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HORO-FLAT SURFACES ALONG CUSPIDAL EDGES IN THE HYPERBOLIC SPACE

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Abstract. There are two important classes of surfaces in the hyperbolic space. One of class consists of extrinsic flat surfaces, which is an analogous notion to developable surfaces in the Euclidean space. Another class consists of horo-flat surfaces, which are given by one-parameter families of horocycles. We use the Legendrian dualities between hyperbolic space, de Sitter space and the lightcone in the Lorentz-Minkowski 4-space in order to study the geometry of flat surfaces defined along the singular set of a cuspidal edge in the hyperbolic space. Such flat surfaces can be considered as flat approximations of the cuspidal edge. We investigate the geometrical properties of a cuspidal edge in terms of the special properties of its flat approximations.

1. Introduction

The tangent plane at a point of a regular surface is a flat approximation of the surface at a point, which is the basic idea to define the curvatures of the surface at the point. In this sense, the curvature at a point measures how far or near is the shape of the surface from a plane at the given point. On the other hand, the normal plane of a surface at a point also provides important information of the surface, for instance, the notion of normal section plays an important role in surface theory. One of the possible generalizations of this viewpoint consists in considering flat surfaces which are tangent or normal to the surface along a given curve. In [12, 18], osculating (and normal) flat surfaces along a curve on a surface in the Euclidean space are investigated, and with the help of these notions, the geometrical behaviour of a curve lying on a given surface was studied in [11, 16].

On the other hand, several articles on the differential geometry of surfaces with singularities have appeared during the two last decades [4, 7–14, 21, 25, 26, 29–34, 36]. An important class of singular surfaces is provided by the wave fronts, on which a smooth unit normal vector field of the surface even at a singular point exists. This means that a tangent and thus normal planes can be defined at any point of a wave front. One of the simplest and generic wave fronts is a *cuspidal edge*, whose set of singular points is a regular space curve. In [23], osculating and normal flat surfaces along the singular points of a cuspidal edge in the Euclidean space are defined and investigated.

In the present paper we analyze the geometry of cuspidal edges in the hyperbolic space. We point out that in the hyperbolic 3-space there exist two notions of flatness of surfaces [19, 22] other than that of flat Gaussian curvature surfaces. We shall consider extrinsic flat surfaces and

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horospherical flat surfaces. The notion of *extrinsic flat surfaces* is a direct analogy to that of flat surfaces in the Euclidean space. However, the notion of *horospherical flat surfaces* has completely different properties [22]. It is a one-parameter family of horocycles, namely, a surface swept by a horocycle. We call them *horocyclic surfaces*. We call each horocycle a *generating horocycle*. It is known that a horospherical flat surface is (at least locally) parametrized as a horocyclic surface [22, Theorem 4.4]. We introduce osculating and normal horospherical flat surfaces along a cuspidal edge and we call them *flat approximations*. The main purpose of this paper is to investigate the geometrical properties of a cuspidal edge in terms of the special properties of its flat approximations. We use in §2 the Legendrian duality theorem obtained in [15] in order to define the flat wave fronts as well as some invariants of cuspidal edges in the hyperbolic space. Moreover, certain families of functions of the cuspidal edge are introduced in §2 as the main tool in this paper. In §3, we quickly review the general theory of horocyclic surfaces given in [22]. The basic properties of the above families of functions are investigated in §4 and §5. In §6.3 we analyze special cuspidal edges depending on special properties of flat approximations. Finally, in §7 we make a remark on the global properties of a curve in the hyperbolic space from the view point of the Legendrian duality.

We shall assume throughout the whole paper that all the maps and manifolds are of class C^∞ unless the contrary is explicitly stated.

2. Flat fronts in the hyperbolic space

The hyperbolic space is realized as a spacelike pseudo-hypersphere with an imaginary radius in the Lorentz-Minkowski 4-space. The first author obtained in [15] a general theory on Legendrian dualities for pseudo-spheres in the Lorentz-Minkowski space leading to a commutative diagram between certain contact manifolds defined by the dual relations. Such dualities have proven to be useful in the study of the differential geometry of submanifolds of the pseudo-spheres and the results obtained have been described in several papers [2,5,17,22,24]. See also [6,27,28].

