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A CLOSEDNESS THEOREM AND APPLICATIONS IN GEOMETRY OF RATIONAL POINTS OVER HENSELIAN VALUED FIELDS

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Dedicated to Goo Ishikawa on the occasion of his 60th birthday

Abstract. We develop geometry of algebraic subvarieties of K^n over arbitrary Henselian valued fields K of equicharacteristic zero. This is a continuation of our previous article concerned with algebraic geometry over rank one valued fields. At the center of our approach is again the closedness theorem to the effect that the projections $K^n \times \mathbb{P}^m(K) \to K^n$ are definably closed maps. It enables, in particular, application of resolution of singularities in much the same way as over locally compact ground fields. As before, the proof of that theorem uses, among others, the local behavior of definable functions of one variable and fiber shrinking, being a relaxed version of curve selection. But now, to achieve the former result, we first examine functions given by algebraic power series. All our previous results will be established here in the general settings: several versions of curve selection (via resolution of singularities) and of the Łojasiewicz inequality (via two instances of quantifier elimination indicated below), extending continuous hereditarily rational functions as well as the theory of regulous functions, sets and sheaves, including Nullstellensatz and Cartan's theorems A and B. Two basic tools are quantifier elimination for Henselian valued fields due to Pas and relative quantifier elimination for ordered abelian groups (in a many-sorted language with imaginary auxiliary sorts) due to Cluckers–Halupczok. Other, new applications of the closedness theorem are piecewise continuity of definable functions, Hölder continuity of functions definable on closed bounded subsets of K^n, the existence of definable retractions onto closed definable subsets of K^n and a definable, non-Archimedean version of the Tietze–Urysohn extension theorem. In a recent paper, we established a version of the closedness theorem over Henselian valued fields with analytic structure along with several applications.

1. Introduction

Throughout the paper, K will be an arbitrary Henselian valued field of equicharacteristic zero with valuation v, value group Γ, valuation ring R and residue field $\Bbbk$. Examples of such fields are the quotient fields of the rings of formal power series and of Puiseux series with coefficients from a field $\Bbbk$ of characteristic zero as well as the fields of Hahn series (maximally complete valued fields also called Malcev–Neumann fields; cf. [27]):

$$\Bbbk((t^\Gamma)) := \left\{ f(t) = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma : \ a_\gamma \in \Bbbk, \ \operatorname{supp} f(t) \text{ is well ordered} \right\}.$$

We consider the ground field K along with the three-sorted language $\mathcal{L}$ of Denef–Pas (cf. [53, 44]). The three sorts of $\mathcal{L}$ are: the valued field K-sort, the value group Γ-sort and the residue field $\Bbbk$-sort. The language of the K-sort is the language of rings; that of the Γ-sort is any augmentation

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of the language of ordered abelian groups (and ∞); finally, that of the $\Bbbk$-sort is any augmentation of the language of rings. The only symbols of $\mathcal{L}$ connecting the sorts are two functions from the main K-sort to the auxiliary Γ-sort and $\Bbbk$-sort: the valuation map and an angular component map.

Every valued field K has a topology induced by its valuation v. Cartesian products K^n are equipped with the product topology, and their subsets inherit a topology, called the K-topology. This paper is a continuation of our paper [44] devoted to geometry over Henselian rank one valued fields, and includes our recent preprints [45, 46, 47]. The main aim is to prove (in Section 8) the closedness theorem stated below, and next to derive several results in the following Sections 9–14.

Theorem 1.1. *Let D be an $\mathcal{L}$-definable subset of K^n. Then the canonical projection*

$$\pi : D \times R^m \longrightarrow D$$

is definably closed in the K-topology, i.e. if $B \subset D \times R^m$ is an $\mathcal{L}$-definable closed subset, so is its image $\pi(B) \subset D$.

Remark 1.2. Not all valued fields K have an angular component map, but it exists if K has a cross section, which happens whenever K is $\aleph_1$-saturated (cf. [7, Chap. II]). Moreover, a valued field K has an angular component map whenever its residue field $\Bbbk$ is $\aleph_1$-saturated (cf. [54, Corollary 1.6]). In general, unlike for p-adic fields and their finite extensions, adding an angular component map does strengthen the family of definable sets. Since the K-topology is definable in the language of valued fields, the closedness theorem is a first order property. Therefore it is valid over arbitrary Henselian valued fields of equicharacteristic zero, because it can be proven using saturated elementary extensions, thus assuming that an angular component map exists.

Two basic tools applied in this paper are quantifier elimination for Henselian valued fields (along with preparation cell decomposition) due to Pas [53] and relative quantifier elimination for ordered abelian groups (in a many-sorted language with imaginary auxiliary sorts) due to Cluckers–Halupczok [8]. In the case where the ground field K is of rank one, Theorem 1.1 was established in our paper [44, Section 7], where instead we applied simply quantifier elimination for ordered abelian groups in the Presburger language. Of course, when K is a locally compact field, it holds by a routine topological argument.

As before, our approach relies on the local behavior of definable functions of one variable and the so-called fiber shrinking, being a relaxed version of curve selection. Over arbitrary Henselian valued fields, the former result will be established in Section 5, and the latter in Section 6. Now, however, in the proofs of fiber shrinking (Proposition 6.1) and the closedness theorem (Theorem 1.1), we also apply relative quantifier elimination for ordered abelian groups, due to Cluckers–Halupczok [8]. It will be recalled in Section 7.

Section 2 contains a version of the implicit function theorem (Proposition 2.5). In the next section, we provide a version of the Artin–Mazur theorem on algebraic power series (Proposition 3.3). Consequently, every algebraic power series over K determines a unique continuous function which is definable in the language of valued fields. Section 4 presents certain versions of the theorems of Abhyankar–Jung (Proposition 4.1) and Newton-Puiseux (Proposition 4.2) for Henselian subalgebras of formal power series which are closed under power substitution and division by a coordinate, given in our paper [43] (see also [52]). In Section 5, we use the foregoing results in analysis of functions of one variable, definable in the language of Denef–Pas, to establish a theorem on existence of the limit (Theorem 5.1).

The closedness theorem will allow us to establish several results as for instance: piecewise continuity of definable functions (Section 9), certain non-archimedean versions of curve selection (Section 10) and of the Łojasiewicz inequality with a direct consequence, Hölder continuity of definable functions on closed bounded subsets of K^n (Section 11) as well as extending hereditarily rational functions (Section 12) and the theory of regulous functions, sets and sheaves, including Nullstellensatz and Cartan's theorems A and B (Section 12). Over rank one valued fields, these results (except piecewise and Hölder continuity) were established in our paper [44]. The theory of hereditarily rational functions on the real and p-adic varieties was developed in the joint paper [30]. Yet another application of the closedness theorem is the existence of definable retractions onto closed definable subsets of K^n and a definable, non-Archimedean version of the Tietze–Urysohn extension theorem. These results are established for the algebraic case and for Henselian fields with analytic structure in our recent papers [49, 50, 51]. It is very plausible that they will also hold in the more general case of axiomatically based structures on Henselian valued fields.

The closedness theorem immediately yields five corollaries stated below. Corollaries 1.6 and 1.7, enable application of resolution of singularities and of transformation to a simple normal crossing by blowing up (cf. [28, Chap. III] for references and relatively short proofs) in much the same way as over locally compact ground fields.

Corollary 1.3. *Let D be an $\mathcal{L}$-definable subset of K^n and $\mathbb{P}^m(K)$ stand for the projective space of dimension m over K. Then the canonical projection $\pi : D \times \mathbb{P}^m(K) \longrightarrow D$ is definably closed.*

Corollary 1.4. *Let A be a closed $\mathcal{L}$-definable subset of $\mathbb{P}^m(K)$ or R^m. Then every continuous $\mathcal{L}$-definable map $f : A \to K^n$ is definably closed in the K-topology.*

Corollary 1.5. *Let ϕ_i, $i = 0, \dots, m$, be regular functions on K^n, D be an $\mathcal{L}$-definable subset of K^n and $\sigma : Y \longrightarrow K\mathbb{A}^n$ the blow-up of the affine space $K\mathbb{A}^n$ with respect to the ideal $(\phi_0, \dots, \phi_m)$. Then the restriction $\sigma : Y(K) \cap \sigma^{-1}(D) \longrightarrow D$ is a definably closed quotient map.*

Proof. Indeed, $Y(K)$ can be regarded as a closed algebraic subvariety of $K^n \times \mathbb{P}^m(K)$ and σ as the canonical projection. □

Corollary 1.6. *Let X be a smooth K-variety, D be an $\mathcal{L}$-definable subset of $X(K)$ and $\sigma : Y \longrightarrow X$ the blow-up along a smooth center. Then the restriction $\sigma : Y(K) \cap \sigma^{-1}(D) \longrightarrow D$ is a definably closed quotient map.*

Corollary 1.7. *(Descent property) Under the assumptions of the above corollary, every continuous $\mathcal{L}$-definable function $g : Y(K) \cap \sigma^{-1}(D) \longrightarrow K$ that is constant on the fibers of the blow-up σ descends to a (unique) continuous $\mathcal{L}$-definable function $f : D \longrightarrow K$.*

2. Some versions of the implicit function theorem

In this section, we give elementary proofs of some versions of the inverse mapping and implicit function theorems; cf. the versions established in the papers [55, Theorem 7.4], [22, Section 9], [36, Section 4] and [21, Proposition 3.1.4]. We begin with a simplest version (H) of Hensel's lemma in several variables, studied by Fisher [20]. Given an ideal $\mathfrak{m}$ of a ring R, let $\mathfrak{m}^{\times n}$ stand for the n-fold Cartesian product of $\mathfrak{m}$ and $R^\times$ for the set of units of R. The origin $(0, \dots, 0) \in R^n$ is denoted by $\mathbf{0}$.

