-like Morse family of hypersurfaces and it depends smoothly on s.

Let

$$i:(\mathcal{L}\times\mathbb{R}^r,(p,0))\subset(PT^*(\mathbb{R}^n\times\mathbb{R}^r\times\mathbb{R}),p_0)\quad\text{and}\quad i':(\mathcal{L}'\times\mathbb{R}^r,(p',0))\subset(PT^*(\mathbb{R}^n\times\mathbb{R}^r\times\mathbb{R}),p'_0)$$

be Legendrian submanifold germs. We say that i and i' are *r-parameter $S.P^+$-Legendrian equivalent* (or, briefly *r-$S.P^+$-Legendrian equivalent*) if there exist diffeomorphism germs

$$\Phi:(\mathbb{R}^n\times\mathbb{R}^r\times\mathbb{R},\overline{\pi}_r(p_0))\to(\mathbb{R}^n\times\mathbb{R}^r\times\mathbb{R},\overline{\pi}_r(p'_0))$$

of the form $\Phi(x,s,t)=(\phi_1(x,s),\varphi(s),t+\alpha(x,s))$ and $\Psi:(\mathcal{L}\times\mathbb{R}^r,p_0)\to(\mathcal{L}'\times\mathbb{R}^r,p'_0)$ of the form $\Psi(u,s)=(\psi_1(u,s),\varphi(s))$ such that $\widehat{\Phi}\circ i=i\circ\Psi$, where

$$\widehat{\Phi}:(PT^*(\mathbb{R}^n\times\mathbb{R}^r\times\mathbb{R}),p_0)\to(PT^*(\mathbb{R}^n\times\mathbb{R}^r\times\mathbb{R}),p'_0)$$

is the unique contact lift of Φ.

Let $\mathcal{F}, \mathcal{G} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be function germs. We say that $\mathcal{F}$ and $\mathcal{G}$ are *r-parameter s-$S.P^+$-$\mathcal{K}$-equivalent* (or, briefly, *r-s-$S.P^+$-$\mathcal{K}$-equivalent*) if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0) \to (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0)$ of the form

$$\Phi(q, x, s, t) = (\phi(q, x, s, t), \phi_1(x, s), \varphi(s), t + \alpha(x, s))$$

such that $\langle \mathcal{F} \circ \Phi \rangle_{\mathcal{E}_{(q,x,s,t)}} = \langle \mathcal{G} \rangle_{\mathcal{E}_{(q,x,s,t)}}$. Then we also have the following theorem.

Theorem 4.1. *Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ and $\mathcal{G} : (\mathbb{R}^{k'} \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be r-parameter families of graph-like Legendrian unfoldings. Then $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ and $\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are r-$S.P^+$-Legendrian equivalent if and only if $\mathcal{F}$ and $\mathcal{G}$ are stably r-s-$S.P^+$-$\mathcal{K}$-equivalent.*

We also consider the stability of r-parameter families of graph-like Legendrian unfoldings with respect to r-$S.P^+$-Legendrian equivalence.

5. Relations between equivalence relations

We consider a relation of the r-parameter version of equivalence relations between r-parameter families of Lagrangian submanifolds and r-parameter families of graph-like Legendrian unfoldings. One of the main results in this paper is as follows:

Theorem 5.1. *Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ and $\mathcal{G} : (\mathbb{R}^{k'} \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be r-parameter families of graph-like Morse families of hypersurfaces of the forms*

$$\mathcal{F}(q, x, s, t) = \lambda(q, x, s, t)(F(q, x, s) - t) \quad \text{and} \quad \mathcal{G}(q', x, s, t) = \mu(q', x, s, t)(G(q', x, s) - t).$$

Then r-parameter families of Lagrangian submanifold germs $L(F)(C(F))$ and $L(G)(C(G))$ are r-Lagrangian equivalent if and only if the r-parameter families of graph-like Legendrian unfoldings $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ and $\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are r-$S.P^+$-Legendrian equivalent.

Proof. By Theorem 3.1, if $L(F)(C(F))$ and $L(G)(C(G))$ are r-Lagrangian equivalent, then F and G are stably r-P-$\mathcal{R}^+$-equivalent. In this case, we may assume that $k = k'$, F and G are r-P-$\mathcal{R}^+$-equivalent, so that there exist a diffeomorphism germ $\Phi : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^r, 0)$ of the form $\Phi(q, x, s) = (\phi_1(q, x, s), \phi_2(x, s), \varphi(s))$ and a function germ $\alpha : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0)$ such that $G(q, x, s) = F(\Phi(q, x, s)) + \alpha(x, s)$. Then we define the diffeomorphism germ

$$\widetilde{\Phi} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0) \to (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}), 0)$$

by $\widetilde{\Phi}(q, x, s, t) = (\phi_1(q, x, s), \phi_2(x, s), \varphi(s), t - \alpha(x, s))$. It follows that

$$\overline{G}(q, x, s, t) = G(q, x, s) - t = F \circ \Phi(q, x, s) - t + \alpha(x, s) = \overline{F} \circ \widetilde{\Phi}(q, x, s, t).$$

This means that $\mathcal{F}$ and $\mathcal{G}$ are r-s-$S.P^+$-$\mathcal{K}$-equivalent. By Theorem 4.1, $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ and $\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are r-$S.P^+$-Legendrian equivalent.

Conversely, we assume that $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F}))$ and $\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G}))$ are r-$S.P^+$-Legendrian equivalent. Since $\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})) = \mathfrak{L}_F(C(F))$, $\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G})) = \mathfrak{L}_G(C(G))$, it follows from the assumption that there exist diffeomorphism germs $\Phi : (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}, 0)$ of the form

$$\Phi(x, s, t) = (\phi_1(x, s), \varphi(s), t + \alpha(x, s))$$

and $\Psi : (C(F), 0) \to (C(G), 0)$ of the form $\Psi(u, s) = (\psi_1(u, s), \varphi(s))$ such that

$$\widehat{\Phi}(\mathfrak{L}_F(C(F))) = \mathfrak{L}_G(C(G) \circ \Psi).$$

Then we have $\Phi^{-1}(x, s, t) = (\phi_1^{-1}(x, s), \varphi^{-1}(s), t - \alpha(x, s))$, where $\phi_1^{-1} : (\mathbb{R}^n \times \mathbb{R}^s, 0) \to (\mathbb{R}^n, 0)$ satisfies the condition $\phi_1^{-1}(\phi_1(x, s), \varphi(s)) = x$ and $\alpha(x, s)$ means $\alpha(\phi_1^{-1}(x, s), \varphi^{-1}(s))$.

