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ON A SINGULARITY APPEARING IN THE MULTIPLICATION OF POLYNOMIALS

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To Cidinha, on her 70th birthday

Abstract. The multiplication of monic polynomials of degrees n and m defines a mapping $R^{n+m} \to R^{n+m}$. Singularities of this mapping at a point corresponding to two polynomials (P, Q) appear when the two polynomials have a common root. In [Ch-LdM] it was shown that, when every such common root is simple in one of the polynomials, the singularity type can be described using swallowtail singularities whose geometry is well understood. In this paper we consider the case where there are common double roots. We start with the minimal possible situation where both polynomials are of degree 2, and give a normal form for the singularity that allows us to describe its geometry quite thoroughly. This normal form is then extended to other polynomial pairs with only one common multiple root which is a double root in one of them. Finally we give a general statement for pairs whose greater common divisor has only single or double roots.

Introduction.

Let $\mathrm{MP}(\mathbf{K}, n)$ be the space of monic polynomials of degree n with coefficients in a field $\mathbf{K}$. We will consider only cases where $\mathbf{K}$ is either the real or the complex field. A polynomial in $\mathrm{MP}(\mathbf{K}, n)$ is given by n coefficients, so the space $\mathrm{MP}(\mathbf{K}, n)$ can be identified with $\mathbf{K}^n$.

Multiplication of polynomials gives a mapping:

$$\mathrm{Mult} : \mathrm{MP}(\mathbf{K}, n) \times \mathrm{MP}(\mathbf{K}, m) \to \mathrm{MP}(\mathbf{K}, n + m),$$

which can then be identified as a mapping from $\mathbf{K}^{n+m}$ to itself.

We are interested in understanding the properties of this differentiable map: at which pairs (P, Q) of polynomials is it a local diffeomorphism? When it is not so, can we describe the type of singularities that may appear, starting with the most simple situations?

In [Ch-LdM] these questions were given some first answers (which were then applied to the theory of deformations of linear operators):

(i) *The points* (P, Q) *where the mapping* Mult *is a local diffeomorphism are characterized as those where the two polynomials are relatively prime.*

(ii) *The singularity type is given at the pairs where the greatest common divisor of them has only simple roots (see Theorem 1 below).*

(iii) *A general normal form for every type of singularity appearing in* Mult.

It is the purpose of this article to study the singularity type of Mult when P and Q have a common double root. First, we give a new normal form for the simplest case that allows us to describe the geometry of its singularity type in the real and complex cases. Surprisingly, in the real case the critical set is not equivalent to, but still related to a well-known swallowtail singularity, typical of the cases where the greatest common divisor has only simple roots.

In the interesting paper [L-W] the Thom-Boardman symbol of the singularities of the mapping Mult at all points (P, Q) is computed. Our approach, following [Ch-LdM], is different: we search for a simple normal form and a complete topological description of the singularity. This objective looks difficult to achieve except in the simplest cases.

1. Known results.

The main result of [Ch-LdM] is the following:

Theorem 1. *For* $(P_0, Q_0) \in \mathrm{MP}(\mathbf{K}, n) \times \mathrm{MP}(\mathbf{K}, m)$*:*

(i) *The corank of the differential* $D\,\mathrm{Mult}(P_0, Q_0)$ *is the degree of* $\gcd(P_0, Q_0)$.

(ii) *In particular,* Mult *is a local diffeomorphism at* (P_0, Q_0) *if and only if* $\gcd(P_0, Q_0) = 1$.

(iii) *The mapping* Mult *is a* $(k+1)$*–swallowtail at* (P_0, Q_0) *for some positive integer* k *if, and only if,* $\deg\gcd(P_0, Q_0) = 1$*, the integer* k *being the maximum of the multiplicities in* P_0 *and* Q_0 *of their common root.*

(iv) *If* $\mathbf{K} = \mathbf{R}$*, the mapping* Mult *is a complex* $(k+1)$*–swallowtail at* (P_0, Q_0) *for some positive integer* k *if, and only if,* $\gcd(P_0, Q_0)$ *is an irreducible polynomial of degree* 2*,* k *being the maximum of the multiplicities in* P_0 *and* Q_0 *of their complex conjugate common roots.*