(H) *Assume that a ring R satisfies Hensel's conditions (i.e. it is linearly topologized, Hausdorff and complete) and that an ideal $\mathfrak{m}$ of R is closed. Let $f = (f_1, \ldots, f_n)$ be an n-tuple of restricted power series $f_1, \ldots, f_n \in R\{X\}$, $X = (X_1, \ldots, X_n)$, J be its Jacobian determinant and $a \in R^n$. If $f(\mathbf{0}) \in \mathfrak{m}^{\times n}$ and $J(\mathbf{0}) \in R^{\times}$, then there is a unique $a \in \mathfrak{m}^{\times n}$ such that $f(a) = \mathbf{0}$.*

Proposition 2.1. *Under the above assumptions, f induces a bijection*

$$\mathfrak{m}^{\times n} \ni x \longrightarrow f(x) \in \mathfrak{m}^{\times n}$$

of $\mathfrak{m}^{\times n}$ onto itself.

Proof. For any $y \in \mathfrak{m}^{\times n}$, apply condition (H) to the restricted power series $f(X) - y$. □

If, moreover, the pair $(R, \mathfrak{m})$ satisfies Hensel's conditions (i.e. every element of $\mathfrak{m}$ is topologically nilpotent), then condition (H) holds by [5, Chap. III, §4.5].

Remark 2.2. Henselian local rings can be characterized both by the classical Hensel lemma and by condition (H): a local ring $(R, \mathfrak{m})$ is Henselian iff $(R, \mathfrak{m})$ with the discrete topology satisfies condition (H) (cf. [20, Proposition 2]).

Now consider a Henselian local ring $(R, \mathfrak{m})$. Let $f = (f_1, \ldots, f_n)$ be an n-tuple of polynomials $f_1, \ldots, f_n \in R[X]$, $X = (X_1, \ldots, X_n)$ and J be its Jacobian determinant.

Corollary 2.3. *Suppose that $f(\mathbf{0}) \in \mathfrak{m}^{\times n}$ and $J(\mathbf{0}) \in R^{\times}$. Then f is a homeomorphism of $\mathfrak{m}^{\times n}$ onto itself in the $\mathfrak{m}$-adic topology. If, in addition, R is a Henselian valued ring with maximal ideal $\mathfrak{m}$, then f is a homeomorphism of $\mathfrak{m}^{\times n}$ onto itself in the valuation topology.*

Proof. Obviously, $J(a) \in R^{\times}$ for every $a \in \mathfrak{m}^{\times n}$. Let $\mathcal{M}$ be the jacobian matrix of f. Then

$$f(a + x) - f(a) = \mathcal{M}(a) \cdot x + g(x) = \mathcal{M}(a) \cdot (x + \mathcal{M}(a)^{-1} \cdot g(x))$$

for an n-tuple $g = (g_1, \ldots, g_n)$ of polynomials $g_1, \ldots, g_n \in (X)^2 R[X]$. Hence the assertion follows easily. □

The proposition below is a version of the inverse mapping theorem.

Proposition 2.4. *If $f(\mathbf{0}) = \mathbf{0}$ and $e := J(\mathbf{0}) \neq 0$, then f is an open embedding of $e \cdot \mathfrak{m}^{\times n}$ onto $e^2 \cdot \mathfrak{m}^{\times n}$.*

Proof. Let $\mathcal{N}$ be the adjugate of the matrix $\mathcal{M}(\mathbf{0})$ and $y = e^2 b$ with $b \in \mathfrak{m}^{\times n}$. Since

$$f(eX) = e \cdot \mathcal{M}(\mathbf{0}) \cdot X + e^2 g(X)$$

for an n-tuple $g = (g_1, \ldots, g_n)$ of polynomials $g_1, \ldots, g_n \in (X)^2 R[X]$, we get the equivalences

$$f(eX) = y \;\Leftrightarrow\; f(eX) - y = \mathbf{0} \;\Leftrightarrow\; e \cdot \mathcal{M}(\mathbf{0}) \cdot (X + \mathcal{N} g(X) - \mathcal{N} b) = \mathbf{0}.$$

Applying Corollary 2.3 to the map $h(X) := X + \mathcal{N} g(X)$, we get

$$f^{-1}(y) = ex \;\Leftrightarrow\; x = h^{-1}(\mathcal{N} b) \;\text{ and }\; f^{-1}(y) = e h^{-1}(\mathcal{N} \cdot y/e^2).$$

This finishes the proof. □

Further, let $0 \leq r < n$, $p = (p_{r+1}, \ldots, p_n)$ be an $(n-r)$-tuple of polynomials

$$p_{r+1}, \ldots, p_n \in R[X], \quad X = (X_1, \ldots, X_n),$$

and

$$J := \frac{\partial(p_{r+1}, \ldots, p_n)}{\partial(X_{r+1}, \ldots, X_n)}, \quad e := J(\mathbf{0}).$$

Suppose that

$$\mathbf{0} \in V := \{x \in R^n : p_{r+1}(x) = \ldots = p_n(x) = 0\}.$$

In a similar fashion as above, we can establish the following version of the implicit function theorem.

Proposition 2.5. *If $e \neq 0$, then there exists a unique continuous map*

$$\phi : (e^2 \cdot \mathfrak{m})^{\times r} \longrightarrow (e \cdot \mathfrak{m})^{\times (n-r)}$$

which is definable in the language of valued fields and such that $\phi(0) = 0$ and the graph map

$$(e^2 \cdot \mathfrak{m})^{\times r} \ni u \longrightarrow (u, \phi(u)) \in (e^2 \cdot \mathfrak{m})^{\times r} \times (e \cdot \mathfrak{m})^{\times (n-r)}$$

is an open embedding into the zero locus V of the polynomials p and, more precisely, onto

$$V \cap \left[(e^2 \cdot \mathfrak{m})^{\times r} \times (e \cdot \mathfrak{m})^{\times (n-r)} \right].$$

Proof. Put $f(X) := (X_1, \ldots, X_r, p(X))$; of course, the jacobian determinant of f at $\mathbf{0} \in R^n$ is equal to e. Keep the notation from the proof of Proposition 2.4, take any $b \in e^2 \cdot \mathfrak{m}^{\times r}$ and put $y := (e^2 b, 0) \in R^n$. Then we have the equivalences

$$f(eX) = y \;\Leftrightarrow\; f(eX) - y = \mathbf{0} \;\Leftrightarrow\; e\mathcal{M}(\mathbf{0}) \cdot (X + \mathcal{N} g(X) - \mathcal{N} \cdot (b, 0)) = \mathbf{0}.$$

Applying Corollary 2.3 to the map $h(X) := X + \mathcal{N} g(X)$, we get

$$f^{-1}(y) = ex \;\Leftrightarrow\; x = h^{-1}(\mathcal{N} \cdot (b, 0)) \;\text{ and }\; f^{-1}(y) = eh^{-1}(\mathcal{N} \cdot y/e^2).$$

Therefore the function

$$\phi(u) := eh^{-1}(\mathcal{N} \cdot (u, 0)/e^2)$$

is the one we are looking for. $\square$

3. Density property and a version of the Artin–Mazur theorem over Henselian valued fields

We say that a topological field K satisfies the *density property* (cf. [30, 44]) if the following equivalent conditions hold.

(1) If X is a smooth, irreducible K-variety and $\emptyset \neq U \subset X$ is a Zariski open subset, then $U(K)$ is dense in $X(K)$ in the K-topology.
(2) If C is a smooth, irreducible K-curve and $\emptyset \neq U$ is a Zariski open subset, then $U(K)$ is dense in $C(K)$ in the K-topology.
(3) If C is a smooth, irreducible K-curve, then $C(K)$ has no isolated points.

(This property is indispensable for ensuring reasonable topological and geometric properties of algebraic subsets of K^n; see [44] for the case where the ground field K is a Henselian rank one valued field.) The density property of Henselian non-trivially valued fields follows immediately from Proposition 2.5 and the Jacobian criterion for smoothness (see e.g. [17, Theorem 16.19]), recalled below for the reader's convenience.

Theorem 3.1. *Let $I = (p_1, \ldots, p_s) \subset K[X]$, $X = (X_1, \ldots, X_n)$ be an ideal, $A := K[X]/I$ and $V := \operatorname{Spec}(A)$. Suppose the origin $\mathbf{0} \in K^n$ lies in V (equivalently, $I \subset (X)K[X]$) and V is of dimension r at $\mathbf{0}$. Then the Jacobian matrix*

$$\mathcal{M} := \left[\frac{\partial p_i}{\partial X_j}(\mathbf{0}) : \; i = 1, \ldots, s, \; j = 1, \ldots, n \right]$$

has rank $\leq (n - r)$ and V is smooth at $\mathbf{0}$ iff $\mathcal{M}$ has exactly rank $(n - r)$. Furthermore, if V is smooth at $\mathbf{0}$ and

$$\mathcal{J} := \frac{\partial(p_{r+1}, \ldots, p_n)}{\partial(X_{r+1}, \ldots, X_n)}(\mathbf{0}) = \det \left[\frac{\partial p_i}{\partial X_j}(\mathbf{0}) : \; i, j = r+1, \ldots, n \right] \neq 0,$$

then $p_{r+1}, \dots, p_n$ generate the localization $I \cdot K[X]_{(X_1,\dots,X_n)}$ of the ideal I with respect to the maximal ideal $(X_1, \dots, X_n)$.