We observe that the flatness of a surface contained in a three dimensional pseudo-sphere is determined by the degeneration of the dual surface. By taking this fact into account, we investigate in the present paper the flat approximations of cuspidal edges contained in the hyperbolic 3-space.

Consider the Lorentz-Minkowski 4-space $\mathbb{R}^4_1 = (\mathbb{R}^4, \langle\ ,\ \rangle)$ with the pseudo-inner product $\langle\ ,\ \rangle = (-{+}{+}{+})$ and the following subspaces

$$H^3 = \{\boldsymbol{v} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{v}, \boldsymbol{v}\rangle = -1\}, \quad S^3_1 = \{\boldsymbol{v} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{v}, \boldsymbol{v}\rangle = 1\}, \quad LC^* = \{\boldsymbol{v} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{v}, \boldsymbol{v}\rangle = 0\}$$

that we call respectively, the *hyperbolic* 3-*space*, the *de Sitter* 3-*space* and the *lightcone*. We take now the submanifolds,

$$\Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \in H^3 \times S^3_1 \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0\},$$
$$\Delta_2 = \{(\boldsymbol{v}, \boldsymbol{w}) \in H^3 \times LC^* \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = -1\},$$

together with their corresponding canonical projections

$$\pi_{11} : \Delta_1 \to H^3, \quad \pi_{12} : \Delta_1 \to S^3_1, \quad \pi_{21} : \Delta_2 \to H^3, \quad \pi_{22} : \Delta_2 \to LC^*.$$

We can consider the 1-forms $\langle d\boldsymbol{v}, \boldsymbol{w}\rangle$ and $\langle \boldsymbol{v}, d\boldsymbol{w}\rangle$ on $\mathbb{R}^4_1 \times \mathbb{R}^4_1$, given by

$$\langle d\boldsymbol{v}, \boldsymbol{w}\rangle = -w_0 dv_0 + w_1 dv_1 + w_2 dv_2 + w_3 dv_3, \quad \langle \boldsymbol{v}, d\boldsymbol{w}\rangle = -v_0 dw_0 + v_1 dw_1 + v_2 dw_2 + v_3 dw_3,$$

for $\boldsymbol{v} = (v_0, v_1, v_2, v_3)$, $\boldsymbol{w} = (w_0, w_1, w_2, w_3) \in \mathbb{R}^4_1$. Clearly, the restrictions

$$\theta_{i1} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|_{\Delta_i}, \quad \theta_{i2} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|_{\Delta_i} \quad (i = 1, 2)$$

determine the same hyperplane field over Δ_i. Moreover, Δ_i is a contact manifold with the contact form $\theta_{i1}(=\theta_{i2})$, and π_{i1}, π_{i2} are Legendrian fibrations [15, Theorem 2.2]. There is a contact diffeomorphism $\Phi_{12}:\Delta_1\to\Delta_2$, given by $\Phi_{12}(\boldsymbol{v},\boldsymbol{w})=(\boldsymbol{v},\boldsymbol{v}\pm\boldsymbol{w})$ [15, page 330].

For a non-zero vector $\boldsymbol{v}\in\mathbb{R}^4_1$ and a real number c, we define a *hyperplane with pseudo normal* $\boldsymbol{v}$ by

$$HP(\boldsymbol{v},c)=\{\boldsymbol{x}\in\mathbb{R}^4_1\mid\langle\boldsymbol{x},\boldsymbol{v}\rangle=c\ \}.$$

We say that $HP(\boldsymbol{v},c)$ is a *spacelike*, a *timelike* or a *lightlike hyperplane* according $\boldsymbol{v}$ satisfies that $\langle\boldsymbol{v},\boldsymbol{v}\rangle<0$, $\langle\boldsymbol{v},\boldsymbol{v}\rangle>0$ or $\langle\boldsymbol{v},\boldsymbol{v}\rangle=0$ respectively. We then have three kinds of totally umbilical surfaces in H^3, given by the intersection of H^3 with the different hyperplanes of $\mathbb{R}^4_1$: A surface $H^3\cap HP(\boldsymbol{v},c)$ is said to be a *sphere*, an *equidistant surface* or a *horosphere* provided $HP(\boldsymbol{v},c)$ is a spacelike, a timelike or a lightlike hyperplane respectively. Moreover, an equidistant surface $H^3\cap HP(\boldsymbol{v},0)$ is called a *hyperbolic plane.*

