Journal *of* Singularities
Volume 22 (2020), 373-384

Proc. of 15th International Workshop on Singularities, São Carlos, 2018
DOI: 10.5427/jsing.2020.22v

TOPOLOGY OF COMPLEMENTS TO REAL AFFINE SPACE LINE ARRANGEMENTS

GOO ISHIKAWA AND MOTOKI OYAMA

ABSTRACT. It is shown that the diffeomorphism type of the complement to a real space line arrangement in any dimensional affine ambient space is determined only by the number of lines and the data on multiple points.

1. INTRODUCTION

Let $\mathscr{A} = \{\ell_1, \ell_2, \dots, \ell_d\}$ be a real space line arrangement, or a configuration, consisting of affine d-lines in $\mathbb{R}^3$. The different lines $\ell_i, \ell_j (i \neq j)$ may intersect, so that the union $\cup_{i=1}^d \ell_i$ is an affine real algebraic curve of degree d in $\mathbb{R}^3$ possibly with multiple points. In this paper we determine the topological type of the complement $M(\mathscr{A}) := \mathbb{R}^3 \setminus (\cup_{i=1}^d \ell_i)$ of $\mathscr{A}$, which is an open 3-manifold. We observe that the topological type $M(\mathscr{A})$ is determined only by the number of lines and the data on multiple points of $\mathscr{A}$. Moreover we determine the diffeomorphism type of $M(\mathscr{A})$.

Set $D^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$, the n-dimensional closed disk. The pair $(D^i \times D^j, D^i \times \partial(D^j))$ with $i + j = n,\ 0 \leq i, 0 \leq j$, is called an n-dimensional *handle of index* j (see [17][1] for instance).

Now take one D^3 and, for any non-negative integer g, attach to it g-number of 3-dimensional handles $(D_k^2 \times D_k^1, D_k^2 \times \partial(D_k^1))$ of index 1 $(1 \leq k \leq g)$, by an attaching embedding

$$\varphi : \bigsqcup_{k=1}^{g} (D_k^2 \times \partial(D_k^1)) \to \partial(D^3) = S^2$$

such that the obtained 3-manifold

$$B_g := D^3 \bigcup\nolimits_{\varphi} (\bigsqcup\nolimits_{k=1}^{g} (D_k^2 \times D_k^1))$$

is orientable. We call B_g the 3-*ball with trivial* g-*handles of index* 1 (Figure 1.)

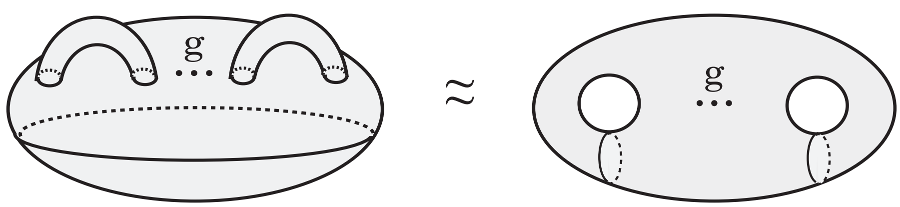

FIGURE 1. 3-ball with trivial g-handles of index 1.

Note that the topological type of B_g does not depend on the attaching map φ and is uniquely determined only by the number g. The boundary of B_g is the orientable closed surface Σ_g of genus g.

2010 *Mathematics Subject Classification.* Primary 57M25; Secondary 57R45, 57M15, 58K15.
Key words and phrases. trivial handle attachment, height function, space graph complement, open three-manifold.
The first author was supported by JSPS KAKENHI No.15H03615 and No.19K03458.

Let $\mathscr{A}$ be any d-line arrangement in $\mathbb{R}^3$. Let $t_i = t_i(\mathscr{A})$ denote the number of multiple points with multiplicity i, $i = 2, \dots, d$. The vector $(t_d, t_{d-1}, \dots, t_2)$ provides a degree of degeneration of the line arrangement $\mathscr{A}$. Set $g := d + \sum_{i=2}^{d}(i-1)t_i$. In this paper we show the following result:

Theorem 1.1. *The complement $M(\mathscr{A})$ is homeomorphic to the interior of* 3*-ball with trivial g-handles of index* 1.

Corollary 1.2. *$M(\mathscr{A})$ is homotopy equivalent to the bouquet $\bigvee_{k=1}^{g} S^1$.*

The above results are naturally generalised to any line arrangements in $\mathbb{R}^n (n \geq 3)$.

Let $\mathscr{A} = \{\ell_1, \ell_2, \dots, \ell_d\}$ be a line arrangement in $\mathbb{R}^n$ and set $M(\mathscr{A}) := \mathbb{R}^n \setminus (\cup_{i=1}^{d} \ell_i)$. Again let t_i denote the number of multiple points of $\mathscr{A}$ of multiplicity i, $i = 2, \dots, d$. Set $g := d + \sum_{i=2}^{d}(i-1)t_i$. Then we have

Theorem 1.3. *$M(\mathscr{A})$ is homeomorphic to the interior of n-ball B_g with trivially attached g-handles of index $n-2$.*

Thus we see that the topology of complements of real space line arrangements is completely determined by the combinational data, the *intersection poset* in particular. Recall that the intersection poset $P = P(\mathscr{A})$ is the partially ordered set which consists of all multiple points, the lines themselves $\ell_1, \ell_2, \dots, \ell_d$ and $T = \mathbb{R}^n$ as elements, endowed with the inclusion order. Then the number t_i is recovered as the number of minimal points x such that $\#\{y \in P \mid x < y, y \neq T\} = i$ and d as the number of maximal points of $P \setminus \{T\}$.

Corollary 1.4. *$M(\mathscr{A})$ is homotopy equivalent to the bouquet $\bigvee_{k=1}^{g} S^{n-2}$.*

In particular $M(\mathscr{A})$ is a minimal space, i.e. it is homotopy equivalent to a *CW* complex such that the number of i-cells is equal to its i-th Betti number for all $i \geq 0$.

Even for semi-algebraic open subsets in $\mathbb{R}^n$, homotopical equivalence does not imply topological equivalence in general. However we see this is the case for complements of real affine line arrangements, as a result of Theorem 1.3 and Corollary 1.4.