Therefore, the Jacobi matrix of Φ^{-1} at $\Phi(x,s,t)$ is given by

$$J_{\Phi(x,s,t)}\Phi^{-1} = \begin{pmatrix} \frac{\partial \phi_1^{-1}}{\partial x}(\phi_1(x,s),\varphi(s)) & \frac{\partial \phi_1^{-1}}{\partial s}(\phi_1(x,s),\varphi(s)) & 0 \\ 0 & \frac{\partial \varphi^{-1}}{\partial s}(\varphi(s)) & 0 \\ -\frac{\partial \alpha}{\partial x}(\phi_1(x,s),\varphi(s)) & -\frac{\partial \alpha}{\partial s}(\phi_1(x,s),\varphi(s)) & 1 \end{pmatrix}.$$

It follows that

$$\begin{aligned}\hat{\Phi}(x,s,t,[p:u:\tau]) = \Big(\Phi(x,s,t), \Big[p\cdot\frac{\partial \phi_1^{-1}}{\partial x}(\phi_1(x,s),\varphi(s)) - \tau\frac{\partial \alpha}{\partial x}(\phi_1(x,s),\varphi(s)) :\\ p\cdot\frac{\partial \phi_1^{-1}}{\partial s}(\phi_1(x,s),\varphi(s)) + u\cdot\frac{\partial \varphi^{-1}}{\partial s}(\varphi(s)) - \tau\frac{\partial \alpha}{\partial s}(\phi_1(x,s),\varphi(s)) : \tau\Big]\Big).\end{aligned}$$

Since $\tau \neq 0$, we have

$$\begin{aligned}&\Big[p\cdot\frac{\partial \phi_1^{-1}}{\partial x}(\phi_1(x,s),\varphi(s)) - \tau\frac{\partial \alpha}{\partial x}(\phi_1(x,s),\varphi(s)) :\\ &\quad p\cdot\frac{\partial \phi_1^{-1}}{\partial s}(\phi_1(x,s),\varphi(s)) + u\cdot\frac{\partial \varphi^{-1}}{\partial s}(\varphi(s)) - \tau\frac{\partial \alpha}{\partial s}(\phi_1(x,s),\varphi(s)) : \tau\Big]\\ &= \Big[-\frac{p}{\tau}\cdot\frac{\partial \phi_1^{-1}}{\partial x}(\phi_1(x,s),\varphi(s)) + \frac{\partial \alpha}{\partial x}(\phi_1(x,s),\varphi(s)) :\\ &\quad -\frac{p}{\tau}\cdot\frac{\partial \phi_1^{-1}}{\partial s}(\phi_1(x,s),\varphi(s)) - \frac{u}{\tau}\cdot\frac{\partial \varphi^{-1}}{\partial s}(\varphi(s)) + \frac{\partial \alpha}{\partial s}(\phi_1(x,s),\varphi(s)) : -1\Big].\end{aligned}$$

We consider the graph-like affine coordinates $(x,s,t,p,u) \in J^1_{GA}(\mathbb{R}^n \times \mathbb{R}^r, \mathbb{R})$, where we denote again $-p/\tau$ by p and $-u/\tau$ by u, respectively. By the form of $\hat{\Phi}$, we have

$$\hat{\Phi}(J^1_{GA}(\mathbb{R}^n \times \mathbb{R}^r, \mathbb{R})) = J^1_{GA}(\mathbb{R}^n \times \mathbb{R}^r, \mathbb{R}).$$

We define $\widetilde{\Phi} : T^*(\mathbb{R}^n \times \mathbb{R}^r) \to T^*(\mathbb{R}^n \times \mathbb{R}^r)$ by

$$\widetilde{\Phi}(x,s,p,u) = (\phi_1(x,s), \varphi(s), \phi_2(x,s,p), \phi_3(x,s,p,u)),$$

where

$$\begin{aligned}\phi_2(x,s,p) &= p\cdot\frac{\partial \phi_1^{-1}}{\partial x}(\phi_1(x,s),\varphi(s)) + \frac{\partial \alpha}{\partial x}(\phi_1(x,s),\varphi(s)),\\ \phi_3(x,s,p,u) &= p\cdot\frac{\partial \phi_1^{-1}}{\partial s}(\phi_1(x,s),\varphi(s)) + u\cdot\frac{\partial \varphi^{-1}}{\partial s}(\varphi(s)) + \frac{\partial \alpha}{\partial s}(\phi_1(x,s),\varphi(s)).\end{aligned}$$

Since $\hat{\Phi}$ is a contact diffeomorphism germ, there exists a non-zero function germ $\lambda : J^1_{GA}(\mathbb{R}^n \times \mathbb{R}^r, \mathbb{R}) \to \mathbb{R}$ such that $\hat{\Phi}^*\theta = \lambda\theta$, where $\theta = dt - \sum_{i=1}^n p_i dx_i - \sum_{j=1}^r u_j ds_j$. Therefore, we have

$$dt + d\alpha - \phi_2 \cdot d\phi_1 - \phi_3 \cdot d\varphi = \lambda(dt - p\cdot dx - u\cdot ds).$$

It follows that $\lambda = 1$ and

$$d\alpha - \phi_2 \cdot d\phi_1 - \phi_3 \cdot d\varphi = -p\cdot dx - u\cdot ds.$$

If we set $\overline{\theta} = -\sum_{i=1}^n p_i dx_i - \sum_{j=1}^r u_j ds_j$, then

$$\widetilde{\Phi}^*\omega = \widetilde{\Phi}^* d\overline{\theta} = d\widetilde{\Phi}^*\overline{\theta} = d(-d\alpha + \overline{\theta}) = -d(d\alpha) + d\overline{\theta} = \omega.$$