Let $U\subset\mathbb{R}^2$ be an open subset. We say that two maps $f:U\to H^3$ and $g:U\to S^3_1$ are Δ_1-*dual* (one to each other) if the map $(f,g):U\to\Delta_1$ is isotropic [15]. Then a map $f:U\to H^3$ is said to be a *frontal* if it has a Δ_1-dual $g:U\to S^3_1$. Moreover, we say that $f:U\to H^3$ is a *front* provided it has a Δ_1-dual $g:U\to S^3_1$, such that $(f,g):U\to\Delta_1$ is an immersion. Analogous concepts for the Δ_2-duality can be introduced too.

A map $f:U\to H^3$ is said to be *flat* (or more precisely, *extrinsically flat*) if its Δ_1-dual $g:U\to S^3_1$ satisfies that $\operatorname{rank} dg_p\le 1$ for any $p\in U$. On the other hand, $f:U\to H^3$ is said to be *horospherically flat* (or *horo-flat*) provided its Δ_2-dual, $g:U\to LC^*$, satisfies that $\operatorname{rank} dg_p\le 1$ for any $p\in U$.

Let M^3 be a 3-dimensional manifold. A singular point p of the map-germ $f:(U,p)\to M^3$ is a *cuspidal edge* if f is $\mathcal{A}$-equivalent to the germ $(u_1,u_2)\mapsto(u_1,u_2^2,u_2^3)$ at 0. If a singular point p of $f:(U,p)\to M^3$ is a cuspidal edge, then we also say that the germ f is a *cuspidal edge.* Here we recall that two map-germs $f,g:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ are $\mathcal{A}$-equivalent provided there exist diffeomorphism germs $\phi:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ and $\Phi:(\mathbb{R}^3,0)\to(\mathbb{R}^3,0)$ such that $\Phi\circ f\circ\phi^{-1}=g$.

It is well-known that a cuspidal edge $f:(U,p)\to H^3$ is a front, namely, there exists a Δ_1 dual $g:(U,p)\to S^3_1$ of f such that (f,g) is an immersion (see [1, 25], for example). Since both, the singular set $S(f)$ of f and its image $f(S(f))$, are regular curves, we can take a local coordinate system (u_1,u_2) centered at p on U such that

$$S(f)=\{(u_1,u_2)|u_2=0\},\quad |\langle f_{u_1}(u_1,0),f_{u_1}(u_1,0)\rangle|=1,\quad\text{and}\quad\det(f_{u_1},f_{u_2u_2},g,f)>0.$$

We set $u_1=u$ and $\boldsymbol{\gamma}(u)=f(u,0)$ and define vector fields along $\boldsymbol{\gamma}$ as follows:

$$\begin{array}{rcl}\boldsymbol{t}(u)&=&f_u(u,0),\\ \boldsymbol{\nu}(u)&=&g(u,0),\\ \boldsymbol{b}(u)&=&\boldsymbol{\gamma}(u)\wedge\boldsymbol{t}(u)\wedge\boldsymbol{\nu}(u),\\ \boldsymbol{l}^{\varepsilon}_{\nu}(u)&=&\boldsymbol{\gamma}(u)+\varepsilon\boldsymbol{\nu}(u),\\ \boldsymbol{l}^{\varepsilon}_{b}(u)&=&\boldsymbol{\gamma}(u)+\varepsilon\boldsymbol{b}(u),\end{array}\tag{2.1}$$

where $\varepsilon=\pm1$. Here, for any $\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3\in\mathbb{R}^4_1$, we define a vector $\boldsymbol{x}_1\wedge\boldsymbol{x}_2\wedge\boldsymbol{x}_3$ by

$$\boldsymbol{x}_1\wedge\boldsymbol{x}_2\wedge\boldsymbol{x}_3=\begin{vmatrix}-\boldsymbol{e}_0&\boldsymbol{e}_1&\boldsymbol{e}_2&\boldsymbol{e}_3\\ x^1_0&x^1_1&x^1_2&x^1_3\\ x^2_0&x^2_1&x^2_2&x^2_3\\ x^3_0&x^3_1&x^3_2&x^3_3\end{vmatrix},$$