By the uniqueness of smoothing of corners, and by careful arguments at all steps of the proof of Theorem 1.3, we see that Theorem 1.3 can be proved in differentiable category.

Theorem 1.5. *$M(\mathscr{A})$ is diffeomorphic to the interior of n-ball B_g with trivially attached g-handles of index $n-2$.*

Note that the relative classification problem of line arrangements $(\mathbb{R}^n, \cup_{i=1}^{d} \ell_i)$ is classical but far from being solved ([6] for instance). Moreover there is a big difference in differentiable category and topological category. In fact even the local classification near multiple points of high multiplicity i, $i \geq n+2$ has moduli in differentiable category while it has no moduli in topological category. The classification of complements turns to be easier and simpler as we observe in this paper.

The real line arrangements on the plane $\mathbb{R}^2$ is one of classical and interesting subjects to study. It is known or easy to show that the number of connected components of the complement to a real plane line arrangement is given exactly by $1+g$ using the number $g = d + \sum_{i=2}^{d}(i-1)t_i$. This can be derived from Corollary 1.4 by just setting $n = 2$. For example, it can be shown from known combinatorial results for line arrangements on projective plane (see [4] for instance). In fact we prove it using our method in the process of the proof of Theorem 1.3. Therefore Theorem 1.3 and Corollary 1.4 are regarded as a natural generalisation of the classical fact.

Though our object in this paper is the class of real affine line arrangements, it is natural to consider also real projective line arrangements consisting of projective lines in the projective space $\mathbb{R}P^n$, or corresponding real linear plane arrangements consisting of 2-dimensional linear subspaces in $\mathbb{R}^{n+1}$. However the topology of complements in both cases are not determined, in general, by the intersection posets, which are defined similarly to the affine case. In fact there exists an example of pairwise transversal linear plane arrangements $\mathscr{B}$ and $\mathscr{B}'$ in $\mathbb{R}^4$ with $d=4$ such that the complements $M(\mathscr{B})$ and $M(\mathscr{B}')$ have non-isomorphic cohomology algebras and therefore they are not homotopy equivalent, so, not homeomorphic to each other ([19], Theorem 2.1).

A linear plane arrangement in $\mathbb{R}^4$ is pairwise transverse if and only if the corresponding projective line arrangement is non-singular (without multiple points) in $\mathbb{R}P^3$. Non-singular line arrangements in $\mathbb{R}P^3$, which are called skew line configurations, are studied in details (see [6, 13, 15, 16] for instance). Moreover, the topology of non-singular real algebraic curves in $\mathbb{R}P^3$ is studied, related to Hilbert's 16th problem, by many authors (see [8] for instance). Also refer to the surveys on the study of real algebraic varieties ([5, 14]).

It is natural to consider also complex line arrangements in $\mathbb{C}^n = \mathbb{R}^{2n}$. The topology of complex subspace arrangements in $\mathbb{C}^n$, in particular, homotopy types of them is studied in detail (see [10, 19] for instance). Then it is known that the intersection poset turns to have more information in complex cases than in real cases. Refer to [12, 20], for instance, on the theory on the homotopy types of complements for general subspace arrangements.

In §2, we define the notion of trivial handle attachments clearly. In §3, we show Theorem 1.3 and Theorem 1.5 in parallel, using an idea of stratified Morse theory ([3]) in a simple situation. We then realize a difference of topological features between the complements to line arrangements and to knots, links, tangles or general space graphs (Remark 3.8). In the last section, related to our results, we discuss briefly the topology of real projective line arrangements and real linear plane arrangements.

The authors thank Professor Masahiko Yoshinaga for his valuable suggestion to turn authors' attention to real space line arrangements. They thank also an anonymous referee for his/her valuable comments.

2. Trivial handle attachments

First we introduce the local model of trivial handle attachments.

Let $j<n$. Let $S^j \subset \mathbb{R}^n$ be the sphere defined by $x_1^2+\cdots+x_j^2+x_n^2=1, x_{j+1}=0,\dots,x_{n-1}=0$, and $\partial(D^j)=S^{j-1}=S^j\cap\{x_n=0\}$. Let $e_\ell\in\mathbb{R}^n$ be the vector defined by $(e_\ell)_i=\delta_{\ell i}$. Then define an embedding $\widetilde{\Phi}: D^{n-j}\times S^j\to\mathbb{R}^n$ by

$$\widetilde{\Phi}(t_1,\dots,t_{n-j-1},t_{n-j},x) := x+t_1e_{n-1}+\cdots+t_{n-j-1}e_{j+1}+t_{n-j}x,$$

which gives a tubular neighbourhood of S^j in $\mathbb{R}^n$. Set

$$\varphi_{\mathrm{st}} := \widetilde{\Phi}|_{D^{n-j}\times\partial(D^j)} : D^{n-j}\times S^{j-1}\to\mathbb{R}^{n-1}\subset\mathbb{R}^n,$$

which gives a tubular neighbourhood of S^{j-1} in $\mathbb{R}^{n-1}=\{x_n=0\}$. We call φ_{st} the *standard* attaching map of the handle of index j. Note that the embedding φ_{st} extends to the *standard* handle $\Phi: D^{n-j}\times D^j\to\mathbb{R}^n$, which is defined by

$$\Phi(t_1,\dots,t_{n-j-1},t_{n-j},x_1,\dots,x_j) := \widetilde{\Phi}\left(t_1,\dots,t_{n-j-1},t_{n-j},x_1,\dots,x_j,0,\dots,0,\sqrt{1-\textstyle\sum_{i=1}^j x_i^2}\right),$$

attached to $\{x_n\le 0\}$ along φ_{st}.

Let M be a topological (resp. differentiable) n-manifold with a *connected* boundary ∂M.

Let $p\in\partial M$. A coordinate neighbourhood (U,ψ), $\psi: U\to\psi(U)\subset\mathbb{R}^{n-1}\times\mathbb{R}$ around p in M is called *adapted* if $\psi: U\to\mathbb{R}^n$ is a homeomorphism of U and $\psi(U)\cap\{x_n\le 0\}$ which maps $U\cap\partial M$ into $\mathbb{R}^{n-1}=\{x_n=0\}$.