This means that $\widetilde{\Phi}$ is a symplectic diffeomorphism germ. Since

$$\Pi_r \circ \hat{\Phi}|_{J^1_{GA}(\mathbb{R}^n\times\mathbb{R}^r,\mathbb{R})} = \widetilde{\Phi} \circ \Pi_r|_{J^1_{GA}(\mathbb{R}^n\times\mathbb{R}^r,\mathbb{R})},$$

we have

$$\begin{aligned} L(G)(C(G)\circ\Psi) &= \Pi_r(\mathfrak{L}_G(C(G)\circ\Psi)) = \Pi_r\circ\hat{\Phi}(\mathfrak{L}_F(C(F))) \\ &= \widetilde{\Phi}\circ\Pi_r(\mathcal{L}_F(C(F))) = \widetilde{\Phi}\circ L(F)(C(F)), \end{aligned}$$

where

$$\Pi_r : J^1_{GA}(\mathbb{R}^n\times\mathbb{R}^r,\mathbb{R}) \to T^*(\mathbb{R}^n\times\mathbb{R}^r)$$

is the canonical projection $\Pi_r(x,s,t,p,u) = (x,s,p,u)$. It follows that $L(F)(C(F))$ and $L(G)(C(G))$ are r-Lagrangian equivalent. This completes the proof. □

Let $i : (\mathcal{L}\times\mathbb{R}^r, p_0) \subset J^1_{GA}(\mathbb{R}^n\times\mathbb{R}^r,\mathbb{R})$ and $i' : (\mathcal{L}'\times\mathbb{R}^r, p'_0) \subset J^1_{GA}(\mathbb{R}^n\times\mathbb{R}^r,\mathbb{R})$ be r-parameter families of graph-like Legendrian unfoldings. We say that $W(\mathcal{L}\times\mathbb{R}^r)$ and $W(\mathcal{L}'\times\mathbb{R}^r)$ are *r-S.P$^+$-diffeomorphic* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R}^n\times\mathbb{R}^r\times\mathbb{R}, \overline{\pi}(p_0)) \to (\mathbb{R}^n\times\mathbb{R}^r\times\mathbb{R}, \overline{\pi}(p'_0))$$

of the form $\Phi(x,s,t) = (\phi_1(x,s), \varphi(s), t+\alpha(x,s))$ such that $\Phi(W(\mathcal{L}\times\mathbb{R}^r)) = W(\mathcal{L}'\times\mathbb{R}^r)$. We have the following direct corollaries of Theorem 5.1.

Corollary 5.2. *With the same notations as those in Theorem 5.1, suppose that*

$$\overline{\pi}_r\circ\mathcal{L}_{\mathcal{F}} \quad\text{and}\quad \overline{\pi}_r\circ\mathcal{L}_{\mathcal{G}}$$

are proper map germs and the regular sets of these map germs are dense respectively. Then r-parameter families of Lagrangian submanifold germs $L(F)(C(F))$ and $L(G)(C(G))$ are r-Lagrangian equivalent if and only if $W(\mathcal{L}_{\mathcal{F}}(\Sigma_*(\mathcal{F})))$ and $W(\mathcal{L}_{\mathcal{G}}(\Sigma_*(\mathcal{G})))$ are r-S.P$^+$-diffeomorphic.

Corollary 5.3. *Suppose that $\mathcal{F}(q,x,s,t) = \lambda(q,x,s,t)(F(q,x,s)-t)$ is an r-parameter family of graph-like Morse families of hypersurfaces. Then $L(F)(C(F))$ is r-Lagrange stable if and only if $\mathcal{L}(\Sigma_*(\mathcal{F}))$ is r-S.P$^+$-Legendre stable.*

6. Classifications of bifurcations of Lagrangian submanifolds

We consider bifurcations of Lagrangian submanifold germs, that is, the case of $r = 1$. As an application of Theorem 5.1, we give generic classifications of bifurcations of Lagrangian submanifold germs for lower dimensions by using one-parameter families of graph-like Legendrian unfoldings.

Theorem 6.1. *Let $1 \le n \le 3$. A generic one-parameter family of Lagrangian submanifold germs $L(F)(C(F))$ of a one-parameter family of Morse families of functions*

$$F : (\mathbb{R}^k\times\mathbb{R}^n\times\mathbb{R}, 0) \to (\mathbb{R}, 0),$$

is one-parameter Lagrangian equivalent to the one-parameter family of Lagrangian submanifold germs of one of the following one-parameter families of Morse families of functions:

$n = 1$;
(1) q_1,
(2) $\pm q_1^2 + x_1$,
(3) $q_1^3 + x_1q_1$,
(4) $\pm q_1^4 + \alpha(x_1,s)q_1^2 + x_1q_1$, $\partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_1(0) = 0$,

$n = 2$;
(1) q_1,
(2) $\pm q_1^2 + x_1q_1$,
(3) $q_1^3 + x_1q_1 + x_2$,

$(4)_1$ $\pm q_1^4 + x_1 q_1^2 + x_2 q_2,$
$(4)_2$ $\pm q_1^4 + \alpha(x_1, x_2, s) q_1^2 + x_1 q_1 + x_2,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_1(0) = \partial\alpha/\partial x_2(0) = 0,$
$(5)_1$ $q_1^5 + \alpha(x_1, x_2, s) q_1^3 + x_1 q_1^2 + x_2 q_1,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_1(0) = \partial\alpha/\partial x_2(0) = 0,$
$(5)_2$ $q_1^5 + x_1 q_1^3 + \alpha(x_1, x_2, s) q_1^2 + x_2 q_1,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_1(0) = \partial\alpha/\partial x_2(0) = 0,$
(6) $q_1^3 \pm q_1 q_2^2 + \alpha(x_1, x_2, s) q_1^2 + x_1 q_1 + x_2 q_2,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_1(0) = \partial\alpha/\partial x_2(0) = 0,$