The proof consists in giving a simple normal form for such mappings. All these mappings are well-known and so is the general description of their singular and critical sets. A reduction lemma shows that the singularity type of Mult at a point (P_0, Q_0) splits into a product of the singularity types of the factors of the polynomials corresponding to the different roots:

Lemma 1. *The singularity type of* Mult *at a pair of polynomials with several common roots is the set-theoretical product of the singularity types ot* Mult *at each of the pairs consisting of the factors of the polynomials involving only one of those roots.*

This is because the multiplication of factors involving different roots is locally invertible by (ii) and so the product can be factored, multiplied separately and then multiplied together again, all the complementary multiplications being local bijections.

Another argument given in [Ch-LdM] can be formulated in general as follows:

Lemma 2. *Assume* $P_0 \in \mathrm{MP}(\mathbf{R}, 2k)$ *has no real roots and let* $P_0 = P_{01}\bar{P}_{01}$ *be a decomposition of* P_0 *such that* P_{01} *and* $\bar{P}_{01}$ *have no common roots. Then the mapping* $P_1 \mapsto P_1\bar{P}_1$ *is a diffeomorphism between a neighborhood of* P_{01} *in* $\mathrm{MP}(\mathbf{C}, k)$ *and a neighborhood of* P_0 *in* $\mathrm{MP}(\mathbf{R}, 2k)$.

This is because in a neighborhood of P_0 in $\mathrm{MP}(\mathbf{C}, 2k)$ every polynomial P can be written in a unique way as P_1P_2 with P_1, P_2 in neighborhoods of P_{01} and $\bar{P}_{01}$, respectively. When $P \in \mathrm{MP}(2k, \mathbf{R})$ then $P = \bar{P} = \bar{P}_1\bar{P}_2$. The uniqueness of the decomposition implies that $P_2 = \bar{P}_1$ and $P = P_1\bar{P}_1$ so the mapping $P \mapsto P_1$ is a local inverse of $P_1 \mapsto P_1\bar{P}_1$.

Also, in [Ch-LdM], Proposition 2, there are normal forms for all possible singularity types of Mult at pairs with only one root which is common. We still do not know how to use these normal forms to obtain a geometric description of the singularity types, so we looked for new normal forms in the cases we study.

2. Polynomials with common double roots.

We will start by describing the minimal case: two polynomials of degree 2 with one single root which is common and double in both of them. We will give a new normal form of the mapping Mult in the neighborhood of such a pair and a detailed description of its singularity type in the case $K = \mathbf{R}$. Section 2.4 treats the case of two real polynomials with a double common complex root.

In section 2.5 we give a new normal form for the case of two polynomials with only one root which is common, double in one of them and of multiplicity $k \geq 2$ in the other one.

In section 2.6 we will combine all the cases known to give a statement about pairs of polynomials whose greater common divisor has only simple and double roots.

2.1. **The minimal case for general K.** We consider now the case where both polynomials are of degree 2 with a common double root α which is in $\mathbf{K}$.

A change of variable $x = y + \alpha$ in those polynomials is an automorphism of $MP(\mathbf{K}, 2)$ that preserves the multiplication and gives us two polynomials in y whose common double root is zero. So we can assume that both $P_0(x)$ and $Q_0(x)$ are equal to x^2 and $\mathrm{Mult}(P_0, Q_0) = x^4$. A variation of the pair (P_0, Q_0) is given by the pair (P, Q) where $P(x) = x^2 + sx + t$ and $Q(x) = x^2 + ux + v$. Their product is then

$$P(x)Q(x) = x^4 + ux^3 + sx^3 + vx^2 + sux^2 + tx^2 + svx + tux + tv.$$

In terms of the parameters s, t, u, v the mapping is

$$F(s, t, u, v) = (u + s, v + su + t, sv + tu, tv).$$

This is a simple mapping of degree 2, but this fact does not give us an idea of its geometry. In several steps we will simplify this map through invertible changes of variables, obtaining a map of degree 4 that can be much better understood.