Now we consider an attaching of several handles of index j to M along ∂M. We call a handle attaching map $\varphi: \bigsqcup_{k=1}^{\ell}(D_k^{n-j}\times\partial(D_k^j))\to\partial M$ *trivial* if there exist disjoint adapted coordinate neighbourhoods

$(U_1,\psi_1),\dots,(U_\ell,\psi_\ell)$ on M such that $\varphi(D_k^{n-j}\times\partial(D_k^j))\subset U_k$ and $\psi_k\circ\varphi : D_k^{n-j}\times\partial(D_k^j)\to\mathbb{R}^{n-1}\times\mathbb{R}$ is the standard attachment for $k=1,\dots,\ell$. (Figure 2)

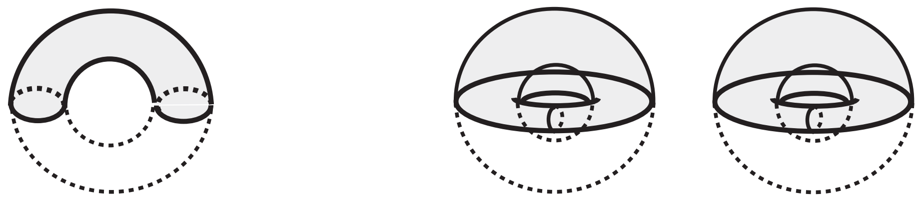

FIGURE 2. Trivial handle attachments: the cases $n=3, j=1, \ell=1$ and $n=4$, j = 2, $\ell=2$.

Then $M\cup_\varphi\left(\bigsqcup_{k=1}^{\ell}(D_k^{n-j}\times D_k^j)\right)$ is called the manifold obtained from M by attaching standard handles and the topological type of M does not depend on the attaching map φ but depends only on j and ℓ. Moreover if M is a differentiable manifold, then the diffeomorphism type of the attached manifold is uniquely determined by the smoothing or straightening of corners (see Proposition 2.6.2 of [17] for instance). Note that the diffeomorphism type of the interior does not change by the smoothing.

Note that, if φ is a trivial handle attaching map, then $\varphi|_{0\times\partial(D_k^j)} : 0\times\partial(D_k^j)\to\partial M$ is unknotted and $\varphi|_{\bigsqcup_{k=1}^{\ell}(0\times\partial(D_k^j))} : \bigsqcup_{k=1}^{\ell}(0\times\partial(D_k^j))\to\partial M$ is unlinked (see Figure 4). Therefore we can slide the trivial attachment mapping $\bigsqcup_{k=1}^{\ell}(D_k^{n-j}\times\partial(D_k^j))$ to an embedding into a disjoint union to an arbitrarily small neighbourhoods of any disjoint ℓ number points on ∂M up to isotopy (cf. Homogeneity Lemma [9]).

Remark 2.1. The assumption that ∂M is connected is essential. For example, let

$$M=\{x\in\mathbb{R}^n \mid -1\le x_n\le 1\}.$$

Then we have at least two non-homeomorphic spaces by different attachments of two trivial handles of index 1 (Figure 3).

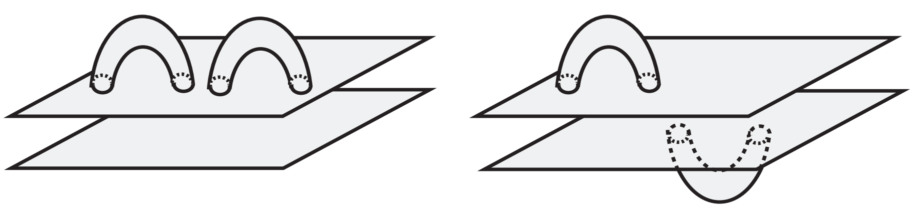

FIGURE 3. Non-homeomorphic attachments of trivial handles $n=3, j=1, \ell=2$.

We see that iterative trivial attachments gives a homeomorphic (resp. differentiable) manifold to the manifold obtained by the simultaneous trivial attachments.

Lemma 2.2. *Let M' be a topological (resp. differentiable) n-manifold with connected boundary $\partial M'$. Suppose M' is homeomorphic (diffeomorphic) to a space $M_1 := M\cup_\varphi\left(\bigsqcup_{k=1}^{\ell}(D_k^{n-j}\times D_k^j)\right)$ obtained, from a topological (differentiable) manifold M with connected boundary, by attaching k number of trivial* *handles of index j. Then the space $M_2 := M' \cup_{\varphi'} \left(\bigsqcup_{k=\ell+1}^{\ell+m}(D_k^{n-j} \times D_k^j)\right)$ obtained from M' by attaching m number of trivial handles of index j is homeomorphic (diffeomorphic) to the space*

$$M_3 := M \cup_{\varphi''} \left(\bigsqcup_{k=1}^{\ell+m}(D_k^{n-j} \times D_k^j)\right)$$

obtained from M by attaching $\ell+m$ number of trivial handles of index j.

See Figure 4 for the case $j = 1$.

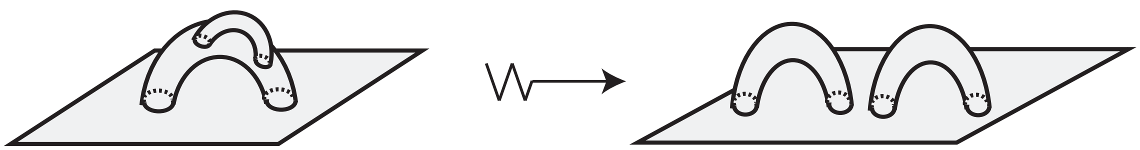

FIGURE 4. Sliding of trivial handle attachments.