Remark 3.2. Under the above assumptions, consider the completion $\widehat{A} = K[[X]]/I \cdot K[[X]]$ of A in the (X)-adic topology. If $\mathcal{J} \neq 0$, it follows from the implicit function theorem for formal power series that there are unique power series

$$\phi_{r+1}, \dots, \phi_n \in (X_1, \dots, X_r) \cdot K[[X_1, \dots, X_r]]$$

such that

$$p_i(X_1, \dots, X_r, \phi_{r+1}(X_1, \dots, X_r), \dots, \phi_n(X_1, \dots, X_r)) = 0$$

for $i = r+1, \dots, n$. Therefore the homomorphism

$$\widehat{\alpha} : \widehat{A} \longrightarrow K[[X_1, \dots, X_r]], \quad X_j \mapsto X_j, \ X_k \mapsto \phi_k(X_1, \dots, X_r),$$

for $j = 1, \dots, r$ and $k = r+1, \dots, n$, is an isomorphism.

Conversely, suppose that $\widehat{\alpha}$ is an isomorphism; this means that the projection from V onto $\operatorname{Spec} K[X_1, \dots, X_r]$ is etale at $\mathbf{0}$. Then the local rings A and $\widehat{A}$ are regular and, moreover, it is easy to check that the determinant $\mathcal{J} \neq 0$ does not vanish after perhaps renumbering the polynomials $p_i(X)$.

We say that a formal power series $\phi \in K[[X]]$, $X = (X_1, \dots, X_n)$, is algebraic if it is algebraic over $K[X]$. The kernel of the homomorphism of K-algebras

$$\sigma : K[X, T] \longrightarrow K[[X]], \quad X_1 \mapsto X_1, \dots, X_n \mapsto X_n, \ T \mapsto \phi(X),$$

is, of course, a principal prime ideal: $\ker \sigma = (p) \subset K[X, T]$, where $p \in K[X, T]$ is a unique (up to a constant factor) irreducible polynomial, called an *irreducible polynomial* of ϕ.

We now state a version of the Artin–Mazur theorem (cf. [3, 4] for the classical versions).

Proposition 3.3. *Let $\phi \in (X)K[[X]]$ be an algebraic formal power series. Then there exist polynomials*

$$p_1, \dots, p_r \in K[X, Y], \quad Y = (Y_1, \dots, Y_r),$$

and formal power series $\phi_2, \dots, \phi_r \in K[[X]]$ such that

$$e := \frac{\partial(p_1, \dots, p_r)}{\partial(Y_1, \dots, Y_r)}(\mathbf{0}) = \det \left[\frac{\partial p_i}{\partial Y_j}(\mathbf{0}) : \ i, j = 1, \dots, r \right] \neq 0,$$

and

$$p_i(X_1, \dots, X_n, \phi_1(X), \dots, \phi_r(X)) = 0, \quad i = 1, \dots, r,$$

where $\phi_1 := \phi$.

Proof. Let $p_1(X, Y_1)$ be an irreducible polynomial of ϕ_1. Then the integral closure B of $A := K[X, Y_1]/(p_1)$ is a finite A-module and thus is of the form

$$B = K[X, Y]/(p_1, \dots, p_s), \quad Y = (Y_1, \dots, Y_r),$$

where $p_1, \dots, p_s \in K[X, Y]$. Obviously, A and B are of dimension n, and the induced embedding $\alpha : A \to K[[X]]$ extends to an embedding $\beta : B \to K[[X]]$. Put

$$\phi_k := \beta(Y_k) \in K[[X]], \quad k = 1, \dots, r.$$

Substituting $Y_k - \phi_k(0)$ for Y_k, we may assume that $\phi_k(0) = 0$ for all $k = 1, \dots, r$. Hence $p_i(\mathbf{0}) = 0$ for all $i = 1, \dots, s$.

The completion $\widehat{B}$ of B in the (X, Y)-adic topology is a local ring of dimension n, and the induced homomorphism

$$\widehat{\beta} : \widehat{B} = K[[X, Y]]/(p_1, \dots, p_s) \longrightarrow K[[X]]$$

is, of course, surjective. But, by the Zariski main theorem (cf. [59, Chap. VIII, § 13, Theorem 32]), $\widehat{B}$ is a normal domain. Comparison of dimensions shows that $\widehat{\beta}$ is an isomorphism. Now, it follows from Remark 3.2 that the determinant $e \neq 0$ does not vanish after perhaps renumbering the polynomials $p_i(X)$. This finishes the proof. □

Propositions 3.3 and 2.5 immediately yield the following

Corollary 3.4. *Let $\phi \in (X)K[[X]]$ be an algebraic power series with irreducible polynomial $p(X,T) \in K[X,T]$. Then there is an $a \in K$, $a \neq 0$, and a unique continuous function*

$$\widetilde{\phi} : a \cdot R^n \longrightarrow K$$

corresponding to ϕ, which is definable in the language of valued fields and such that $\widetilde{\phi}(0) = 0$ and $p(x, \widetilde{\phi}(x)) = 0$ for all $x \in a \cdot R^n$. □

For simplicity, we shall denote the induced continuous function by the same letter ϕ. This abuse of notation will not lead to confusion in general.

Remark 3.5. Clearly, the ring $K[[X]]_{alg}$ of algebraic power series is the henselization of the local ring $K[X]_{(X)}$ of regular functions. Therefore the implicit functions $\phi_{r+1}(u), \dots, \phi_n(u)$ from Proposition 2.5 correspond to unique algebraic power series

$$\phi_{r+1}(X_1, \dots, X_r), \dots, \phi_n(X_1, \dots, X_r)$$

without constant term. In fact, one can deduce by means of the classical version of the implicit function theorem for restricted power series (cf. [5, Chap. III, §4.5] or [20]) that $\phi_{r+1}, \dots, \phi_n$ are of the form

$$\phi_k(X_1, \dots, X_r) = e \cdot \omega_k(X_1/e^2, \dots, X_r/e^2), \quad k = r+1, \dots, n,$$

where $\omega_k(X_1, \dots, X_r) \in R[[X_1, \dots, X_r]]$ and $e \in R$.

4. The Newton–Puiseux and Abhyankar–Jung Theorems

Here we are going to provide a version of the Newton–Puiseux theorem, which will be used in analysis of definable functions of one variable in the next section.

We call a polynomial

$$f(X;T) = T^s + a_{s-1}(X)T^{n-1} + \cdots + a_0(X) \in K[[X]][T],$$

$X = (X_1, \dots, X_s)$, quasiordinary if its discriminant $D(X)$ is a normal crossing:

$$D(X) = X^\alpha \cdot u(X) \quad \text{with} \quad \alpha \in \mathbb{N}^s,\ u(X) \in k[[X]],\ u(0) \neq 0.$$

Let K be an algebraically closed field of characteristic zero. Consider a henselian $K[X]$-subalgebra $K\langle X\rangle$ of the formal power series ring $K[[X]]$ which is closed under reciprocal (whence it is a local ring), power substitution and division by a coordinate. For positive integers $r_1, \dots, r_n$ put

$$K\langle X_1^{1/r_1}, \dots, X_n^{1/r_n}\rangle := \left\{ a(X_1^{1/r_1}, \dots, X_n^{1/r_n}) : a(X) \in K\langle X\rangle \right\};$$

when $r_1 = \ldots = r_m = r$, we denote the above algebra by $K\langle X^{1/r}\rangle$.

In our paper [43] (see also [52]), we established a version of the Abhyankar–Jung theorem recalled below. This axiomatic approach to that theorem was given for the first time in our preprint [42].

Proposition 4.1. *Under the above assumptions, every quasiordinary polynomial*

$$f(X;T) = T^s + a_{s-1}(X)T^{s-1} + \cdots + a_0(X) \in K\langle X\rangle[T]$$

has all its roots in $K\langle X^{1/r}\rangle$ for some $r \in \mathbb{N}$; actually, one can take $r = s!$.

A particular case is the following version of the Newton-Puiseux theorem.

Corollary 4.2. *Let X denote one variable. Every polynomial*

$$f(X;T) = T^s + a_{s-1}(X)T^{s-1} + \cdots + a_0(X) \in K\langle X\rangle[T]$$

has all its roots in $K\langle X^{1/r}\rangle$ for some $r \in \mathbb{N}$; one can take $r = s!$. Equivalently, the polynomial $f(X^r, T)$ splits into T-linear factors. If $f(X,T)$ is irreducible, then $r = s$ will do and

$$f(X^s, T) = \prod_{i=1}^{s} (T - \phi(\epsilon^i X)),$$

where $\phi(X) \in K\langle X\rangle$ and ϵ is a primitive root of unity.

Remark 4.3. Since the proof of these theorems is of finitary character, it is easy to check that if the ground field K of characteristic zero is not algebraically closed, they remain valid for the Henselian subalgebra $\overline{K} \otimes_K K\langle X\rangle$ of $\overline{K}[[X]]$, where $\overline{K}$ denotes the algebraic closure of K.

The ring $K[[X]]_{alg}$ of algebraic power series is a local Henselian ring closed under power substitutions and division by a coordinate. Thus the above results apply to the algebra

$$K\langle X\rangle = K[[X]]_{alg}.$$

5. Definable functions of one variable

At this stage, we can readily to proceed with analysis of definable functions of one variable over arbitrary Henselian valued fields of equicharacteristic zero. We wish to establish a general version of the theorem on existence of the limit stated below. It was proven in [44, Proposition 5.2] over rank one valued fields. Now the language $\mathcal{L}$ under consideration is the three-sorted language of Denef–Pas.

Theorem 5.1. *(Existence of the limit) Let $f : A \to K$ be an $\mathcal{L}$-definable function on a subset A of K and suppose 0 is an accumulation point of A. Then there is a finite partition of A into $\mathcal{L}$-definable sets $A_1, \ldots, A_r$ and points $w_1 \ldots, w_r \in \mathbb{P}^1(K)$ such that*

$$\lim_{x \to 0} f|A_i\,(x) = w_i \quad \textit{for} \ \ i = 1, \ldots, r.$$

Moreover, there is a neighborhood U of 0 such that each definable set

$$\{(v(x), v(f(x))) : \ x \in (A_i \cap U) \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \ \{\infty\}), \ i = 1, \ldots, r,$$

is contained in an affine line with rational slope $q \cdot l = p_i \cdot k + \beta_i, \quad i = 1, \ldots, r,$ with $p_i, q \in \mathbb{Z}$, $q > 0$, $\beta_i \in \Gamma$, or in $\Gamma \times \{\infty\}$.