We begin by taking the first two components of F as new independent variables, through changes of coordinates:

$$s = s_1 - u, \quad t = t_1 - s_1 u - v + u^2$$

to obtain the equivalent map

$$F_1(s_1, t_1, u, v) \left(s_1, t_1, vs_1 - 2vu + ut_1 - s_1 u^2 + u^3, (t_1 - s_1 u - v + u^2)v\right).$$

To simplify the third component we use the changes of coordinates:

$$v = \frac{t_1}{2} - \frac{s_1 u}{2} + \frac{u^2}{2} - \frac{v_1}{2}, \quad u = u_1 + \frac{s_1}{2}$$

giving the new equivalent function

$$F_2(s_1, t_1, u_1, v_1) = \left(s_1, t_1, \frac{-s_1^3 + (4u_1^2 + 4t_1)s_1}{8} + u_1 v_1, \frac{(s_1^2 - 4u_1^2 - 4t_1 - 4v_1)^2}{64}\right).$$

Now we operate on the target space by substracting two functions of the first two components: $-\frac{s_1^3}{8} + \frac{s_1 t_1}{2}$ from the third component and $(-\frac{s_1^2}{8} + \frac{t_1}{2})^2$ from the fourth one.

Another change of variables finishes the simplification of the third coordinate:

$$v_1 = v_2 - \frac{s_1 u_1}{2},$$

$$F_3(s_1, t_1, u_1, v_2) = \left(s_1, t_1, u_1 v_2, \frac{u_1^4 + s_1 u_1 v_2 - v_2^2}{4} + \frac{(-3s_1^2 + 8t_1)u_1^2}{16}\right).$$

Now it is time to simplify the fourth coordinate through the substitutions

$$t_1 = \frac{t_2}{2} + \frac{3s_1^2}{8}, \quad s_1 = 4s_2,$$

$$F_4(s_2, t_2, u_1, v_2) = \left(4s_2, \frac{t_2}{2} + 6s_2^2, u_1 v_2, \frac{1}{4}u_1^4 + s_2 u_1 v_2 + \frac{1}{4}t_2 u_1^2 - \frac{1}{4}v_2^2\right).$$

We have messed with the first two components, but we can fix them back easily by acting on the target: divide the first component by 4 and then substract from the second one the function $6s_2^2$ of the first one. Then multiply the second component by 2 to make it again equal to t_2.

Finally, one can substract the product of the first and third components from the last one to obtain a remarkable simplification of the original mapping[1]:

$$F_5(s_2, t_2, u_1, v_2) = \left(s_2, t_2, u_1 v_2, \frac{1}{4}(u_1^4 + t_2 u_1^2 - v_2^2)\right).$$

Seen as an unfolding, we observe that the coordinate s_2 plays no role in the deformation of the mapping, so we can omit it from both sides and need only study the one-parameter unfolding, which in new coordinates can be written as:

$$f(a, x, y) = \left(a, xy, x^4 + ax^2 - y^2\right).$$

So Mult at (P_0, P_0) is equivalent to the *suspension* of f.

f is an unfolding of the mapping

$$f_0(x, y) = (xy, x^4 - y^2),$$

which for $\mathbf{K} = \mathbf{R}$ reminds us of the *square of a complex variable* mapping $(x, y) \mapsto (x^2 - y^2, 2xy)$ and, actually, the two mappings are topologically equivalent (see section 2.3).

The unfolding $f(a, x, y)$ is based on the deformation

$$f_a(x, y) = \left(xy, x^4 + ax^2 - y^2\right).$$

To obtain the singular points of f we compute its Jacobian matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & y & x \\ x^2 & 4x^3 + 2ax & -2y \end{pmatrix}$$

so the singular set is given by:

$$J = -4x^4 - 2ax^2 - 2y^2 = 0,$$

which gives also the singular set of f_a for each fixed a.

2.2. **The minimal case for $\mathbf{K} = \mathbf{C}$.** When $\mathbf{K} = \mathbf{C}$ it turns out that for all $a \neq 0$, the deformations f_a are equivalent: the substitutions $x = \sqrt{a}X, y = aY$, followed by multiplication of the components by adequate constants, gives $\left(XY, X^4 + X^2 - Y^2\right)$, which is the case $a = 1$.