Proof of Lemma 2.2. Let $f : M_1 \to M'$ be a homeomorphism (resp. a diffeomorphism). Then

$$f\big(\bigsqcup_{k=1}^{\ell}(D_k^{n-j} \times D_k^j)\big)$$

is not contained in $\partial M'$. Then we slide, up to isotopy, the attaching map $\varphi' : \bigsqcup_{k=\ell+1}^{\ell+m}(D_k^{n-j} \times \partial D_k^j) \to \partial M'$ to $\varphi''' : \bigsqcup_{k=\ell+1}^{\ell+m}(D_k^{n-j} \times \partial D_k^j) \to \partial M'$ such that

$$f\left(\varphi\left(\bigsqcup_{k=1}^{\ell}\left(D_k^{n-j} \times \partial D_k^j\right)\right)\right) \cap \varphi'''\left(\bigsqcup_{k=\ell+1}^{\ell+m}\left(D_k^{n-j} \times \partial D_k^j\right)\right) = \emptyset.$$

Consider $\varphi'' := \varphi \bigsqcup f^{-1} \circ \varphi''' : \bigsqcup_{k=1}^{\ell+m}(D_k^{n-j} \times \partial D_k^j) \to \partial M$. Then M_2 is homeomorphic (resp. diffeomorphic) to M_3. $\square$

3. AFFINE LINE ARRANGEMENTS

Let $n \geq 2$.

We consider line arrangements in $\mathbb{R}^n$ or more generally consider a subset X in $\mathbb{R}^n$ which is a union of finite number of closed line segments and half lines. Then X may be regarded as a finite graph (with compact and non-compact edges) embedded as a closed set in $\mathbb{R}^n$ (Figure 5). Here we admit vertices of valency 1.

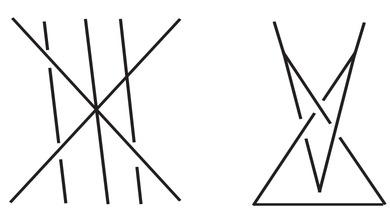

FIGURE 5. A line arrangement and a space graph

Take a unit vector $v \in S^{n-1} \subset \mathbb{R}^n$ and define the height function $h : \mathbb{R}^n \to \mathbb{R}$ by $h(x) := x \cdot v$ using Euclidean inner product. Choose v so that

(i) v is neither perpendicular to any line segments nor half lines in X.

(ii) For each c, the hyperplane $h(x) = c$ of level c contains at most one vertex of X.

Note that there exists a union Σ of finite number of great hyperspheres such that any unit vector in $S^{n-1} \setminus \Sigma$ satisfies the conditions (i) and (ii).

After a rotation of $\mathbb{R}^n$, we may suppose $h(x) = x_n$. We write $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1})$. Set $M = \mathbb{R}^n \setminus X$ and, for any $c \in \mathbb{R}$,

$$M_{\leq c} := \{x \in M \mid x_n \leq c\}, \quad M_{<c} := \{x \in M \mid x_n < c\}.$$

Let $V \subset X$ be the set of vertices of X. Set $V = \{u_1, u_2, \dots, u_r\}, c_i = h(u_i)$ and $C = h(V) = \{c_1, c_2, \dots, c_r\}$ with $c_1 < c_2 < \cdots < c_r$.

Though the following lemma is clear intuitively, we give a proof to make sure.

Lemma 3.1. *The topological (resp. diffeomorphism) type of $M_{\leq c}$ is constant on $c_i < c < c_{i+1}$ and the topological (diffeomorphism) type of $M_{<c}$ is constant on $c_i < c \leq c_{i+1}$, $i = 0, 1, \dots, r$, with $c_0 = -\infty, c_{r+1} = \infty$. Here $M_{<\infty}$ means M itself.*

Proof: First we treat the case $i < r$. Take a sufficiently large $R > 0$ such that

$$\{x \in X \mid c_i < x_n < c_{i+1}, \|x'\| > R/2\} = \emptyset.$$

Consider the cylinder

$$C := \{x \in \mathbb{R}^n \mid c_i < x_n < c_{i+1}, \|x'\| \leq R\}.$$

Then $\mathscr{C} := \{\mathrm{Int}C \setminus X, X \cap C, \partial C\}$ is a Whitney stratification of C. The function $h : C \to (c_i, c_{i+1})$ is proper and the restriction of h to each stratum is a submersion. Now we follow the standard method (the proof of Thom's first isotopy lemma [11, 7]) to show differentiable triviality of mappings. Note that the flow used in the proof of isotopy lemma is differentiable in each stratum. For any $\varepsilon > 0$, take a vector field η over (c_i, c_{i+1}) such that $\eta = 0$ on $(c_i, c_i + \varepsilon/2)$ and $\eta = \partial/\partial y$ on $(c_i + \varepsilon, c_{i+1})$, where y is the coordinate on $\mathbb{R}$. Then η lifts to a controlled vector field ξ over C such that ξ tangents to each stratum. We extend $\xi|_{\partial C}$ to $\{x \in \mathbb{R}^n \mid c_i < x_n < c_{i+1}, \|x'\| \geq R\}$ via the retraction $x = (x', x_n) \mapsto (\frac{1}{\|x'\|} R x', x_n)$ and to $\{x \in \mathbb{R}^n \mid x_n < c_i + \varepsilon/2\}$ by letting it 0, and we have an integrable vector field ξ on $\{x \in \mathbb{R}^n \mid x_n < c_{i+1}\}$. By integrating ξ, we have a homeomorphism of $M_{\leq c}$ and $M_{\leq c'}$ for any $c, c' \in (c_i, c_{i+1})$ and a diffeomorphism of $M_{<c}$ and $M_{<c'}$ for any $c, c' \in (c_i, c_{i+1}]$. Note that the differentiable flow of the vector field may not be defined through $x_n = c_{i+1}$ but it gives a diffeomorphism of $M_{<c}$ and $M_{<c_{i+1}}$.