Proof. Having the Newton–Puiseux theorem for algebraic power series at hand, we can repeat mutatis mutandis the proof from loc. cit. as briefly outlined below. In that paper, the field L is the completion of the algebraic closure $\overline{K}$ of the ground field K. Here, in view of Corollary 4.3, the K-algebras $L\{X\}$ and $\widehat{K}\{X\}$ should be just replaced with $\overline{K} \otimes_K K[[X]]_{alg}$ and $K[[X]]_{alg}$, respectively. Then the reasonings follow almost verbatim. Note also that Lemma 5.1 (to the effect that K is a closed subspace of $\overline{K}$) holds true for arbitrary Henselian valued fields of equicharacteristic zero. This follows directly from that the field K is algebraically maximal (as it is Henselian and finitely ramified; see e.g. [18, Chap. 4]). □

We conclude with the following comment. The above proposition along with the technique of fiber shrinking from [44, Section 6] were two basic tools in the proof of the closedness theorem [44, Theorem 3.1] over Henselian rank one valued fields, which plays an important role in Henselian geometry.

6. Fiber shrinking

Consider a Henselian valued field K of equicharacteristic zero along with the three-sorted language $\mathcal{L}$ of Denef–Pas. In this section, we remind the reader the concept of fiber shrinking introduced in our paper [44, Section 6].

Let A be an $\mathcal{L}$-definable subset of K^n with accumulation point $a = (a_1, \dots, a_n) \in K^n$ and E an $\mathcal{L}$-definable subset of K with accumulation point a_1. We call an $\mathcal{L}$-definable family of sets $\Phi = \bigcup_{t \in E} \{t\} \times \Phi_t \subset A$ an $\mathcal{L}$-definable x_1-fiber shrinking for the set A at a if

$$\lim_{t \to a_1} \Phi_t = (a_2, \dots, a_n),$$

i.e. for any neighborhood U of $(a_2, \dots, a_n) \in K^{n-1}$, there is a neighborhood V of $a_1 \in K$ such that $\emptyset \neq \Phi_t \subset U$ for every $t \in V \cap E$, $t \neq a_1$. When $n = 1$, A is itself a fiber shrinking for the subset A of K at an accumulation point $a \in K$.

Proposition 6.1. *(Fiber shrinking) Every $\mathcal{L}$-definable subset A of K^n with accumulation point $a \in K^n$ has, after a permutation of the coordinates, an $\mathcal{L}$-definable x_1-fiber shrinking at a.*

In the case where the ground field K is of rank one, the proof of Proposition 6.1 was given in [44, Section 6]. In the general case, it can be repeated verbatim once we demonstrate the following result on definable subsets in the value group sort Γ.

Lemma 6.2. *Let Γ be an ordered abelian group and P be a definable subset of Γ^n. Suppose that $(\infty, \dots, \infty)$ is an accumulation point of P, i.e. for any $\delta \in \Gamma$ the set*

$$\{x \in P : x_1 > \delta, \dots, x_n > \delta\} \neq \emptyset$$

is non-empty. Then there is an affine semi-line

$$L = \{(r_1 k + \gamma_1, \dots, r_n k + \gamma_n) : \, k \in \Gamma, \; k \geq 0\} \quad \textit{with} \;\; r_1, \dots, r_n \in \mathbb{N},$$

passing through a point $\gamma = (\gamma_1, \dots, \gamma_n) \in P$ and such that $(\infty, \dots, \infty)$ is an accumulation point of the intersection $P \cap L$ too.

In [44, Section 6], Lemma 6.2 was established for archimedean groups by means of quantifier elimination in the Presburger language. Now, in the general case, it follows in a similar fashion by means of relative quantifier elimination for ordered abelian groups in the language $\mathcal{L}_{qe}$ due to Cluckers–Halupczok [8], outlined in the next section. Indeed, applying Theorem 7.1 along with Remarks 7.2 and 7.3), it is not difficult to see that the parametrized congruence conditions which occur in the description of the set P are not an essential obstacle to finding the line L we are looking for. Therefore the lemma reduces, likewise as it was in [44, Section 6], to a problem of semi-linear geometry.

7. Quantifier elimination for ordered abelian groups

It is well known that archimedean ordered abelian groups admit quantifier elimination in the Presburger language. Much more complicated are quantifier elimination results for non-archimedean groups (especially those with infinite rank), going back as far as Gurevich [24]. He established a transfer of sentences from ordered abelian groups to so-called coloured chains (i.e. linearly ordered sets with additional unary predicates), enhanced later to allow arbitrary formulas. This was done in his doctoral dissertation "The decision problem for some algebraic theories" (Sverdlovsk, 1968), and by Schmitt in his habilitation dissertation "Model theory of ordered abelian groups" (Heidelberg, 1982); see also the paper [56]. Such a transfer is a kind of relative quantifier elimination, which allows Gurevich–Schmitt [25], in their study of the NIP property, to lift model theoretic properties from ordered sets to ordered abelian groups or, in other words, to transform statements on ordered abelian groups into those on coloured chains.

Instead Cluckers–Halupczok [8] introduce a suitable many-sorted language $\mathcal{L}_{qe}$ with main group sort Γ and auxiliary imaginary sorts (with canonical parameters for some definable families of convex subgroups) which carry the structure of a linearly ordered set with some additional unary predicates. They provide quantifier elimination relative to the auxiliary sorts, where each definable set in the group sort is a union of a family of quantifier free definable sets with parameter running a definable (with quantifiers) set of the auxiliary sorts.

Fortunately, sometimes it is possible to directly deduce information about ordered abelian groups without any deeper knowledge of the auxiliary sorts. For instance, this may be illustrated by their theorem on piecewise linearity of definable functions [8, Corollary 1.10] as well as by Proposition 6.2 and application of quantifier elimination in the proof of the closedness theorem in Section 4.

Now we briefly recall the language $\mathcal{L}_{qe}$ taking care of points essential for our applications. The main group sort Γ is with the constant 0, the binary function $+$ and the unary function $-$. The collection $\mathcal{A}$ of auxiliary sorts consists of certain imaginary sorts:

$$\mathcal{A} := \{\mathcal{S}_p, \mathcal{T}_p, \mathcal{T}_p^+ : p \in \mathbb{P}\};$$

here $\mathbb{P}$ stands for the set of prime numbers. By abuse of notation, $\mathcal{A}$ will also denote the union of the auxiliary sorts. In this section, we denote Γ-sort variables by $x, y, z, \ldots$ and auxiliary sorts variables by $\eta, \theta, \zeta, \ldots$.

Further, the language $\mathcal{L}_{qe}$ consists of some unary predicates on $\mathcal{S}_p$, $p \in \mathbb{P}$, some binary order relations on $\mathcal{A}$, a ternary relation

$$x \equiv_{m,\alpha}^{m'} y \ \text{ on } \ \Gamma \times \Gamma \times \mathcal{S}_p \ \text{ for each } \ p \in \mathbb{P}, \ m, m' \in \mathbb{N},$$

and finally predicates for the ternary relations $x \diamond_\alpha y + k_\alpha$ on $\Gamma \times \Gamma \times \mathcal{A}$, where $\diamond \in \{=, <, \equiv_m\}$, $m \in \mathbb{N}$, $k \in \mathbb{Z}$ and α is the third operand running any of the auxiliary sorts $\mathcal{A}$.

We now explain the meaning of the above ternary relations, which are defined by means of certain definable convex subgroups Γ_α and $\Gamma_\alpha^{m'}$ of Γ with $\alpha \in \mathcal{A}$ and $m' \in \mathbb{N}$. Namely we write

$$x \equiv_{m,\alpha}^{m'} y \ \text{ iff } \ x - y \in \Gamma_\alpha^{m'} + m\Gamma.$$

Further, let 1_α denote the minimal positive element of Γ/Γ_α if Γ/Γ_α is discrete and $1_\alpha := 0$ otherwise, and set $k_\alpha := k \cdot 1_\alpha$ for all $k \in \mathbb{Z}$. By definition we write

$$x \diamond_\alpha y + k_\alpha \quad \text{iff} \quad x \,(\mathrm{mod}\, \Gamma_\alpha) \diamond y \ (\mathrm{mod}\, \Gamma_\alpha) + k_\alpha.$$

(Thus the language $\mathcal{L}_{qe}$ incorporates the Presburger language on all quotients Γ/Γ_α.) Note also that the ordinary predicates $<$ and $\equiv_m$ on Γ are Γ-quantifier-free definable in the language $\mathcal{L}_{qe}$.

Now we can readily formulate quantifier elimination relative to the auxiliary sorts ([8, Theorem 1.8]).

Theorem 7.1. *In the theory T of ordered abelian groups, each $\mathcal{L}_{qe}$-formula $\phi(\bar{x}, \bar{\eta})$ is equivalent to an $\mathcal{L}_{qe}$-formula $\psi(\bar{x}, \bar{\eta})$ in family union form, i.e.*

$$\psi(\bar{x}, \bar{\eta}) = \bigvee_{i=1}^{k} \exists \bar{\theta} \ \left[\chi_i(\bar{\eta}, \bar{\theta}) \wedge \omega_i(\bar{x}, \bar{\theta}) \right],$$

where $\bar{\theta}$ are $\mathcal{A}$-variables, the formulas $\chi_i(\bar{\eta}, \bar{\theta})$ live purely in the auxiliary sorts $\mathcal{A}$, each $\omega_i(\bar{x}, \bar{\theta})$ is a conjunction of literals (i.e. atomic or negated atomic formulas) and T implies that the

$\mathcal{L}_{qe}(\mathcal{A})$-formulas

$$\{\chi_i(\bar{\eta},\bar{\alpha}) \wedge \omega_i(\bar{x},\bar{\alpha}) : \ i=1,\dots,k, \ \bar{\alpha} \in \mathcal{A}\}$$

are pairwise inconsistent.