However, f_0 is not equivalent to f_a for $a \neq 0$. The jacobian determinant of f_a is in general $-4x^4 - 2ax^2 - 2y^2$, so the origin is always a zero and a singular point of J. Under those circumstances, equivalent maps must have jacobians with equivalent 2-jets, but for $a = 0$ the 2-jet of the jacobian determinant is degenerate, which is not the case for $a \neq 0$. Also, the singular sets are not equivalent.

In this case f_a is, for all a, surjective and, generically, four-to-one, since the corresponding equations have always a solution and generically four different ones (cf. the computations in the next section).

[1]For the record, it will be useful for section 2.5 to take note now of the global substitution suffered by the coordinates s, t:

$$s = 2s_2 - u_1, \quad t = s_2^2 - s_2 u_1 + \frac{1}{2}u_1^2 + \frac{1}{4}t_2 + \frac{1}{2}v_2.$$

2.3. **The minimal case for $\mathbf{K} = \mathbf{R}$.** In the case $\mathbf{K} = \mathbf{R}$ at polynomials of degree 2 with two common double real roots, the computation in section 2.1 gives again that Mult is equivalent to the suspension of the mapping

$$f(a, x, y) = \left(a, xy, x^4 + ax^2 - y^2\right),$$

which is an unfolding of

$$f_0(x, y) = \left(xy, x^4 - y^2\right).$$

The mapping f_0 appears in Mather's classification of stable germs as being of the type $II_{2,4}$ with algebra $\mathbf{R}[[x, y]]/(xy, x^2 - y^4)$. See [M], p. 240. Its jacobian determinant is $-4x^4 - 2y^2$; so the origin is the only critical point of f_0.

Consider now f_0 as a (non-holomorphic) function of the complex variable $z = x + iy$. Since this function takes the same values for z and $-z$, it can be written as a function of z^2; so we can express f_0 as a composition

$$f_0(x, y) = g_0(x^2 - y^2, 2xy),$$

where g_0 is a differentiable function outside the origin. It follows from the computations below that g_0 is a homeomorphism of R^2 which is a diffeomorphism outside the origin.

As for f_a, its differentiable type now depends on the sign of a: For $a > 0$, the substitutions in the previous section show that f_a is equivalent to f_1. For $a < 0$ we have to use instead the substitutions $x = \sqrt{-a}X, y = aY$ to obtain in the same way that f_a is equivalent to f_{-1}.

By the same argument as in the case $\mathbf{K} = \mathbf{C}$ we obtain that f_0 is not equivalent to f_a for any $a \neq 0$.

We shall prove now that f_a is 2 to 1 outside the origin for $a \geq 0$ and surjective for all a:

If $a \geq 0$, take a point (x, y) and another point (x_1, y_1) with the same image:

$$f_a(x, y) = f_a(x_1, y_1);$$

so

$$xy = x_1 y_1, \quad x^4 + ax^2 - y^2 = x_1^4 + ax_1^2 - y_1^2.$$

If $x = 0$ then one of x_1, y_1 is zero.

If $x_1 = 0$ then $y_1 = \pm y$ and there is only one more point with the same image as (x, y).

If $x = 0$ and $y_1 = 0$ then the second equation gives

$$-y^2 = x_1^4 + ax_1^2,$$

which is only possible for $y = x_1 = 0$ and there is no other point with the same image as (x, y).

If $x \neq 0$ we can solve for y in the first equation and substitute its value in the second one. After multiplying by x^2 and factoring the resulting polynomial we get

$$(x - x_1)(x + x_1)(x^4 + x_1^2 x^2 + ax^2 + y_1^2) = 0.$$

The third factor must be positive since $x \neq 0$ and $a \geq 0$ so we must have $x_1 = \pm x$ and therefore $y_1 = \pm y$, with the same sign. So there is only one more point with the same image as (x, y). So f_a is 2-to-1 outside the origin.

To see that f_a is surjective for every a, we need to solve the equations

$$xy = \chi, \quad x^4 + ax^2 - y^2 = \eta$$

for a given $(\chi, \eta) \in \mathbf{R}^2$.