Second we treat the case $i = r$. Consider the quadratic cone $\|x'\|^2 - R x_n^2 = 0$ in $\mathbb{R}^n$. Supposing $c_{r+1} > 0$ after a translation along x_n-axis in necessary, and taking R sufficiently large, we have that $X \cap \{x \in \mathbb{R}^n \mid c_{r+1} < x_n\}$ lies inside of the cone $\|x'\|^2 - R x_n^2 < 0$. Now set

$$D := \{x \in \mathbb{R}^n \mid c_{r+1} < x_n, \|x'\|^2 - R x_n^2 \leq 0\},$$

and consider the proper map $h : D \to (c_{r+1}, \infty)$ with the Whitney stratification

$$\mathscr{D} := \{\mathrm{Int}D \setminus X, X \cap D, \partial D\}.$$

For any $\varepsilon > 0$, take a (non-complete) vector field η over (c_{r+1}, ∞) such that $\eta = 0$ on $(c_{r+1}, c_{r+1} + \varepsilon/2)$ and $\eta = (1 + y^2)\partial/\partial y$ on (c_{r+1}, ∞). We lift η to a controlled vector filed ξ over D and then over $\mathbb{R}^n$. Then, using the integration of ξ, we have a diffeomorphism of $M_{\leq c}$ and $M_{\leq c'}$ for any $c, c' \in (c_i, c_{i+1})$, and a diffeomorphism of $M_{<c}$ and $M_{<c'}$ for any $c, c' \in (c_i, c_{i+1}]$. In particular we have that $M_{<c}$ for $c_{r+1} < c$ is diffeomorphic to M itself. $\square$

Remark 3.2. The topological (resp. diffeomorphism) type of $M_{\leq c}$ (resp. $h^{-1}(c)\setminus X$) is not necessarily constant at $c = c_{i+1}$.

We observe the topological change of $M_{<c}$ when c moves across a critical value c_i as follows:

Lemma 3.3. *Let u be a vertex of X and let $c = h(u)$. Let $s = s(u)$ denote the number of edges of X which are adjacent to u from above with respect to h.*

Then, for a sufficiently small $\varepsilon > 0$, the open set $M_{<c+\varepsilon}$ is diffeomorphic to the interior of

$$M_{\leq c-\varepsilon}\bigcup_{\varphi}\Big(\bigsqcup_{i=1}^{s-1}(D_i^2\times D_i^{n-2})\Big),$$

obtained by an attaching map

$$\varphi:\bigsqcup_{i=1}^{s-1}D^2\times\partial(D^{n-2})\longrightarrow h^{-1}(c-\varepsilon)\setminus X\ =\partial(M_{\leq c-\varepsilon})\subset M_{\leq c-\varepsilon},$$

of $(s-1)$ number of trivial handles of index $n-2$, provided $s\geq 1$.

In particular $M_{<c+\varepsilon}$ is diffeomorphic to $M_{<c-\varepsilon}$ if $s=1$.

If $s=0$ then $M_{<c+\varepsilon}$ is diffeomorphic to the interior of $M_{\leq c-\varepsilon}\bigcup_{\varphi}(D^1\times D^{n-1})$ obtained by an attaching map $\varphi: D^1\times\partial(D^{n-1})\to h^{-1}(c-\varepsilon)\setminus X$ of a (not necessarily trivial) handle of index $n-1$. (See Figure 6.)

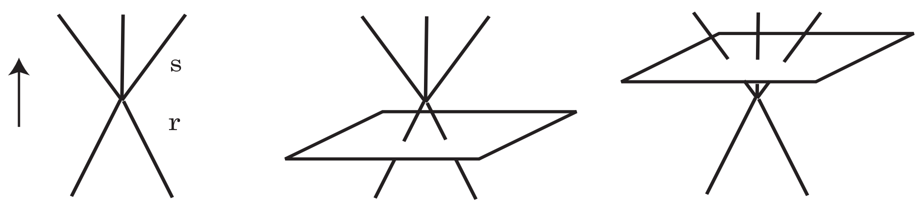

FIGURE 6. Topological bifurcations.

Remark 3.4. In the case $s=0$, the handle attachment is not necessarily trivial since the core of the attachment does not necessarily bounds a disk. (See Figure 13.)

Remark 3.5. Note that if $r = r(u)$ denotes the number of edges of X which are adjacent to p from *below* with respect to h, then the intersection $X\cap h^{-1}(c-\varepsilon)$ consists of r-points in the hyperplane $h^{-1}(c-\varepsilon)$ and thus $h^{-1}(c-\varepsilon)\setminus X$ is a punctured hyperplane by r-points.

Remark 3.6. Note that locally in a neighbourhood of each vertex u of X, the topological equivalence class of the germ of a generic height function $h:(\mathbb{R}^n,X,u)\to(\mathbb{R},c)$ is determined only by s and r, the numbers of branches. This can be shown by using Thom's isotopy lemma ([7]).

Proof of Lemma 3.3. For sufficiently small $0<\varepsilon<\varepsilon'$, $M_{<c-\varepsilon}\setminus M_{\leq c-\varepsilon'}$ is a space

$$\{x\in\mathbb{R}^n \mid c-\varepsilon' < h(x) < c-\varepsilon\}$$

deleted r-half-lines. We may suppose the intersection $X\cap h^{-1}(c-\varepsilon)$ lies on a line, up to a diffeomorphism of $M_{\leq c-\varepsilon}$. We delete r-small tubular neighbourhoods of the half-lines from the half space, then still we have a diffeomorphic space to $M_{<c-\varepsilon}\setminus M_{\leq c-\varepsilon'}$. Then we connect the r-holes by boring a sequence of canals without changing the diffeomorphism type of complements. See Figures 7 and 8. The boring a canal means, in general dimension, to delete $D^1\times D^{n-1}$ along the line segment connecting the holes.

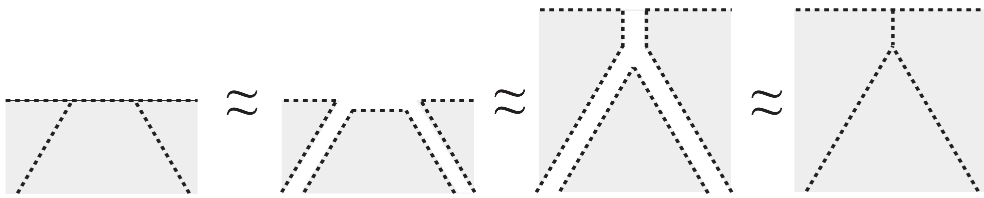

FIGURE 7. No topological changes of complements occur when $s = 1$.