Remark 7.2. The sets definable (or, definable with parameters) in the main group sort Γ resemble to some extent the sets which are definable in the Presburger language. Indeed, the atomic formulas involved in the formulas $\omega_i(\bar{x},\bar{\theta})$ are of the form $t(\bar{x}) \diamond_{\theta_j} k_{\theta_j}$, where $t(\bar{x})$ is a $\mathbb{Z}$-linear combination (respectively, a $\mathbb{Z}$-linear combination plus an element of Γ), the predicates

$$\diamond \in \{=, <, \equiv_m, \equiv_m^{m'}\} \ \text{ with some } \ m, m' \in \mathbb{N},$$

θ_j is one of the entries of $\bar{\theta}$ and $k \in \mathbb{Z}$; here $k=0$ if $\diamond$ is $\equiv_m^{m'}$. Clearly, while linear equalities and inequalities define polyhedra, congruence conditions define sets which consist of entire cosets of $m\Gamma$ for finitely many $m \in \mathbb{N}$.

Remark 7.3. Note also that the sets given by atomic formulas $t(\bar{x}) \diamond_{\theta_j} k_{\theta_j}$ consist of entire cosets of the subgroups Γ_{θ_j}. Therefore, the union of those subgroups Γ_{θ_j} which essentially occur in a formula in family union form, describing a proper subset of Γ^n, is not cofinal with Γ. This observation is often useful as, for instance, in the proofs of fiber shrinking and Theorem 1.1.

8. Proof of the closedness theorem

In the proof of Theorem 1.1, we shall generally follow the ideas from our previous paper [44, Section 7]. We must show that if B is an $\mathcal{L}$-definable subset of $D \times (K^{\circ})^n$ and a point a lies in the closure of $A := \pi(B)$, then there is a point b in the closure of B such that $\pi(b) = a$. Again, the proof reduces easily to the case $m=1$ and next, by means of fiber shrinking (Proposition 6.1), to the case $n=1$. We may obviously assume that $a = 0 \notin A$.

Whereas in the paper [44] preparation cell decomposition (due to Pas; see [53, Theorem 3.2] and [44, Theorem 2.4]) was combined with quantifier elimination in the Γ sort in the Presburger language, here it is combined with relative quantifier elimination in the language $\mathcal{L}_{qe}$ considered in Section 7. In a similar manner as in [44], we can now assume that B is a subset F of a cell C of the form presented below. Let $a(x,\xi), b(x,\xi), c(x,\xi) : D \longrightarrow K$ be three $\mathcal{L}$-definable functions on an $\mathcal{L}$-definable subset D of $K^2 \times \Bbbk^m$ and let $\nu \in \mathbb{N}$ is a positive integer. For each $\xi \in \Bbbk^m$ set

$$C(\xi) := \Bigg\{ (x,y) \in K_x^n \times K_y : \ (x,\xi) \in D,$$
$$v(a(x,\xi)) \lhd_1 v((y-c(x,\xi))^{\nu}) \lhd_2 v(b(x,\xi)), \ \overline{ac}(y-c(x,\xi)) = \xi_1 \Bigg\},$$

where $\lhd_1, \lhd_2$ stand for $<, \leq$ or no condition in any occurrence. A cell C is by definition a disjoint union of the fibres $C(\xi)$. The subset F of C is a union of fibers $F(\xi)$ of the form

$$F(\xi) := \Bigg\{ (x,y) \in C(\xi) : \ \exists \ \bar{\theta} \ \chi(\bar{\theta}) \wedge$$
$$\bigwedge_{i \in I_a} v(a_i(x,\xi)) \lhd_{1,\theta_{j_i}} v((y-c(x,\xi))^{\nu_i}), \ \bigwedge_{i \in I_b} v((y-c(x,\xi))^{\nu_i}) \lhd_{2,\theta_{j_i}} v(b_i(x,\xi))$$
$$\wedge \bigwedge_{i \in I_f} v((y-c(x,\xi))^{\nu_i}) \diamond_{\theta_{j_i}} v(f_i(x,\xi)) \Bigg\},$$

where I_a, I_b, I_f are finite (possibly empty) sets of indices, a_i, b_i, f_i are $\mathcal{L}$-definable functions, $\nu_i, M \in \mathbb{N}$ are positive integers, $\lhd_1$, $\lhd_2$ stand for $<$ or $\leq$, the predicates

$$\diamond \in \{\equiv_M, \neg \equiv_M, \equiv_M^{m'}, \neg \equiv_M^{m'}\} \text{ with some } m' \in \mathbb{N},$$

and θ_{j_i} is one of the entries of $\bar{\theta}$.

As before, since every $\mathcal{L}$-definable subset in the Cartesian product $\Gamma^n \times \Bbbk^m$ of auxiliary sorts is a finite union of the Cartesian products of definable subsets in Γ^n and in $\Bbbk^m$, we can assume that B is one fiber $F(\xi')$ for a parameter $\xi' \in \Bbbk^m$. For simplicity, we abbreviate

$$c(x, \xi'), a(x, \xi'), b(x, \xi'), a_i(x, \xi'), b_i(x, \xi'), f_i(x, \xi')$$

to

$$c(x), a(x), b(x), a_i(x), b_i(x), f_i(x)$$

with $i \in I_a$, $i \in I_b$ and $i \in I_f$. Denote by $E \subset K$ the common domain of these functions; then 0 is an accumulation point of E.

By the theorem on existence of the limit (Theorem 5.1), we can assume that the limits

$$c(0), a(0), b(0), a_i(0), b_i(0), f_i(0)$$

of the functions

$$c(x), a(x), b(x), a_i(x), b_i(x), f_i(x)$$

when $x \to 0$ exist in R. Moreover, there is a neighborhood U of 0 such that, each definable set

$$\{(v(x), v(f_i(x))) : x \in (E \cap U) \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \{\infty\}), \quad i \in I_f,$$

is contained in an affine line with rational slope

$$q \cdot l = p_i \cdot k + \beta_i, \quad i \in I_f, \tag{8.1}$$

with $p_i, q \in \mathbb{Z}$, $q > 0$, $\beta_i \in \Gamma$, or in $\Gamma \times \{\infty\}$.

The role of the center $c(x)$ is, of course, immaterial. We may assume, without loss of generality, that it vanishes, $c(x) \equiv 0$, for if a point $b = (0, w) \in K^2$ lies in the closure of the cell with zero center, the point $(0, w + c(0))$ lies in the closure of the cell with center $c(x)$.

Observe now that If $\lhd_1$ occurs and $a(0) = 0$, the set $F(\xi')$ is itself an x-fiber shrinking at $(0, 0)$ and the point $b = (0, 0)$ is an accumulation point of B lying over $a = 0$, as desired. And so is the point $b = (0, 0)$ if $\lhd_{1,\theta_{j_i}}$ occurs and $a_i(0) = 0$ for some $i \in I_a$, because then the set $F(\xi')$ contains the x-fiber shrinking

$$F(\xi') \cap \{(x, y) \in E \times K : v(a_i(x)) \lhd_1 v(y^{\nu_i})\}.$$

So suppose that either only $\lhd_2$ occur or $\lhd_1$ occur and, moreover, $a(0) \neq 0$ and $a_i(0) \neq 0$ for all $i \in I_a$. By elimination of K-quantifiers, the set $v(E)$ is a definable subset of Γ. Further, it is easy to check, applying Theorem 7.1 ff. likewise as it was in Lemma 6.2, that the set $v(E)$ is given near infinity only by finitely many parametrized congruence conditions of the form

$$v(E) = \left\{ k \in \Gamma : \ k > \beta \ \wedge \ \exists \ \bar{\theta} \ \omega(\bar{\theta}) \ \wedge \ \bigwedge_{i=1}^{s} m_i k \diamond_{N,\theta_{j_i}} \gamma_i \right\}. \tag{8.2}$$

where $\beta, \gamma_i \in \Gamma$, $m_i, N \in \mathbb{N}$ for $i = 1, \ldots, s$, the predicates

$$\diamond \in \{\equiv_N, \neg \equiv_N, \equiv_N^{m'}, \neg \equiv_N^{m'}\} \text{ with some } m' \in \mathbb{N},$$

and θ_{j_i} is one of the entries of $\bar{\theta}$. Obviously, after perhaps shrinking the neighborhood of zero, we may assume that

$$v(a(x)) = v(a(0)) \text{ and } v(a_i(x)) = v(a_i(0))$$

for all $i \in I_a$ and $x \in E \setminus \{0\}$, $v(x) > \beta$.

Now, take an element $(u, w) \in F(\xi')$ with $u \in E \setminus \{0\}$, $v(u) > \beta$. In order to complete the proof, it suffices to show that $(0, w)$ is an accumulation point of $F(\xi')$. To this end, observe that, by equality 8.2, there is a point $x \in E$ arbitrarily close to 0 such that

$$v(x) \in v(u) + qMN \cdot \Gamma.$$

By equality 8.1, we get $v(f_i(x)) \in v(f_i(u)) + p_i MN \cdot \Gamma,\ i \in I_f$, and hence

$$v\left(f_i(x)\right) \equiv_M v(f_i(u)), \quad i \in I_f. \tag{8.3}$$

Clearly, in the vicinity of zero we have

$$v(y^{\nu}) \lhd_2 v(b(x, \xi)) \quad \text{and} \quad \bigwedge_{i \in I_b} v(y^{\nu_i}) \lhd_{2, \theta_{j_i}} v(b_i(x, \xi)).$$

Therefore equality 8.3 along with the definition of the fibre $F(\xi')$ yield $(x, w) \in F(\xi')$, concluding the proof of the closedness theorem.