If $\chi = 0$ there is always a solution: $x = 0$, $y = \sqrt{-\eta}$ for $\eta \leq 0$; $y = 0$ and x a solution $x^4 + ax^2 = \eta$ for $\eta > 0$.

If $\chi \neq 0$ then x and y are non-zero. Then we can proceed as before: solve for y in the first equation, substitute its value in the second one and multiply by x^2. We obtain:

$$x^6 + ax^4 - \eta x^2 - \chi^2 = 0.$$

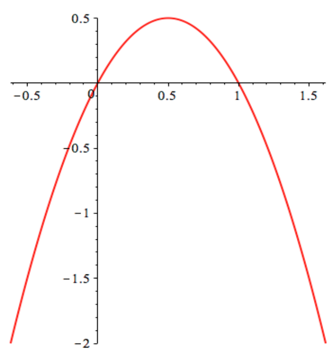
For $x = 0$ this polynomial is negative, while it tends to $+\infty$ when x tends to $+\infty$. Therefore there is a positive solution of this equation (and a negative one, too).

So we have shown that f_a is surjective for all a.

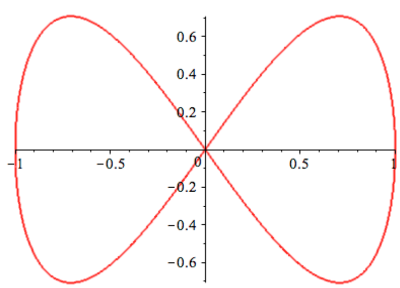
For $a > 0$ the jacobian determinant is again 0 only at the origin.

For $a < 0$ we can see the singular set as follows: Substituting $X = x^2$ and $Y = y^2$ in the jacobian determinant we obtain a parabola:

$$4X^2 - 2aX - 2Y = 0$$



The singular set is then the pre-image of the part of this parabola in the first quadrant under the mapping $(x, y) \mapsto (x^2, y^2)$ so it is the lemniscate:



This lemniscate is actually, up to linear changes of coordinates, the variant known as *Geromo's lemniscate*:

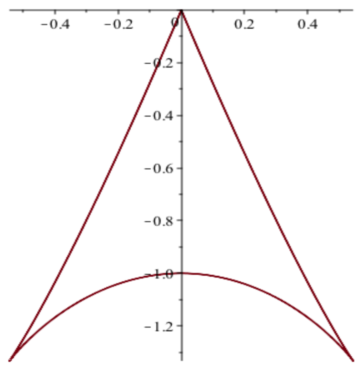
$$x^4 - x^2 + y^2 = 0.$$

A parametrization of this lemniscate is known (see [Wik]), which adapted to ours becomes

$$\gamma(\phi) = (\sqrt{-a/2}\,\cos(\phi), a\,\sin(\phi)\cos(\phi)/\sqrt{2})$$

as can be directly verified. We will use this parametrization to obtain the image of the singular set:

$$\sigma(\phi) = (a\sqrt{-a}\,\cos(\phi)^2\sin(\phi)/2, a^2\cos(\phi)^4/4 - a^2\cos(\phi)^2/2 - a^2\sin(\phi)^2\cos(\phi)^2/2)$$



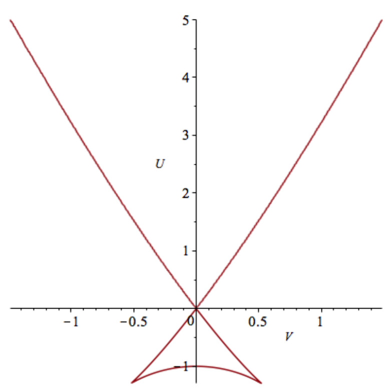
This figure has three singular points, two simple cusps (as can easily verified) at the lower level and a strange angle at the origin.

Let us call U, V the coordinates in the target plane containing this critical set. One can find the equations satisfied by the critical set by using the parameters $X = x^2, Y = y^2$ as before and eliminating the variables X, Y from the components of the mapping and the equation of the singular set. Alternatively, one can parametrize algebraically the intersections of the lemniscate with the four quadrants to carry out this elimination.