$n = 3$;
(1) $q_1,$
(2) $\pm q_1^2 + x_1 q_1,$
(3) $q_1^3 + x_1 q_1 + x_2,$
$(4)_1$ $\pm q_1^4 + x_1 q_1^2 + x_2 q_2 + x_3,$
$(4)_2$ $\pm q_1^4 + \alpha(x_1, x_2, x_3, s) q_1^2 + x_1 q_1 + x_2,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0\ i = 1, 2, 3,$
$(5)_1$ $q_1^5 + x_1 q_1^3 + x_2 q_1^2 + x_3 q_1,$
$(5)_2$ $q_1^5 + \alpha(x_1, x_2, x_3, s) q_1^3 + x_1 q_1^2 + x_2 q_1 + x_3,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0,$
$(5)_3$ $q_1^5 + x_1 q_1^3 + \alpha(x_1, x_2, x_3, s) q_1^2 + x_2 q_1 + x_3,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0,$
$(6)_1$ $q_1^3 \pm q_1 q_2^2 + x_1 q_1^2 + x_2 q_1 + x_3 q_2,$
$(6)_2$ $q_1^3 \pm q_1 q_2^2 + \alpha(x_1, x_2, x_3, s) q_1^2 + x_1 q_1 + x_2 q_2 + x_3,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0,$
$(7)_1$ $\pm q_1^6 + \alpha(x_1, x_2, x_3, s) q_1^4 + x_1 q_1^3 + x_2 q_1^2 + x_3 q_1,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0,$
$(7)_2$ $\pm q_1^6 + x_1 q_1^4 + \alpha(x_1, x_2, x_3, s) q_1^3 + x_2 q_1^2 + x_3 q_1,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0,$
$(7)_3$ $\pm q_1^6 + x_1 q_1^4 + x_1 q_1^3 + \alpha(x_1, x_2, x_3, s) q_1^2 + x_3 q_1,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0,$
$(8)_1$ $\pm(q_1^2 q_2 + q_2^4) + \alpha(x_1, x_2, x_3, s) q_1^2 + x_1 q_2^2 + x_2 q_1 + x_3 q_2,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0,$
$(8)_2$ $\pm(q_1^2 q_2 + q_2^4) + x_1 q_1^2 + \alpha(x_1, x_2, x_3, s) q_2^2 + x_2 q_1 + x_3 q_2,\ \partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0,$
where $i = 1, 2, 3$.

The function germs α are called *functional moduli*. By definition of the one-parameter $S.P^+$-$\mathcal{K}$-equivalence relation, functional moduli must satisfy some extra conditions; however, we do not argue about such conditions here (cf. [17]).

In order to prove Theorem 6.1, we prepare some notations and results for the classification of function germs. We use a method for the classification of function germs in [26, 27, 28].

Let $\mathcal{F} : (\mathbb{R}^k \times (\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}), 0) \to (\mathbb{R}, 0)$ be a one-parameter family of graph-like Morse families of hypersurfaces of the form

$$\mathcal{F}(q, x, s, t) = \lambda(q, x, s, t)(F(q, x, s) - t).$$

We write $\overline{F}(q, x, s, t) = F(q, x, s) - t$. For an unfolding $\overline{F} : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ of $\overline{f}(q, x, t) = f(q, x) - t$, $\overline{F}$ is a 1-$S.P^+$-$\mathcal{K}$-versal deformation of $\overline{f}$ if

$$\mathcal{E}_{(q,x,t)} = \left\langle \frac{\partial f}{\partial q}(q, x), f(q, x) - t \right\rangle_{\mathcal{E}_{(q,x,t)}} + \left\langle \frac{\partial f}{\partial x}(q, x), 1 \right\rangle_{\mathcal{E}_x} + \left\langle \frac{\partial F}{\partial s}|_{s=0} \right\rangle_{\mathbb{R}}.$$

It follows that if

$$\dim_{\mathbb{R}} \mathcal{E}_{(q,x,t)} / \left(\left\langle \frac{\partial f}{\partial q}(q, x), f(q, x) - t \right\rangle_{\mathcal{E}_{(q,x,t)}} + \left\langle \frac{\partial f}{\partial x}(q, x), 1 \right\rangle_{\mathcal{E}_x} \right) \leq 1,$$

then

$$\dim_{\mathbb{R}} \mathcal{E}_{(q,t)} / \left(\left\langle \frac{\partial f}{\partial q}(q), f(q) - t \right\rangle_{\mathcal{E}_{(q,t)}} + \langle 1 \rangle_{\mathbb{R}} \right) \leq n + 1.$$

However, the condition of 1-$S.P^+$-$\mathcal{K}$-versal deformations (that is, 1-$S.P^+$-Legendrian stability for corresponding Legendrian submanifold germs) is too strong for giving the classification. We assume that $\overline{F}(q, x, s, t)$ is an $S.P^+$-$\mathcal{K}$-versal deformation of $\overline{f}(q, t)$, namely,

$$\mathcal{E}_{(q,t)} = \left\langle \frac{\partial f}{\partial q}(q,x), f(q,x) - t \right\rangle_{\mathcal{E}_{(q,t)}} + \langle 1 \rangle_{\mathbb{R}} + \left\langle \frac{\partial F}{\partial x}|_{x=s=0}, \frac{\partial F}{\partial s}|_{x=s=0} \right\rangle_{\mathbb{R}}.$$