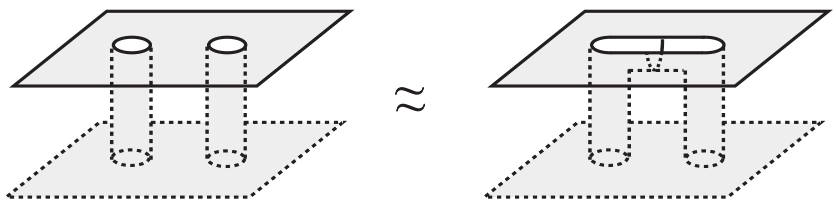

FIGURE 8. Boring a canal does not change the topology of ground.

First let $s = 1$. Then the resulting space is diffeomorphic to $M_{<c+\varepsilon} \setminus M_{\le c-\varepsilon'}$. The diffeomorphism is taken to be the identity on $M_{\le c-\varepsilon'}$ and it extends to a diffeomorphism between $M_{<c-\varepsilon}$ and $M_{<c+\varepsilon}$. This shows Lemma 3.3 in the case $s = 1$.

Next we teat the case $s = 2, r = 0$. The topological change from $M_{c-\varepsilon}$ to $M_{c+\varepsilon}$ is give by digging a tunnel, which is, equivalently, given by a handle attaching of index $n-2$. In fact, we examine the topological change of the complement to

$$\sqcup = \{(0, x_{n-1}, x_n) \in \mathbb{R}^n \mid (-2 \le x_{n-1} \le 2, x_n = 0) \text{ or } (x_{n-1} = -2, x_n \ge 0) \text{ or } (x_{n-1} = 2, x_n \ge 0)\},$$

in $\mathbb{R}^n$ when x_n goes across $x_n = c = 0$. Take the closed tube T of radius 1 of $\sqcup$. Then for the complement $M = \mathbb{R}^n \setminus T$, $M_{<\varepsilon}$ is diffeomorphic to the interior of the half space $\{x_n \le 0\}$ attached the handle

$$H = \{x \in \mathbb{R}^n \mid -1 \le x_{n-1} \le 1,\ \frac{1}{2} \le x_1^2 + \cdots + x_{n-2}^2 + x_n^2 \le 2,\ x_n \ge 0\}.$$

along

$$H \cap \{x_n \le 0\} = \{x \in \mathbb{R}^n \mid -1 \le x_{n-1} \le 1,\ \frac{1}{2} \le x_1^2 + \cdots + x_{n-2}^2 \le 2\}.$$

The pair $(H, H \cap \{x_n \le 0\})$ is diffeomorphic to the pair $(D^2 \times D^{n-2}, D^2 \times \partial D^{n-2})$, where the core $(0 \times D^{n-2}, \partial D^{n-2})$ corresponds to

$$\{x_1^2 + \cdots + x_{n-2}^2 + x_n^2 = 1, x_{n-1} = 0, x_n \ge 0\} \quad \text{and} \quad \{x_1^2 + \cdots + x_{n-2}^2 = 1, x_{n-1} = 0, x_n = 0\}.$$

Note that the latter bounds an $n-1$-dimensional disk $\{x_1^2 + \cdots + x_{n-2}^2 \le 1, x_{n-1} = 0, x_n = 0\}$, which does not touch the boundary $\partial M_{<\varepsilon}$. See Figures 9 and 10.

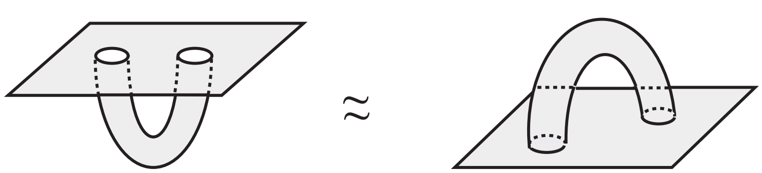

FIGURE 9. Digging a tunnel is same as bridging for the topology of ground.

The same argument works for any r. See Figure 10 for the case $s = 2, r = 2$. Note that complements to “X” and “H” are diffeomorphic. See Figures 10, 11 and 12.

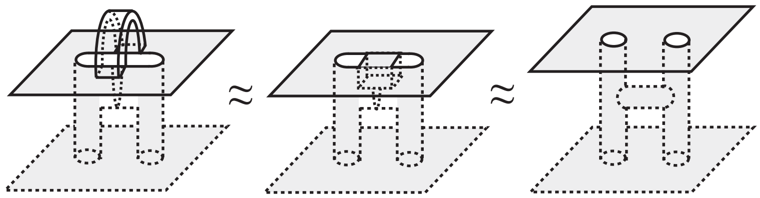

FIGURE 10. The case $s = 2, r = 2$.

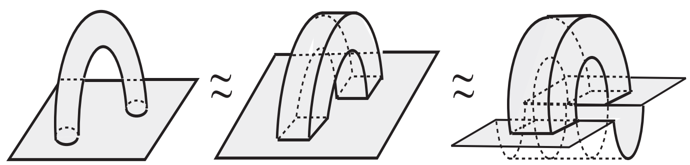

FIGURE 11. Trivial handle attachment and topological bifurcation.

In general, for any $s \geq 2$, the topological change is obtained by attaching trivial $s-1$ handles of index $n-2$. See Figure 12.

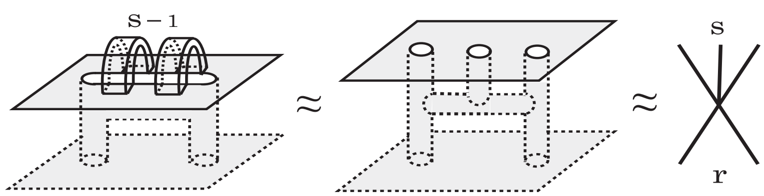

FIGURE 12. The case $s = 3, r = 2$.

In the case $s = 0$, contrarily to above, the change of diffeomorphism type is obtained by an attaching not necessarily trivial handle. See Figure 13.

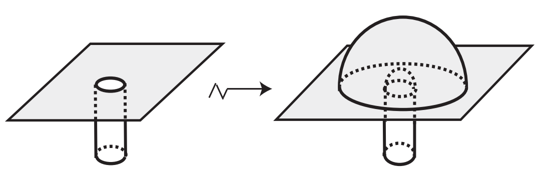

FIGURE 13. Topological change in the case $s = 0$.