In any case, it can be verified directly that the points in the critical set satisfy the following equation:

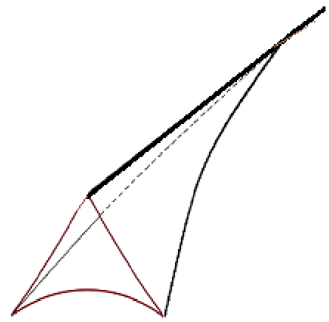
$$108\,a^3U^2 - 729\,U^4 + 486\,a\,U^2V + 27\,a^2V^2 + 108\,V^3 = 0$$

Drawing the zero set of this polynomial for a negative value of a, one obtains the following figure:

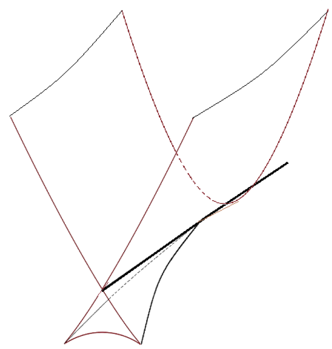


So the critical set of our mapping is just a semi-algebraic subset of this well-known swallowtail curve! (And this explains the angle).

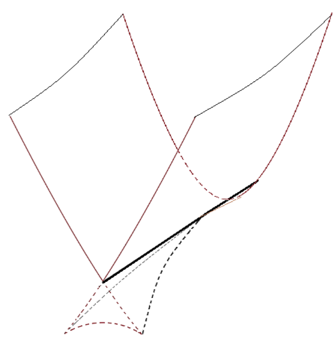
We can also draw the unfolding of the critical set by considering all values of a: for negative values of a it is the previous figure, where the triangular lower part shrinks to a single point when a approaches 0 and continues to be a single point when a is positive (we have highlighted the a axis):



Again, this is only the lower part of the swallowtail unfolding:



It is a curious fact that the complementary upper part of the swallowtail:



appears also as the singularity of a minimax solution of a Hamilton-Jacobi partial differential equation. See [Ch2] section 2.5 for the theory and [Ch1], appendix, for a specific example (the explicit figure appears in page 431).

2.4. **The minimal case of two real polynomials with a double complex root.** In this case we will have actually two conjugate double roots $\alpha, \bar{\alpha}$.

Here we apply Lemma 2 of section 1 to obtain that at such point Mult is equivalent to the suspension of the **complex** mapping

$$f : \mathbf{C}^2 \to \mathbf{C}^2,$$
$$f(a, x, y) = \left(a, xy, x^4 + ax^2 - y^2\right).$$

This is an unfolding of the mapping, in real variables:

$$f(x_1, x_2, y_1, y_2) = (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1, x_1^4 - 6x_1^2x_2^2 + x_2^4 - y_1^2 + y_2^2, 4x_1^3x_2 - 4x_1x_2^3 - 2y_1y_2).$$

2.5. **The case $P_0(x) = (x-a)^2$, $Q_0(x) = (x-a)^k$.** We give now a formula for the general case of two polynomials with a single root which is common, double in one of them and of degree $k \geq 2$ in the other one. So we can assume as before that $P_0(x) = x^2, Q_0(x) = x^k$. The method consists in applying the same changes of variables as in the minimal case $k = 2$ and is valid for any field $\mathbf{K}$. This gives a reasonable closed normal form, while other methods we have tried do not seem to produce one.

First, we illustrate it with small values of k. For $k = 3$ the mapping is given by the coefficients of $(x^2+sx+t)(x^3+ux^2+vx+w)$. After applying the sequence of changes of variable of section 2.1, adjusting factors and renaming the variables, one obtains the following normal form:

$$(a, b, x, y, w) \mapsto \big(a, b, xy + w, x^4 + bx^2 - y^2 + (2a - x)w, (4a^2 - 4ax + 2x^2 + b + 2y)w\big)\,.$$

One could also linearize the third component by using the coordinate $w_1 = xy + w$, thus obtaining a normal form which would be an unfolding of f_0. This would, however, increase the complexity of the expressions of the following components (without much hope of simplification).

Observe that here both parameters a, b appear in the formula, so there are no mute parameters. Also, that the new coordinate w appears only with degree 1 multiplied by factors of degrees 0 to 2 and increasing complexity. It does not seem easy to simplify them with new changes of coordinates.