We give a quick review of the classification of $S.P^+$-$\mathcal{K}$-versal deformations with $S.P^+$-$\mathcal{K}$-cod ≤ 4. For details see [10]. Let F and $F' : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be germs of unfoldings of f and $f' : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$, respectively. We say that F and F' are $S.P^+$-$\mathcal{K}$ (respectively, $S.P$-$\mathcal{K}$)-*equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0)$ of the form $\Phi(q,u,t) = (\phi_1(q,u,t), \phi_2(u), t + \alpha(u))$ (respectively, $\Phi(q,u,t) = (\phi_1(q,u,t), \phi_2(u), t))$ such that $\langle F \circ \Phi \rangle_{\mathcal{E}_{(q,u,t)}} = \langle F' \rangle_{\mathcal{E}_{(q,u,t)}}$. We also say that $F(q,u,t)$ is an $S.P^+$-$\mathcal{K}$ (respectively, $S.P$-$\mathcal{K}$)-*versal deformation* of $f = F|_{\mathbb{R}^k \times 0 \times \mathbb{R}}$ if

$$\mathcal{E}_{(q,t)} = \left\langle f, \frac{\partial f}{\partial q_1}, \dots, \frac{\partial g}{\partial q_k} \right\rangle_{\mathcal{E}_{(q,t)}} + \left\langle \frac{\partial f}{\partial t} \right\rangle_{\mathbb{R}} + \left\langle \frac{\partial F}{\partial u_1}|_{\mathbb{R}^k \times 0 \times \mathbb{R}}, \dots, \frac{\partial F}{\partial u_r}|_{\mathbb{R}^k \times 0 \times \mathbb{R}} \right\rangle_{\mathbb{R}}$$

$$\Bigg(\text{respectively, } \mathcal{E}_{(q,t)} = \left\langle f, \frac{\partial f}{\partial q_1}, \dots, \frac{\partial g}{\partial q_k} \right\rangle_{\mathcal{E}_{(q,t)}} + \left\langle \frac{\partial F}{\partial u_1}|_{\mathbb{R}^k \times 0 \times \mathbb{R}}, \dots, \frac{\partial F}{\partial u_r}|_{\mathbb{R}^k \times 0 \times \mathbb{R}} \right\rangle_{\mathbb{R}}\Bigg).$$

We say that f and f' are S-$\mathcal{K}$-*equivalent* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}^k \times \mathbb{R}, 0)$$

of the form $\Phi(q,t) = (\phi(q,t), t)$ such that $\langle f \circ \Phi \rangle_{\mathcal{E}_{(q,t)}} = \langle f' \rangle_{\mathcal{E}_{(q,t)}}$.

For each germ of a function $f : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$, we set

$$S.P\text{-}\mathcal{K}\text{-cod}(f) = \dim_{\mathbb{R}} \mathcal{E}_{(q,t)} \Big/ \left\langle f, \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k} \right\rangle_{\mathcal{E}_{(q,t)}},$$

$$S.P^+\text{-}\mathcal{K}\text{-cod}(f) = \dim_{\mathbb{R}} \mathcal{E}_{(q,t)} \Big/ \left(\left\langle f, \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k} \right\rangle_{\mathcal{E}_{(q,t)}} + \left\langle \frac{\partial f}{\partial t} \right\rangle_{\mathbb{R}} \right).$$

Then we have the following classifications:

Theorem 6.2 ([10, Theorem 4.2]). *Let $f : (\mathbb{R}^k \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a function germ with $S.P$-$\mathcal{K}$-cod $(f) \leq 5$. Then f is stably S-$\mathcal{K}$-equivalent to one of the germs in the following list:*

(1)	q_1,	$S.P$-$\mathcal{K}$-cod$(f) = 0$;	A_0,
(2)	$\pm t \pm q_1^2$,	$S.P$-$\mathcal{K}$-cod$(f) = 1$;	A_1,
(3)	$\pm t \pm q_1^3$,	$S.P$-$\mathcal{K}$-cod$(f) = 2$;	A_2,
(4)	$\pm t^2 \pm q_1^2$,	$S.P$-$\mathcal{K}$-cod$(f) = 2$;	B_2,
(5)	$\pm t \pm q_1^4$,	$S.P$-$\mathcal{K}$-cod$(f) = 3$;	A_3,
(6)	$\pm t^3 \pm q_1^2$,	$S.P$-$\mathcal{K}$-cod$(f) = 3$;	B_3,
(7)	$q_1^3 \pm tq_1$,	$S.P$-$\mathcal{K}$-cod$(f) = 3$;	C_3,
(8)	$\pm t + q_1^5$,	$S.P$-$\mathcal{K}$-cod$(f) = 4$;	A_4,
(9)	$\pm t + (q_1^3 \pm q_1 q_2^2)$,	$S.P$-$\mathcal{K}$-cod$(f) = 4$;	D_4,
(10)	$\pm t^2 + q_1^3$,	$S.P$-$\mathcal{K}$-cod$(f) = 4$;	F_4,
(11)	$\pm t^4 \pm q_1^2$,	$S.P$-$\mathcal{K}$-cod$(f) = 4$;	B_4,
(12)	$q_1^4 \pm tq_1$,	$S.P$-$\mathcal{K}$-cod$(f) = 4$;	C_4,
(13)	$\pm t + q_1^6$,	$S.P$-$\mathcal{K}$-cod$(f) = 5$;	A_5,
(14)	$\pm t \pm (q_1^4 + q_1 q_2^2)$,	$S.P$-$\mathcal{K}$-cod$(f) = 5$;	D_5,
(15)	$\pm t^5 \pm q_1^2$,	$S.P$-$\mathcal{K}$-cod$(f) = 5$;	B_5,
(16)	$q_1^5 \pm tq_1$,	$S.P$-$\mathcal{K}$-cod$(f) = 5$;	C_5.

We can construct an $S.P$-$\mathcal{K}$ (respectively, $S.P^+$-$\mathcal{K}$)-versal deformation for each normal form by the usual method (cf. [3]). Then the corresponding list is as follows:

$S.P$-$\mathcal{K}$-versal deformations:
(1) q_1,
(2) $\pm t \pm q_1^2 + u_1$,
(3) $\pm t \pm q_1^3 + u_1 q_1 + u_2$,
(4) $\pm t^2 \pm q_1^2 + u_1 t + u_2$,
(5) $\pm t \pm q_1^4 + u_1 q_1^2 + u_2 q_1 + u_3$,
(6) $\pm t^3 \pm q_1^2 + u_1 t^2 + u_2 t + u_3$,
(7) $q_1^3 \pm t q_1 + u_1 q_1^2 + u_2 q_1 + u_3$,
(8) $\pm t + q_1^5 + u_1 q_1^3 + u_2 q_1^2 + u_3 q_1^3 + u_4$,
(9) $\pm t + (q_1^3 \pm q_1 q_2^2) + u_1 q_1^2 + u_2 q_2 + u_3 q_1 + u_4$,
(10) $\pm t^2 + q_1^3 + u_1 t q_1 + u_2 q_1 + u_3 s + u_4$,
(11) $\pm t^4 \pm q_1^2 + u_1 t^3 + u_2 t^2 + u_3 t + u_4$,
(12) $q_1^4 \pm t q_1 + u_1 q_1^3 + u_2 q_1^2 + u_3 q_1 + u_4$,
(13) $\pm t \pm q_1^6 + u_1 q_1^4 + u_2 q_1^3 + u_3 q_1^2 + u_4 q_1 + u_5$,
(14) $\pm t \pm (q_1^4 + q_1 q_2^2) + u_1 q_1^2 + u_2 q_2^2 + u_3 q_1 + u_4 q_2 + u_5$,
(15) $\pm t^5 \pm q_1^2 + u_1 t^4 + u_2 t^3 + u_3 t^2 + u_4 t + u_5$,
(16) $q_1^5 \pm t q_1 + u_1 q_1^4 + u_2 q_1^3 + u_3 q_1^2 + u_4 q_1 + u_5$.
$S.P^+$-$\mathcal{K}$-versal deformations:

(1) q_1,
(2) $\pm t \pm q_1^2$,
(3) $\pm t \pm q_1^3 + v_1 q_1$,
(4) $\pm t^2 \pm q_1^2 + v_1$,
(5) $\pm t \pm q_1^4 + v_1 q_1^2 + v_2 q_1$,
(6) $\pm t^3 \pm q_1^2 + v_1 t + v_2$,
(7) $q_1^3 \pm t q_1 + v_1 q_1^2 + v_2$,
(8) $\pm t \pm q_1^5 + v_1 q_1^3 + v_2 q_1^2 + v_3 q_1^3$,
(9) $\pm t + (q_1^3 \pm q_1 q_2^2) + v_1 q_1^2 + v_2 q_2 + v_3 q_1$,
(10) $\pm t^2 + q_1^3 + v_1 t q_1 + v_2 q_1 + v_3$,
(11) $\pm t^4 \pm q_1^2 + v_1 t^2 + v_2 t + v_3$,
(12) $q_1^4 \pm t q_1 + v_1 q_1^3 + v_2 q_1^2 + v_3$,
(13) $\pm t \pm q_1^6 + v_1 q_1^4 + v_2 q_1^3 + v_3 q_1^2 + v_4 q_1$,
(14) $\pm t \pm (q_1^4 + q_1 q_2^2) + v_1 q_1^2 + v_2 q_2^2 + v_3 q_1 + v_4 q_2$,
(15) $\pm t^5 \pm q_1^2 + v_1 t^4 + v_2 t^3 + v_3 t^2 + v_4 t$,
(16) $q_1^5 \pm t q_1 + v_1 q_1^4 + v_2 q_1^3 + v_3 q_1^2 + v_4 q_1$.

We remark that the relation between $S.P^+$-$\mathcal{K}$-cod and $S.P$-$\mathcal{K}$-cod is given by

$$S.P^+\text{-}\mathcal{K} - \mathrm{cod}(f) \;=\; S.P\text{-}\mathcal{K} - \mathrm{cod}(f) + 1$$

by [10, Proposition 3.5].

The following theorem is useful and important for our purpose (cf. [3]).

Theorem 6.3. *Let F and $F' : (\mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be germs of functions which are $S.P^+$-$\mathcal{K}$ (respectively, $S.P$-$\mathcal{K}$)-versal deformations of $f = F|_{\mathbb{R}^k \times 0 \times \mathbb{R}}$ and $f' = F'|_{\mathbb{R}^k \times 0 \times \mathbb{R}}$ respectively. Then F and F' are $S.P^+$-$\mathcal{K}$ (respectively, $S.P$-$\mathcal{K}$)-equivalent if and only if f and f' are S-$\mathcal{K}$-equivalent.*

Proof of Theorem 6.1. Let $1 \leq n \leq 3$. We denote the set of one-parameter families of Lagrangian submanifolds by $L(U \times V, T^*(\mathbb{R}^n \times \mathbb{R}))$, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}$ are open domains around the origin. The set of Lagrangian stable one-parameter families of Lagrangian submanifolds is an open and dense subset in $L(U \times V, T^*(\mathbb{R}^n \times \mathbb{R}))$ (cf. [1, 2, 30]).

Therefore, by Corollary 2.10 and Theorem 5.1, we can give a classification of an $S.P^+$-$\mathcal{K}$-versal deformation of one-parameter graph-like Legendrian unfoldings under the one-parameter s-$S.P^+$-$\mathcal{K}$ equivalence.

We consider the case of $n = 3$. Since the classifications in the cases $n = 1$ and $n = 2$ are given by the similar method, we omit it. By Theorems 6.2, 6.3 and the form of

$$\overline{F}(q, x, s, t) = F(q, x, s) - t,$$

$\overline{F}$ is stably $S.P^+$-$\mathcal{K}$-equivalent to one of the germs in the following list:

$$\begin{array}{ll}
(1) & -t + q_1 + v_1 + v_2 + v_3 + v_4, \\
(2) & -t \pm q_1^2 + v_1 + v_2 + v_3 + v_4, \\
(3) & -t + q_1^3 + v_1 q_1 + v_2 + v_3 + v_4, \\
(4) & -t \pm q_1^4 + v_1 q_1^2 + v_2 q_1 + v_3 + v_4, \\
(5) & -t + q_1^5 + v_1 q_1^3 + v_2 q_1^2 + v_3 q_1 + v_4, \\
(6) & -t + (q_1^3 \pm q_1 q_2^2) + v_1 q_1^2 + v_2 q_2 + v_3 q_1 + v_4, \\
(7) & -t \pm q_1^6 + v_1 q_1^4 + v_2 q_1^3 + v_3 q_1^2 + v_4 q_1, \\
(8) & -t \pm (q_1^4 \pm q_1 q_2^2) + v_1 q_1^2 + v_2 q_2^2 + v_3 q_2 + v_4 q_1,
\end{array}$$