9. Piecewise continuity of definable functions

Further, let $\mathcal{L}$ be the three-sorted language $\mathcal{L}$ of Denef–Pas. The main purpose of this section is to prove the following

Theorem 9.1. *Let $A \subset K^n$ and $f : A \to \mathbb{P}^1(K)$ be an $\mathcal{L}$-definable function. Then f is piecewise continuous, i.e. there is a finite partition of A into $\mathcal{L}$-definable locally closed subsets $A_1, \ldots, A_s$ of K^n such that the restriction of f to each A_i is continuous.*

We immediately obtain

Corollary 9.2. *The conclusion of the above theorem holds for any $\mathcal{L}$-definable function $f : A \to K$.*

The proof of Theorem 9.1 relies on two basic ingredients. The first one is concerned with a theory of algebraic dimension and decomposition of definable sets into a finite union of locally closed definable subsets we begin with. It was established by van den Dries [13] for certain expansions of rings (and Henselian valued fields, in particular) which admit quantifier elimination and are equipped with a topological system. The second one is the closedness theorem (Theorem 1.1).

Consider an infinite integral domain D with quotient field K. One of the fundamental concepts introduced by van den Dries [13] is that of a *topological system* on a given expansion $\mathcal{D}$ of a domain D in a language $\widetilde{\mathcal{L}}$. That concept incorporates both Zariski-type and definable topologies. We remind the reader that it consists of a topology τ_n on each set D^n, $n \in \mathbb{N}$, such that:

1) For any n-ary $\widetilde{\mathcal{L}}_D$-terms $t_1, \ldots, t_s$, $n, s \in \mathbb{N}$, the induced map

$$D^n \ni a \longrightarrow (t_1(a), \ldots, t_s(a)) \in D^s$$

is continuous.

2) Every singleton $\{a\}$, $a \in D$, is a closed subset of D.

3) For any n-ary relation symbol R of the language $\widetilde{\mathcal{L}}$ and any sequence $1 \le i_1 < \ldots < i_k \le n$, $1 \le k \le n$, the two sets

$$\{(a_{i_1}, \ldots, a_{i_k}) \in D^k :\ \mathcal{D} \models R((a_{i_1}, \ldots, a_{i_k})^{\&}),\ a_{i_1} \neq 0, \ldots, a_{i_k} \neq 0\},$$
$$\{(a_{i_1}, \ldots, a_{i_k}) \in D^k :\ \mathcal{D} \models \neg R((a_{i_1}, \ldots, a_{i_k})^{\&}),\ a_{i_1} \neq 0, \ldots, a_{i_k} \neq 0\}$$

are open in D^k; here $(a_{i_1}, \ldots, a_{i_k})^{\&}$ denotes the element of D^n whose i_j-th coordinate is a_{i_j}, $j = 1, \ldots, k$, and whose remaining coordinates are zero.

Finite intersections of closed sets of the form $\{a \in D^n : t(a) = 0\}$, where t is an n-ary $\widetilde{\mathcal{L}}_D$-term, will be called *special closed subsets* of D^n. Finite intersections of open sets of the form

$$\{a \in D^n : t(a) \neq 0\},$$

$$\{a \in D^n : \ \mathcal{D} \models R((t_{i_1}(a), \ldots, t_{i_k}(a))^{\&}), \, t_{i_1}(a) \neq 0, \ldots, t_{i_k}(a) \neq 0\}$$

or

$$\{a \in D^n : \ \mathcal{D} \models \neg R((t_{i_1}(a), \ldots, t_{i_k}(a))^{\&}), \, t_{i_1}(a) \neq 0, \ldots, t_{i_k}(a) \neq 0\},$$

where t, t_{i_1}, t_{i_k} are $\widetilde{\mathcal{L}}_D$-terms, will be called *special open subsets* of D^n. Finally, an intersection of a special open and a special closed subsets of D^n will be called a *special locally closed* subset of D^n. Every quantifier-free $\widetilde{\mathcal{L}}$-definable set is a finite union of special locally closed sets.

Suppose now that the language $\widetilde{\mathcal{L}}$ extends the language of rings and has no extra function symbols of arity > 0 and that an $\widetilde{\mathcal{L}}$-expansion $\mathcal{D}$ of the domain D under study admits quantifier elimination and is equipped with a topological system such that every non-empty special open subset of D is infinite. These conditions ensure that $\mathcal{D}$ is algebraically bounded and algebraic dimension is a dimension function on $\mathcal{D}$ ([13, Proposition 2.15 and 2.7]). Algebraic dimension is the only dimension function on $\mathcal{D}$ whenever, in addition, D is a non-trivially valued field and the topology τ_1 is induced by its valuation. Then, for simplicity, the algebraic dimension of an $\widetilde{\mathcal{L}}$-definable set E will be denoted by $\dim E$.

Now we recall the following two basic results from the paper [13, Propositions 2.17 and 2.23]:

Proposition 9.3. *Every $\widetilde{\mathcal{L}}$-definable subset of D^n is a finite union of intersections of Zariski closed with special open subsets of D^n and, a fortiori, a finite union of locally closed $\widetilde{\mathcal{L}}$-definable subsets of D^n.*

Proposition 9.4. *Let E be an $\widetilde{\mathcal{L}}$-definable subset of D^n, and let $\overline{E}$ stand for its closure and $\partial E := \overline{E} \setminus E$ for its frontier. Then*

$$\text{alg.dim}\,(\partial E) < \text{alg.dim}\,(E).$$

It is not difficult to strengthen the former proposition as follows.

Corollary 9.5. *Every $\widetilde{\mathcal{L}}$-definable set is a finite disjoint union of locally closed sets.*

Quantifier elimination due to Pas [53, Theorem 4.1] (more precisely, elimination of K-quantifiers) enables translation of the language $\mathcal{L}$ of Denef–Pas on K into a language $\widetilde{\mathcal{L}}$ described above, which is equipped with the topological system wherein τ_n is the K-topology on K^n, $n \in \mathbb{N}$. Indeed, we must augment the language of rings by adding extra relation symbols for the inverse images under the valuation and angular component map of relations on the value group and residue field, respectively. More precisely, we must add the names of sets of the form

$$\{a \in K^n : (v(a_1), \ldots, v(a_n)) \in P\} \quad \text{and} \quad \{a \in K^n : (\overline{ac}\, a_1, \ldots, \overline{ac}\, a_n) \in Q\},$$

where P and Q are definable subsets of Γ^n and $\Bbbk^n$ (as the auxiliary sorts of the language $\mathcal{L}$), respectively.

Summing up, the foregoing results apply in the case of Henselian non-trivially valued fields with the three-sorted language $\mathcal{L}$ of Denef–Pas. Now we can readily prove Theorem 9.1.

Proof. Consider an $\mathcal{L}$-definable function $f : A \to \mathbb{P}^1(K)$ and its graph

$$E := \{(x, f(x)) : x \in A\} \subset K^n \times \mathbb{P}^1(K).$$

We shall proceed with induction with respect to the dimension $d = \dim A = \dim E$ of the source and graph of f. By Corollary 9.5, we can assume that the graph E is a locally closed subset of $K^n \times \mathbb{P}^1(K)$ of dimension d and that the conclusion of the theorem holds for functions with source and graph of dimension $< d$.

Let F be the closure of E in $K^n \times \mathbb{P}^1(K)$ and $\partial E := F \setminus E$ be the frontier of E. Since E is locally closed, the frontier ∂E is a closed subset of $K^n \times \mathbb{P}^1(K)$ as well. Let

$$\pi : K^n \times \mathbb{P}^1(K) \longrightarrow K^n$$

be the canonical projection. Then, by virtue of the closedness theorem, the images $\pi(F)$ and $\pi(\partial E)$ are closed subsets of K^n. Further,

$$\dim F = \dim \pi(F) = d \quad \text{and} \quad \dim \pi(\partial E) \leq \dim \partial E < d;$$

the last inequality holds by Proposition 9.4. Putting $B := \pi(F) \setminus \pi(\partial E) \subset \pi(E) = A$, we thus get $\dim B = d$ and $\dim (A \setminus B) < d$. Clearly, the set

$$E_0 := E \cap (B \times \mathbb{P}^1(K)) = F \cap (B \times \mathbb{P}^1(K))$$

is a closed subset of $B \times \mathbb{P}^1(K)$ and is the graph of the restriction $f_0 : B \longrightarrow \mathbb{P}^1(K)$ of f to B. Again, it follows immediately from the closedness theorem that the restriction $\pi_0 : E_0 \longrightarrow B$ of the projection π to E_0 is a definably closed map. Therefore f_0 is a continuous function. But, by the induction hypothesis, the restriction of f to $A \setminus B$ satisfies the conclusion of the theorem, whence so does the function f. This completes the proof. $\square$

10. Curve selection

We now pass to curve selection over non-locally compact ground fields under study. While the real version of curve selection goes back to the papers [6, 58] (see also [40, 41, 4]), the p-adic one was achieved in the papers [57, 12].

In this section we give two versions of curve selection which are counterparts of the ones from our paper [44, Proposition 8.1 and 8.2] over rank one valued fields. The first one is concerned with valuative semialgebraic sets and we can repeat verbatim its proof which relies on transformation to a normal crossing by blowing up and the closedness theorem.

By a valuative semialgebraic subset of K^n we mean a (finite) Boolean combination of elementary valuative semialgebraic subsets, i.e. sets of the form $\{x \in K^n : \ v(f(x)) \leq v(g(x))\}$, where f and g are regular functions on K^n. We call a map φ semialgebraic if its graph is a valuative semialgebraic set.