The good news is that for greater values of k the coefficients of the new coordinates not only do not increase in complexity, but are actually exactly the same as for $k = 3$. It will be therefore convenient to use a short notation for them:

$$\sigma(a, x) = 2a - x, \quad \tau(a, b, x, y) = 4a^2 - 4ax + 2x^2 + b + 2y.$$

Then, for $k = 4$ we get by the same method the following map:

$$(a, b, x, y, w_3, w_4) \mapsto (a, b, xy + w_3, x^4 + bx^2 - y^2 + \sigma w_3 + w_4, \tau w_3 + \sigma w_4, \tau w_4).$$

For $w = 0$ we obtain essentially the normal form for $k = 2$. This shows that this mapping is a deformation of the mapping f_0 we studied before, and is the basis of the proof by induction of the general normal form for every k:

Let $P(x) = x^2+sx+t$ and $Q_k(x) = x^k+ux^{k-1}+vx^{k-2}+\Sigma_{i=3}^k w_i x^{k-i}$ and $F_k(x) = P(x)Q_k(x)$. Then, clearly

$$F_{k+1}(x) = xF_k(x) + P(x)w_{k+1}.$$

In terms of the coordinates $(s, t, u, v, w_3, \ldots, w_k, w_{k+1})$, this is expressed as

$$F_{k+1}(s, t, u, v, w_3, \ldots, w_k, w_{k+1}) = (F_k(s, t, u, v, w_3, \ldots, w_k), 0) + (0, \ldots, 0, w_{k+1}, w_{k+1}s, w_{k+1}t).$$

Passing to the coordinates (s_2, t_2, u_1, v_2) as in section 2.1, we obtain

$$\begin{aligned} &F_{k+1}(s_2, t_2, u_1, v_2, w_3, \ldots, w_k, w_{k+1}) = \\ &\big(F_k(s_2, t_2, u_1, v_2, w_3, \ldots, w_k), 0\big) + \\ &\qquad \Big(0, \ldots, 0, w_{k+1}, (2s_2 - u_1)w_{k+1}, \Big(s_2^2 - s_2u_1 + \tfrac{1}{2}u_1^2 + \tfrac{1}{4}t_2 + \tfrac{1}{2}v_2\Big)w_{k+1}\Big), \end{aligned}$$

since the coefficients of w_{k+1} are precisely the results of applying the coordinate changes of section 2.1 to the variables s, t (cf. footnote 1).

Starting with $k = 2$ this gives the inductive proof that the mapping Mult at P, P_k is equivalent to the mapping (in new coordinates):

$$\begin{aligned} &G_k(a, b, x, y, w_3, \ldots, w_k) = \\ &(a, b, xy, x^4 + bx^2 - y^2, 0, \ldots, 0) + \\ &\qquad (0, 0, \tau(a, b, x, y)w_1 + \sigma(a, x)w_2 + w_3, \ldots, \tau(a, b, x, y)w_k + \sigma(a, x)w_{k+1} + w_{k+2}), \end{aligned}$$

where σ and τ are as above and it is understood that $w_1 = w_2 = w_{k+1} = w_{k+2} = 0$.

As before, the components $\tau(a,b,x,y)w_i + \sigma(a,x)w_{i+1} + w_{i+2}$ can, in principle, be linearized for $i = 3$ to $k-2$ to present G as an unfolding of $f_0(x,y) = (xy, x^4 - y^2)$ with k parameters.

2.6. **The general result.** Putting together the previous results we can conclude that:

If $P_0 \in \mathrm{MP}(\mathbf{K}, n)$ and $Q_0 \in \mathrm{MP}(\mathbf{K}, m)$ are two polynomials such that their greatest common divisor has only simple and double roots then:

1)*If $\mathbf{K} = \mathbf{C}$ then at (P_0, Q_0),* Mult *is equivalent to the suspension of a product of complex swallowtails and complex mappings G_k.*

2)*If $\mathbf{K} = \mathbf{R}$ then at (P_0, Q_0),* Mult *is equivalent to the suspension of a product of real complex swallowtails and real and complex mappings G_k.*

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