When $n = 2$, the topological bifurcation occurs just as putting $s-1$ number of disjoint open disks. Thus we have Lemma 3.3. □

First let us apply Lemma 3.1 and Lemma 3.3 to the case $n = 2$.

For a $c \in \mathbb{R}$ of sufficiently large $|c|$, supposing a generic height function is given by $h = x_2$ as above. Then $M_{\leq c}$ (resp. $M_{<c}$) is diffeomorphic to the half plane $\{x_2 \leq c\}$ (resp. $\{x_n < c\}$ deleted d number of half lines. The number of connected components is equal to $1+d$. By passing a multiple point of multiplicity i, then by Lemma 3.3, we see that the number of connected components of $M_{\leq c}$ (resp. $M_{<c}$) increases exactly by $(i-1)$. Thus, after passing all multiple points, the number of connected components of $M_{<c}$, which is homeomorphic to $M(\mathscr{A})$, is given by $1+d+\sum_{i=2}^{d}(i-1)t_i$.

Proof of Theorem 1.5. For a $c \in \mathbb{R}$ with $c \ll 0$, the space $M_{\leq c}$ (resp. $M_{<c}$) is diffeomorphic to the half space $\{x_n \leq c\}$ (resp. $\{x_n < c\}$ deleted d number of half lines. By passing a multiple point of multiplicity i, for a sufficiently large c, the space $M_{\leq c}$ is obtained by attaching $i-1$ number of trivial handles of index $n-2$, by Lemma 3.3. After passing all multiple points, the space $M_{\leq c}$ is diffeomorphic to the space obtained by attaching $\sum_{i=2}^{d}(i-1)t_i$ number of trivial handles of index $n-2$ to the half space deleted d number of half lines. Then $M_{<c}$ is diffeomorphic to the interior of B_g with $g = d + \sum_{i=2}^{d}(i-1)t_i$. By Lemma 3.1, for $c \in \mathbb{R}$ with $0 \ll c$, $M_{<c}$ is diffeomorphic to $M(\mathscr{A})$. Hence we have Theorem 1.5. □

Proofs of Theorem 1.3 and Theorem 1.1. Theorem 1.3 follows from Theorem 1.5 and Theorem 1.1 follows from Theorem 1.3 by setting $n = 3$. □

Remark 3.7. Let X be a subset of $\mathbb{R}^n$ which is a union of finite number of closed line segments and half lines. Then similarly to the proof of Theorem 1.1 using Lemma 3.3, we see that, if there exists a height function $h : \mathbb{R}^n \to \mathbb{R}$ satisfying (i)(ii) such that $h|_X : X \to \mathbb{R}$ has no local maximum, then the complement $\mathbb{R}^n \setminus X$ is diffeomorphic to the interior of n-ball with trivially attached g-handles of index $n-2$, for some g. If $X \subset \mathbb{R}^n$ is compact, then any height function has a maximum, so non-trivial attachments may occur.

Remark 3.8. The knot complements have more information than line arrangement complements. For example, it is known that, for knots $K, K' \subset S^3$, if $S^3 \setminus K$ and $S^3 \setminus K'$ are homeomorphic, then the pairs (S^3, K) and (S^3, K') are homeomorphic ([2]). Taking account of it, consider $(\mathbb{R}^3, X)$ for a line arrangement $\mathscr{A} = \{\ell_1, \dots, \ell_d\}$ in $\mathbb{R}^3$ and $X := \bigcup_{i=1}^{d} \ell_i \subset \mathbb{R}^3$ and its one-point compactification $(S^3, \overline{X})$. Then the complement $S^3 \setminus \overline{X}$ is homeomorphic to $M(\mathscr{A})$ and to B_g, which depends only on the number

$$g = d + \sum_{i=1}^{d}(i-1)t_i,$$

while g does not determine the topological type of the pair $(S^3, \overline{X})$ in general.

4. Projective line and linear plane arrangements

Let $\widetilde{\mathscr{A}} = \{\widetilde{\ell}_1, \dots, \widetilde{\ell}_2, \dots, \widetilde{\ell}_d\}$ be a real projective line arrangement in the projective space $\mathbb{R}P^n$ and let $\mathscr{B} = \{L_1, L_2, \dots, L_d\}$ be the real linear plane arrangement in $\mathbb{R}^{n+2}$ corresponding to $\widetilde{\mathscr{A}}$. Then the complement $M(\mathscr{B})$ of $\mathscr{B}$ is homeomorphic to the link complement $S^n \cap M(\mathscr{B})$ times $\mathbb{R}_{>0}$, where S^n is a sphere in $\mathbb{R}^{n+1}$ centred at the origin. Moreover $S^n \cap M(\mathscr{B})$ is a double cover of $M(\widetilde{\mathscr{A}})$ for the corresponding projective line arrangement $\widetilde{\mathscr{A}}$ in $\mathbb{R}P^n$.

Take a projective hyperplane $H \subset \mathbb{R}P^n$ such that H intersects transversely to all lines $\widetilde{\ell}_i, 1 \leq i \leq d$, and that H does not pass through any multiple point of $\widetilde{\mathscr{A}}$. Then identify $\mathbb{R}P^n \setminus H$ with the affine space $\mathbb{R}^n$ and the affine line arrangement $\mathscr{A}$ obtained by setting $\ell_i := \widetilde{\ell}_i \setminus H \subset \mathbb{R}^n$. Take a ball

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq r\} \subset \mathbb{R}^n$$

for a sufficiently large radius r such that interior of D^n contains all multiple points of $\mathscr{A}$ and the boundary $\partial(D^n) = S^{n-1}$ intersects transversally to all lines $\ell_i, 1 \leq i \leq d$. Then the closure $\overline{U}$ of $U := \mathbb{R}P^n \setminus D^n$

is regarded as a tubular neighbourhood of H in $\mathbb{R}P^n$. The closure $\overline{U}$ is homeomorphic to the space $(S^{n-1}\times[-1,1])/\sim$, where $(x,t)\sim(-x,-t)$. Let $a_1,\dots,a_{2d}$ be disjoint $2d$ points in S^{n-1}.