Proposition 10.1. *Let A be a valuative semialgebraic subset of K^n. If a point $a \in K^n$ lies in the closure (in the K-topology) of $A \setminus \{a\}$, then there is a semialgebraic map $\varphi : R \longrightarrow K^n$ given by algebraic power series such that*

$$\varphi(0) = a \quad \textit{and} \quad \varphi(R \setminus \{0\}) \subset A \setminus \{a\}.$$

We now turn to the general version of curve selection for $\mathcal{L}$-definable sets. Under the circumstances, we apply relative quantifier elimination in a many-sorted language due to Cluckers–Halupczok rather than simply quantifier elimination in the Presburger language for rank one valued fields. The passage between the two corresponding reasonings for curve selection is similar to that for fiber shrinking. Nevertheless we provide a detailed proof for more clarity and the reader's convenience. Note that both fiber shrinking and curve selection apply Lemma 6.2.

Proposition 10.2. *Let A be an $\mathcal{L}$-definable subset of K^n. If a point $a \in K^n$ lies in the closure (in the K-topology) of $A \setminus \{a\}$, then there exist a semialgebraic map $\varphi : R \longrightarrow K^n$ given by algebraic power series and an $\mathcal{L}$-definable subset E of R with accumulation point 0 such that*

$$\varphi(0) = a \quad \textit{and} \quad \varphi(E \setminus \{0\}) \subset A \setminus \{a\}.$$

Proof. As before, we proceed with induction with respect to the dimension of the ambient space n. The case $n = 1$ being evident, suppose $n > 1$. By elimination of K-quantifiers, the set $A \setminus \{a\}$ is a finite union of sets defined by conditions of the form

$$(v(f_1(x)), \dots, v(f_r(x))) \in P, \quad (\overline{ac}\, g_1(x), \dots, \overline{ac}\, g_s(x)) \in Q,$$

where $f_i, g_j \in K[x]$ are polynomials, and P and Q are definable subsets of Γ^r and $\Bbbk^s$, respectively. Without loss of generality, we may assume that A is such a set and $a = 0$.

Take a finite composite $\sigma : Y \longrightarrow K\mathbb{A}^n$ of blow-ups along smooth centers such that the pull-backs $f_1^\sigma, \dots, f_r^\sigma$ and $g_1^\sigma, \dots, g_s^\sigma$ are normal crossing divisors unless they vanish. Since the restriction $\sigma : Y(K) \longrightarrow K^n$ is definably closed (Corollary 1.6), there is a point $b \in Y(K) \cap \sigma^{-1}(a)$ which lies in the closure of the set $B := Y(K) \cap \sigma^{-1}(A \setminus \{a\})$. Take local coordinates $y_1 \dots, y_n$ near b in which $b = 0$ and every pull-back above is a normal crossing. We shall first select a semialgebraic map $\psi : R \longrightarrow Y(K)$ given by restricted power series and an $\mathcal{L}$-definable subset E of R with accumulation point 0 such that $\psi(0) = b$ and $\psi(E \setminus \{0\}) \subset B$.

Since the valuation map and the angular component map composed with a continuous function are locally constant near any point at which this function does not vanish, the conditions which describe the set B near b are of the form

$$(v(y_1), \dots, v(y_n)) \in \widetilde{P}, \quad (\overline{ac}\, y_1, \dots, \overline{ac}\, y_n) \in \widetilde{Q},$$

where $\widetilde{P}$ and $\widetilde{Q}$ are definable subsets of Γ^n and $\Bbbk^n$, respectively.

The set B_0 determined by the conditions

$$(v(y_1), \dots, v(y_n)) \in \widetilde{P},$$

$$(\overline{ac}\, y_1, \dots, \overline{ac}\, y_n) \in \widetilde{Q} \cap \bigcup_{i=1}^{n} \{\xi_i = 0\},$$

is contained near b in the union of hyperplanes $\{y_i = 0\}$, $i = 1, \dots, n$. If b is an accumulation point of the set B_0, then the desired map ψ exists by the induction hypothesis. Otherwise b is an accumulation point of the set $B_1 := B \setminus B_0$.

Now we are going to apply relative quantifier elimination in the value group sort Γ. Similarly, as in the proof of Lemma 6.2, the parametrized congruence conditions which occur in the description of the definable subset $\widetilde{P}$ of Γ^n, achieved via quantifier elimination, are not an essential obstacle to finding the desired map ψ, but affect only the definable subset E of R. Neither are the conditions

$$\widetilde{Q} \setminus \bigcup_{i=1}^{n} \{\xi_i = 0\}$$

imposed on the angular components of the coordinates $y_1, \dots, y_n$, because none of them vanishes here. Therefore, in order to select the map ψ, we must first of all analyze the linear conditions (equalities and inequalities) which occur in the description of the set $\widetilde{P}$.

The set $\widetilde{P}$ has an accumulation point $(\infty, \dots, \infty)$ as $b = 0$ is an accumulation point of B. By Lemma 6.2, there is an affine semi-line

$$L = \{(r_1 t + \gamma_1, \dots, r_n t + \gamma_n) : \ t \in \Gamma, \ t \geq 0\} \quad \text{with} \quad r_1, \dots, r_n \in \mathbb{N},$$

passing through a point $\gamma = (\gamma_1, \dots, \gamma_n) \in P$ and such that $(\infty, \dots, \infty)$ is an accumulation point of the intersection $P \cap L$ too.

Now, take some elements

$$(\xi_1, \ldots, \xi_n) \in \widetilde{Q} \setminus \bigcup_{i=1}^{n} \{\xi_i = 0\}$$

and next some elements $w_1, \ldots, w_n \in K$ for which

$$v(w_1) = \gamma_1, \ldots, v(w_n) = \gamma_n \quad \text{and} \quad \overline{ac}\, w_1 = \xi_1, \ldots, \overline{ac}\, w_n = \xi_n.$$

It is not difficult to check that there exists an $\mathcal{L}$-definable subset E of R which is determined by a finite number of parametrized congruence conditions (in the many-sorted language $\mathcal{L}_{qe}$ described in Section 7) imposed on $v(t)$ and the conditions $\overline{ac}\, t = 1$ such that the subset

$$F := \{(w_1 \cdot t^{r_1}, \ldots, w_n \cdot t^{r_n}) : \ t \in E\}$$

of the arc

$$\psi : R \to Y, \quad \psi(t) = (w_1 \cdot t^{r_1}, \ldots, w_n \cdot t^{r_n})$$

is contained in B_1. Then $\varphi := \sigma \circ \psi$ is the map we are looking for. This completes the proof. $\square$

11. The Łojasiewicz inequalities

In this section we provide certain two versions of the Łojasiewicz inequality which generalize the ones from [44, Propositions 9.1 and 9.2] to the case of arbitrary Henselian valued fields. Moreover, the first one is now formulated for several functions $g_1, \ldots, g_m$. For its proof we still need the following easy consequence of the closedness theorem.

Proposition 11.1. *Let $f : A \to K$ be a continuous $\mathcal{L}$-definable function on a closed bounded subset $A \subset K^n$. Then f is a bounded function, i.e. there is an $\omega \in \Gamma$ such that $v(f(x)) \geq \omega$ for all $x \in A$.*

We adopt the following notation:

$$v(x) = v(x_1, \ldots, x_n) := \min \{v(x_1), \ldots, v(x_n)\}$$

for $x = (x_1, \ldots, x_n) \in K^n$.

Theorem 11.2. *Let $f, g_1, \ldots, g_m : A \to K$ be continuous $\mathcal{L}$-definable functions on a closed (in the K-topology) bounded subset A of K^m. If*

$$\{x \in A : g_1(x) = \ldots = g_m(x) = 0\} \subset \{x \in A : f(x) = 0\},$$

then there exist a positive integer s and a constant $\beta \in \Gamma$ such that

$$s \cdot v(f(x)) + \beta \geq v((g_1(x), \ldots, g_m(x)))$$

for all $x \in A$.

Proof. Put $g = (g_1, \ldots, g_m)$. It is easy to check that the set $A_\gamma := \{x \in A : \ v(f(x)) = \gamma\}$ is a closed $\mathcal{L}$-definable subset of A for every $\gamma \in \Gamma$. By the hypothesis and the closedness theorem, the set $g(A_\gamma)$ is a closed $\mathcal{L}$-definable subset of $K^m \setminus \{0\}$, $\gamma \in \Gamma$. The set $v(g(A_\gamma))$ is thus bounded from above, i.e. $v(g(A_\gamma)) \leq \alpha(\gamma)$ for some $\alpha(\gamma) \in \Gamma$. By elimination of K-quantifiers, the set

$$\Lambda := \{(v(f(x)), v(g(x))) \in \Gamma^2 : \ x \in A, \ f(x) \neq 0\} \ \subset \ \{(\gamma, \delta) \in \Gamma^2 : \ \delta \leq \alpha(\gamma)\}$$

is a definable subset of Γ^2 in the many-sorted language $\mathcal{L}_{qe}$ from Section 7. Applying Theorem 7.1 ff., we see that this set is described by a finite number of parametrized linear equalities and inequalities, and of parametrized congruence conditions. Hence

$$\Lambda \cap \{(\gamma, \delta) \in \Gamma^2 : \ \gamma > \gamma_0\} \subset \{(\gamma, \delta) \in \Gamma^2 : \ \delta \leq s \cdot \gamma\}$$

for a positive integer s and some $\gamma_0 \in \Gamma$. We thus get

$$v(g(x)) \leq s \cdot v(f(x)) \text{ if } x \in A, v(f(x)) > \gamma_0.$$

Again, by the hypothesis, we have $g(\{x \in A : v(f(x)) \leq \gamma_0\}) \subset K^m \setminus \{0\}$. Therefore it follows from the closedness theorem that the set $\{v(g(x)) \in \Gamma : v(f(x)) \leq \gamma_0\}$ is bounded from above, say, by a $\theta \in \Gamma$. Taking an $\omega \in \Gamma$ as in Proposition 11.1 and putting $\beta := \max\{0, \theta - s \cdot \omega\}$, we get

$$s \cdot v(f(x)) - v(g(x)) + \beta \geq 0, \quad \text{for all } x \in A,$$

as desired. □

A direct consequence of Theorem 11.2 is the following result on Hölder continuity of definable functions.