where $(v_1, v_2, v_3, v_4) \in (\mathbb{R}^4, 0)$. We would like to classify these germs by the one-parameter s-$S.P^+$-$\mathcal{K}$-equivalence. By the above normal forms, there exists a germ of a diffeomorphism $\phi : (\mathbb{R}^3 \times \mathbb{R}, 0) \to (\mathbb{R}^4, 0)$ such that $\overline{F}$ is stably one-parameter s-$S.P^+$-$\mathcal{K}$-equivalent to one of the germs in the following list:

$$\begin{array}{ll}
(1) & -t + q_1 + v_1(x,s) + v_2(x,s) + v_3(x,s) + v_4(x,s), \\
(2) & -t \pm q_1^2 + v_1(x,s) + v_2(x,s) + v_3(x,s) + v_4(x,s), \\
(3) & -t + q_1^3 + v_1(x,s) q_1 + v_2(x,s) + v_3(x,s) + v_4(x,s), \\
(4) & -t \pm q_1^4 + v_1(x,s) q_1^2 + v_2(x,s) q_1 + v_3(x,s) + v_4(x,s), \\
(5) & -t + q_1^5 + v_1(x,s) q_1^3 + v_2(x,s) q_1^2 + v_3(x,s) q_1 + v_4(x,s), \\
(6) & -t + (q_1^3 \pm q_1 q_2^2) + v_1(x,s) q_1^2 + v_2(x,s) q_2 + v_3(x,s) q_1 + v_4(x,s), \\
(7) & -t + \pm q_1^6 + v_1(x,s) q_1^4 + v_2(x,s) q_1^3 + v_3(x,s) q_1^2 + v_4(x,s) q_1, \\
(8) & -t \pm (q_1^4 \pm q_1 q_2^2) + v_1(x,s) q_1^2 + v_2(x,s) q_2^2 + v_3(x,s) q_2 + v_4(x,s) q_1,
\end{array}$$

where $x = (x_1, x_2, x_3) \in (\mathbb{R}^3, 0)$. Since $\overline{F}$ is a one-parameter family of graph-like Morse families of hypersurfaces, $\partial F / \partial q : (\mathbb{R}^k \times \mathbb{R}^3 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ is non-singular for each fixed $s \in (\mathbb{R}, 0)$, that is, we have a rank condition

$$\operatorname{rank}\left(\frac{\partial^2 F}{\partial q^2}, \frac{\partial^2 F}{\partial q \partial x}\right)(0) = k.$$

By the rank condition, (1), (2) and (3) are one-parameter s-$S.P$-$\mathcal{K}$-equivalent to

$$(1)\ -t+q_1,\quad (2)\ -t\pm q_1^2+x_1q_1,\quad (3)\ -t+q_1^3+x_1q_1+x_2,$$

respectively. In the case (4), we divide it into four cases: $(\partial v_1/\partial x_1)(0)\neq 0$, $(\partial v_1/\partial x_2)(0)\neq 0$, $(\partial v_1/\partial x_3)(0)\neq 0$ or $(\partial v_1/\partial s)(0)\neq 0$. In the first, second and third cases, $\overline{F}$ is one-parameter s-$S.P$-$\mathcal{K}$-equivalent to

$$(4)_1\ -t\pm q_1^4+x_1q_1^2+x_2q_1+x_3.$$

In the fourth case, $\overline{F}$ is one-parameter s-$S.P$-$\mathcal{K}$-equivalent to

$$(4)_2\ -t\pm q_1^4+\alpha(x,s)q_1^2+x_2q_1+x_3,$$

where $\alpha:(\mathbb{R}^3\times\mathbb{R},0)\to(\mathbb{R},0)$ is a smooth function with the conditions

$$(\partial\alpha/\partial s)(0)\neq 0, (\partial\alpha/\partial x_i)(0)=0, i=1,2,3.$$

In the case (5), $\overline{F}$ is one-parameter s-$S.P^+$-$\mathcal{K}$-equivalent to

$$\begin{aligned}&(5)_1\quad -t+q_1^5+x_1q_1^3+x_2q_1^2+x_3q_1,\\&(5)_2\quad -t+q_1^5+\alpha(x,s)q_1^3+x_1q_1^2+x_2q_1+x_3,\\&(5)_3\quad -t+q_1^5+x_1q_1^3+\alpha(x,s)q_1^2+x_2q_1+x_3,\end{aligned}$$

where $\alpha:(\mathbb{R}^3\times\mathbb{R},0)\to(\mathbb{R},0)$ is a smooth function with the conditions

$$(\partial\alpha/\partial s)(0)\neq 0, (\partial\alpha/\partial x_i)(0)=0, i=1,2,3.$$

In the cases (6) and (8), we can give the normal forms by the similar methods to those of the case (4). Moreover, in the case (7), we can also give the normal forms by the similar methods to those of the case (5). This completes the proof. □

Remark 6.4. In the generic classifications under one-parameter caustic equivalence in [1, 2, 30], the functional moduli have a special form. For instance, the functional moduli of the type $(7)_1$ in Theorem 6.1 are equivalent to the form $\alpha(x,s)=s$. Moreover, types $(7)_2$ and $(7)_3$ in Theorem 6.1 do not appear in the generic classifications under one-parameter caustic equivalence.

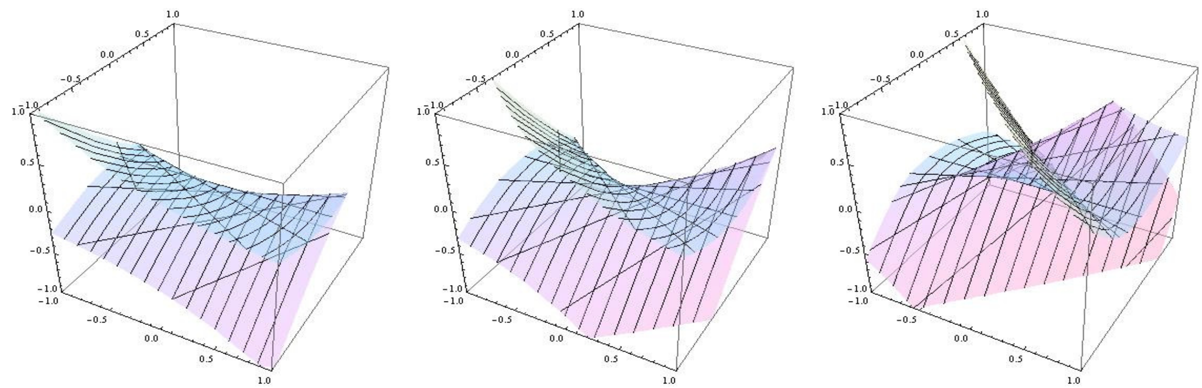
We give concrete examples of bifurcations of caustics for the types $(7)_1$ and $(7)_2$.