Let $W_k^{n-1}\subset S^{n-1}$ be a sufficiently small open disk neighbourhood of $a_k, (1\le k\le 2d)$. Set

$$N := S^{n-1}\setminus W_k^{n-1} \quad \text{and} \quad \widetilde{N} := (N\times[-1,1])/\sim \ (\subset\ (S^{n-1}\times[-1,1])/\sim).$$

Then $\widetilde{N}$ is an n-dimensional manifold with boundary N, which is doubly covered by a "punctured shell" $N\times[-1,1]$ (see Figure 14).

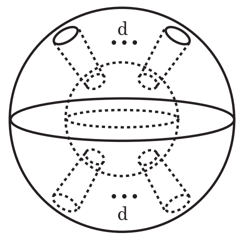

FIGURE 14. Punctured shell.

Thus we observe

Proposition 4.1. *The intersection $U\cap M(\widetilde{\mathscr{A}})$ is homeomorphic to the interior of $\widetilde{N}$. The complement $M(\widetilde{\mathscr{A}})\subset\mathbb{R}P^n$ is homeomorphic to the interior of $B_g\bigcup_\varphi\widetilde{N}$ for an attaching embedding $\varphi: N\to\partial(B_g)$. The homeomorphism class of $M(\widetilde{\mathscr{A}})$ is determined by the isotopy class of the embedding φ. The embedding φ is determined by the intersection of $M(\mathscr{A})$ and a hypersphere of sufficiently large radius in $\mathbb{R}^n$.*

Proof: We see that the intersection of $M(\mathscr{A})$ and a hypersphere of sufficiently large radius in $\mathbb{R}^n$ is homeomorphic to the sphere deleted $2d$-points. Then we have Proposition 4.1 by Theorem 1.3. □

REFERENCES

[1] R.E. Gompf, A.I. Stipsicz, *4-manifolds and Kirby calculus,* Graduate Studies in Math. **20**, Amer. Math. Soc., (1999). DOI: 10.1090/gsm/020

[2] C. McA. Gordon, J. Luecke, *Knots are determined by their complements,* Bull. Amer. Math. Soc., **20-1** (1989), 83–87. DOI: 10.1090/s0273-0979-1989-15706-6

[3] M. Goresky, R. MacPherson, *Stratified Morse theory,* Springer-Verlag, Berlin, Heidelberg, New York, (1988).

[4] B. Grünbaum, *Arrangements and Spreads,* Regional Conference Series in Math. **10**, Amer. Math. Soc. (1972).

[5] D.A. Gudkov, *The topology of real projective algebraic varieties,* Uspekhi Mat. Nauk, **29-4** (1974), 3–79; transl. Russ. Math. Surveys, **29-4** (1974), 1–79. DOI: 10.1070/rm1974v029n04abeh001288

[6] J. W. P. Hirschfeld, *A configuration of lines in three dimensions,* Proc. Edinburgh Math. Soc., **18-2** (1972), 105–123.

[7] J.N. Mather, *Stratifications and mappings,* Dynamical Systems, Academic Press, (1973), pp. 195–232.

[8] G. Mikhalkin, S. Orevkov, *Real algebraic knots and links of low degree,* J. Knot Theory Ramifications **25** (2016), 34 pages. DOI: 10.1142/S0218216516420104

[9] J.W. Milnor, *Topology from the differentiable view point,* Princeton Univ. Press (1997).

[10] P. Orlik, H. Terao, *Arrangements of Hyperplanes,* Grundlehren Math. Wiss. **300**, Springer-Verlag, New York, (1992).

[11] R.Thom, *Ensembles et morphismes stratifies,* Bull. Amer. Math. Soc., **75** (1969), 240–284. DOI: 10.1090/s0002-9904-1969-12138-5

[12] V.A. Vasiliev *Topology of plane arrangements and their complements,* Uspekhi Mat. Nauk, **56-2** (2001) 167–203; transl. Russ. Math. Surv., **56-2** (2001), 365–401. DOI: 10.1070/rm2001v056n02abeh000384

[13] O.Ya. Viro, *Topological problems on lines and points of three-dimensional space,* Dokl. Akad. Nauk SSSR **284-5** (1985), 1049–1052; transl. Soviet Math. Dokl. **32-2** (1985), 528–531.

[14] O.Ya. Viro, *Progress in the topology of real algebraic varieties over the last six years,* Uspekhi Mat. Nauk, **41-3** (1986), 45–67; transl. Russ. Math. Surveys, **41-3** (1986), 55–82. DOI: 10.1070/rm1986v041n03abeh003317

[15] O.Ya. Viro, Yu.V. Drobotukhina, *Configurations of skew-lines,* Algebra i Analiz **1-4** (1989), 222–246; transl. Leningrad Math. J. **1-4**(1990), 1027–1050.

[16] J. Viro, O. Viro, *Configurations of skew lines.* arχiv: math/0611374 [math.GT]

[17] C.T.C. Wall, *Differential Topology,* Cambridge studies in advanced math., **156**, Cambridge Univ. Press, (2016).

[18] M. Yoshinaga, *Minimality of hyperplane arrangements and basis of local system cohomology,* Singularities in geometry and topology, IRMA Lect. Math. Theor. Phys., **20**, Eur. Math. Soc., Zürich, (2012), pp. 345–362. DOI: 10.4171/118-1/16

[19] G.M. Ziegler, *On the difference between real and complex arrangements,* Math. Z. **212-1** (1993), 1–11.

[20] G.M. Ziegler, R.T. Zivaljevi, *Homotopy types of subspace arrangements via diagrams of spaces,* Math. Ann., **295**(1993), 527–548. DOI: 10.1007/bf01444901

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES, HOKKAIDO UNIVERSITY, SAPPORO 060-0810, JAPAN
Email address: ishikawa@math.sci.hokudai.ac.jp

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES, HOKKAIDO UNIVERSITY, SAPPORO 060-0810, JAPAN
Email address: oyama.motoki@gmail.com