Proposition 11.3. *Let $f : A \to K$ be a continuous $\mathcal{L}$-definable function on a closed bounded subset $A \subset K^n$. Then f is Hölder continuous with a positive integer s and a constant $\beta \in \Gamma$, i.e.*

$$s \cdot v(f(x) - f(z)) + \beta \geq v(x - z)$$

for all $x, z \in A$.

Proof. Apply Theorem 11.2 to the functions

$$f(x) - f(y) \text{ and } g_i(x, y) = x_i - y_i, \ i = 1, \ldots, n.$$

□

We immediately obtain

Corollary 11.4. *Every continuous $\mathcal{L}$-definable function $f : A \to K$ on a closed bounded subset $A \subset K^n$ is uniformly continuous.*

Now we state a version of the Łojasiewicz inequality for continuous definable functions of a locally closed subset of K^n.

Theorem 11.5. *Let $f, g : A \to K$ be two continuous $\mathcal{L}$-definable functions on a locally closed subset A of K^n. If*

$$\{x \in A : g(x) = 0\} \subset \{x \in A : f(x) = 0\},$$

then there exist a positive integer s and a continuous $\mathcal{L}$-definable function h on A such that $f^s(x) = h(x) \cdot g(x)$ for all $x \in A$.

Proof. It is easy to check that the set A is of the form $A := U \cap F$, where U and F are two $\mathcal{L}$-definable subsets of K^n, U is open and F is closed in the K-topology.

We shall adapt the foregoing arguments. Since the set U is open, its complement $V := K^n \setminus U$ is closed in K^n and A is the following union of open and closed subsets of K^n and of $\mathbb{P}^n(K)$:

$$X_\beta := \{x \in K^n : v(x_1), \ldots, v(x_n) \geq -\beta, \quad v(x - y) \leq \beta \quad \text{for all } y \in V\},$$

where $\beta \in \Gamma$, $\beta \geq 0$. As before, we see that the sets

$$A_{\beta,\gamma} := \{x \in X_\beta : v(f(x)) = \gamma\} \text{ with } \beta, \gamma \in \Gamma$$

are closed $\mathcal{L}$-definable subsets of $\mathbb{P}^n(K)$, and next that the sets $g(A_{\beta,\gamma})$ are closed $\mathcal{L}$-definable subsets of $K \setminus \{0\}$ for all $\beta, \gamma \in \Gamma$. Likewise, we get

$$\Lambda := \{(\beta, v(f(x)), v(g(x))) \in \Gamma^3 : x \in X_\beta, \ f(x) \neq 0\} \subset \{(\beta, \gamma, \delta) \in \Gamma^3 : \delta < \alpha(\beta, \gamma)\}$$

for some $\alpha(\beta, \gamma) \in \Gamma$.

Λ is a definable subset of Γ^3 in the many-sorted language $\mathcal{L}_{qe}$, and thus is described by a finite number of parametrized linear equalities and inequalities, and of parametrized congruence conditions. Again, the above inclusion reduces to an analysis of those linear equalities and inequalities. Consequently, there exist a positive integer $s \in \mathbb{N}$ and elements $\gamma_0(\beta) \in \Gamma$ such that

$$\Lambda \cap \{(\beta, \gamma, \delta) \in \Gamma^3 : \ \gamma > \gamma_0(\beta)\} \subset \{(\beta, \gamma, \delta) \in \Gamma^3 : \ \delta < s \cdot \gamma\}.$$

Since A is the union of the sets X_β, it is not difficult to check that the quotient f^s/g extends by zero through the zero set of the denominator to a (unique) continuous $\mathcal{L}$-definable function on A, which is the desired result. $\square$

We conclude this section with a theorem which is much stronger than its counterpart, [44, Proposition 12.1], concerning continuous rational functions. The proof we give now resembles the above one, without applying transformation to a normal crossing. Put

$$\mathcal{D}(f) := \{x \in A : f(x) \neq 0\} \ \text{ and } \ \mathcal{Z}(f) := \{x \in A : f(x) = 0\}.$$

Theorem 11.6. *Let $f : A \to K$ be a continuous $\mathcal{L}$-definable function on a locally closed subset A of K^n and $g : \mathcal{D}(f) \to K$ a continuous $\mathcal{L}$-definable function. Then $f^s \cdot g$ extends, for $s \gg 0$, by zero through the set $\mathcal{Z}(f)$ to a (unique) continuous $\mathcal{L}$-definable function on A.*

Proof. As in the proof of Theorem 11.5, let $A = U \cap F$ and consider the same sets $X_\beta \subset K^n$, $\beta \in \Gamma$, and $\Lambda \subset \Gamma^3$. Under the assumptions, we get

$$\Lambda \subset \{(\beta, \gamma, \delta) \in \Gamma^3 : \ \delta > \alpha(\beta, \gamma)\}$$

for some $\alpha(\beta, \gamma) \in \Gamma$. Now, in a similar fashion as before, we can find an integer $r \in \mathbb{Z}$ and elements $\gamma_0(\beta) \in \Gamma$ such that

$$\Lambda \cap \{(\beta, \gamma, \delta) \in \Gamma^3 : \ \gamma > \gamma_0(\beta)\} \subset \{(\beta, \gamma, \delta) \in \Gamma^3 : \ \delta > r \cdot \gamma\}.$$

Take a positive integer $s \in \mathbb{N}$ such that $s + r > 0$. Then, as in the proof of Theorem 11.5, it is not difficult to check that the function $f^s \cdot g$ extends by zero through the zero set of f to a (unique) continuous $\mathcal{L}$-definable function on A, which is the desired result. $\square$

Remark 11.7. Note that Theorem 11.6 is, in fact, a strengthening of Theorem 11.5, and has many important applications. In particular, it plays a crucial role in the proof of the Nullstellensatz for regulous (i.e. continuous and rational) functions on K^n.

12. Continuous hereditarily rational functions and regulous functions and sheaves

Continuous rational functions on singular real algebraic varieties, unlike those on non-singular real algebraic varieties, often behave quite unusually. This is illustrated by many examples from the paper [30, Section 1], and gives rise to the concept of hereditarily rational functions. We shall assume that the ground field K is not algebraically closed. Otherwise, the notion of a continuous rational function on a normal variety coincides with that of a regular function and, in general, the study of continuous rational functions leads to the concept of *seminormality* and *seminormalization*; cf. [1, 2] or [29, Section 10.2] for a recent treatment. Let K be topological field with the density property. For a K-variety Z, let $Z(K)$ denote the set of all K-points on Z. We say that a continuous function $f : Z(K) \longrightarrow K$ is *hereditarily rational* if for every irreducible subvariety $Y \subset Z$ there exists a Zariski dense open subvariety $Y^0 \subset Y$ such that $f|_{Y^0(K)}$ is regular. Below we recall an extension theorem, which plays a crucial role in the theory of continuous rational functions. It says roughly that continuous rational extendability to the non-singular ambient space is ensured by (and in fact equivalent to) the intrinsic property to be continuous hereditarily rational. This theorem was first proven for real and p-adic varieties in [30], and next over Henselian rank one valued fields in [44, Section 10]. The proof of the latter result relied on the closedness theorem (Theorem 1.1), the descent property (Corollary 1.7) and the Łojasiewicz inequality (Theorem 11.5), and can now be repeated verbatim for the case where K is an arbitrary Henselian valued field K of equicharacteristic zero.

Theorem 12.1. *Let X be a non-singular K-variety and $W \subset Z \subset X$ closed subvarieties. Let f be a continuous hereditarily rational function on $Z(K)$ that is regular at all K-points of $Z(K) \setminus W(K)$. Then f extends to a continuous hereditarily rational function F on $X(K)$ that is regular at all K-points of $X(K) \setminus W(K)$.*

The corresponding theorem for hereditarily rational functions of class $\mathcal{C}^k$, $k \in \mathbb{N}$, remains an open problem as yet. This leads to the concept of k-regulous functions, $k \in \mathbb{N}$, on a subvariety $Z(K)$ of a non-singular K-variety $X(K)$, i.e. those functions on $Z(K)$ which are the restrictions to $Z(K)$ of rational functions of class $\mathcal{C}^k$ on $X(K)$.

In real algebraic geometry, the theory of regulous functions, varieties and sheaves was developed by Fichou–Huisman–Mangolte–Monnier [19]. Regulous geometry over Henselian rank one valued fields was studied in our paper [44, Sections 11, 12, 13]. The basic tools we applied are the closedness theorem, descent property, the Łojasiewicz inequalities and transformation to a normal crossing by blowing up. We should emphasize that all those our results, including the Nullstellensatz and Cartan's theorems A and B for regulous quasi-coherent sheaves, remain true over arbitrary Henselian valued fields (of equicharacteristic zero) with almost the same proofs.

We conclude this paper with the following comment.

Remark 12.2. In our recent paper [48], we established a definable, non-Archimedean version of the closedness theorem over Henselian valued fields (of equicharacteristic zero) with analytic structure along with several applications. Let us mention, finally, that the theory of analytic structures goes back to the work of many mathematicians (see e.g. [12, 14, 37, 16, 15, 38, 39, 9, 10, 11]).

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