Example 6.5. Let $\overline{F}:(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R},0)\to(\mathbb{R},0)$ be given by

$$\overline{F}(q,x,s)=-t+q^6+\alpha(x_1,x_2,x_3,s)q^4+x_1q^3+x_2q^2+x_3q,$$

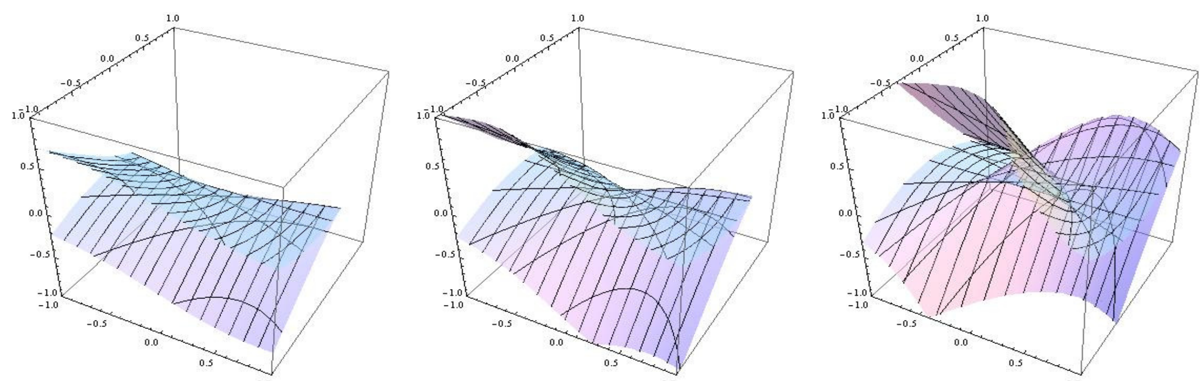
where $\partial\alpha/\partial s(0)\neq 0, \partial\alpha/\partial x_i(0)=0, i=1,2,3$. The one-parameter family of Lagrangian submanifold germs $L(F):(C(F),0)\to T^*\mathbb{R}^3$ is given by $L(F)(q,x,s)=(x,\partial F/\partial x(q,x,s))$.

If we take $\alpha(x,s) = s$, then the one-parameter family of caustics is given by the image of $(u,v,s) \mapsto (v, -15u^4 - 6su^2 - 3uv, 24u^5 + 8su^3 + 3vu^2)$; see Figure 1 (cf. [1, 2, 30]). If we take $\alpha(x,s) = s + x_1^2$, then the the one-parameter family of caustics is given by the image of $(u,v,s) \mapsto (v, -15u^4 - 6(s+v^2)u^2 - 3uv, 24u^5 + 8(s+v^2)u^3 + 3vu^2)$; see Figure 2.



$s > 0$ $s = 0$ $s < 0$

Figure 1. Type $(7)_1$ with $\alpha(x,s) = s$.



$s > 0$ $s = 0$ $s < 0$

Figure 2. Type $(7)_1$ with $\alpha(x,s) = s + x_1^2$.

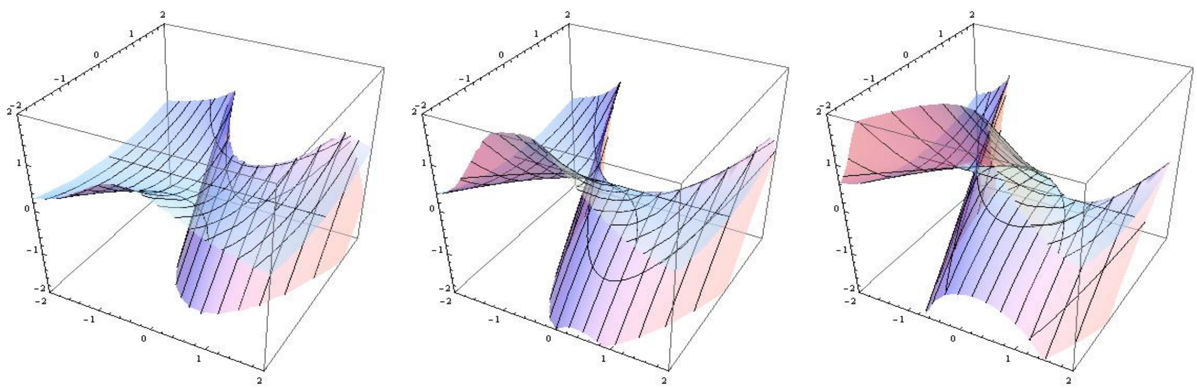
Example 6.6. Let $\overline{F} : (\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be given by

$$\overline{F}(q,x,s) = -t + q^6 + x_1 q^4 + \alpha(x_1,x_2,x_3,s)q^3 + x_2 q^2 + x_3 q,$$

where $\partial\alpha/\partial s(0) \neq 0, \partial\alpha/\partial x_i(0) = 0, i = 1,2,3$. If we take $\alpha(x,s) = s + x_1^2$, then the one-parameter family of caustics is given by the image of

$$(u,v,s) \mapsto (v, -15u^4 - 6vu^2 - 3(s+v^2)u, 24u^5 + 8vu^3 + 3(s+v^2)u^2);$$

see Figure 3.



$s > 0$ $s = 0$ $s < 0$

Figure 3. Type $(7)_2$ with $\alpha(x, s) = s + x_1^2$.

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