where $\{\boldsymbol{e}_0,\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ is the canonical basis of $\mathbb{R}^4_1$ and $\boldsymbol{x}_i=(x^i_0,x^i_1,x^i_2,x^i_3)$. Then $\{\boldsymbol{\gamma},\boldsymbol{t},\boldsymbol{\nu},\boldsymbol{b}\}$ is a pseudo-orthonormal frame satisfying $\det(\boldsymbol{\gamma},\boldsymbol{t},\boldsymbol{\nu},\boldsymbol{b})=1$, and $\{\boldsymbol{\gamma},\boldsymbol{t},\boldsymbol{l}^{\varepsilon}_{\nu},\boldsymbol{b}\}$, $\{\boldsymbol{\gamma},\boldsymbol{t},\boldsymbol{l}^{\varepsilon}_{b},\boldsymbol{\nu}\}$ are moving frames along $\boldsymbol{\gamma}$. We then have the following Frenet-Serret type formulae:

$$\begin{pmatrix}\boldsymbol{\gamma}'\\ \boldsymbol{t}'\\ \boldsymbol{\nu}'\\ \boldsymbol{b}'\end{pmatrix}=\begin{pmatrix}0&1&0&0\\ 1&0&\kappa_\nu^h&\kappa_b^h\\ 0&-\kappa_\nu^h&0&\kappa_t^h\\ 0&-\kappa_b^h&-\kappa_t^h&0\end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}\\ \boldsymbol{t}\\ \boldsymbol{\nu}\\ \boldsymbol{b}\end{pmatrix}, \tag{2.2}$$

$$\begin{pmatrix}\boldsymbol{\gamma}'\\ \boldsymbol{t}'\\ (\boldsymbol{l}_\nu^\varepsilon)'\\ \boldsymbol{b}'\end{pmatrix}=\begin{pmatrix}0&1&0&0\\ \mathfrak{h}_\nu^\varepsilon&0&\varepsilon\kappa_\nu^h&\kappa_b^h\\ 0&\mathfrak{h}_\nu^\varepsilon&0&\varepsilon\kappa_t^h\\ \varepsilon\kappa_t^h&-\kappa_b^h&-\varepsilon\kappa_t^h&0\end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}\\ \boldsymbol{t}\\ \boldsymbol{l}_\nu^\varepsilon\\ \boldsymbol{b}\end{pmatrix}, \tag{2.3}$$

and

$$\begin{pmatrix}\boldsymbol{\gamma}'\\ \boldsymbol{t}'\\ (\boldsymbol{l}_b^\varepsilon)'\\ \boldsymbol{\nu}'\end{pmatrix}=\begin{pmatrix}0&1&0&0\\ \mathfrak{h}_b^\varepsilon&0&\varepsilon\kappa_b^h&\kappa_\nu^h\\ 0&\mathfrak{h}_b^\varepsilon&0&-\varepsilon\kappa_t^h\\ -\varepsilon\kappa_t^h&-\kappa_\nu^h&\varepsilon\kappa_t^h&0\end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}\\ \boldsymbol{t}\\ \boldsymbol{l}_b^\varepsilon\\ \boldsymbol{\nu}\end{pmatrix}, \tag{2.4}$$

where

$$\begin{array}{rcl}\kappa_\nu^h&=&\langle\boldsymbol{\gamma}'',\boldsymbol{\nu}\rangle,\\ \kappa_b^h&=&-\det(\boldsymbol{\gamma},\boldsymbol{\gamma}',\boldsymbol{\gamma}'',\boldsymbol{\nu}),\\ \kappa_t^h&=&\det(\boldsymbol{\gamma},\boldsymbol{\gamma}',\boldsymbol{\nu},\boldsymbol{\nu}'),\\ \mathfrak{h}_\nu^\varepsilon&=&1-\varepsilon\kappa_\nu^h,\\ \mathfrak{h}_b^\varepsilon&=&1-\varepsilon\kappa_b^h.\end{array} \tag{2.5}$$

Here, we call κ_ν^h the *normal curvature*, κ_b^h the *geodesic curvature*, κ_t^h the *cuspidal torsion*, $\mathfrak{h}_\nu^\varepsilon$ the *horospherical normal curvature*, $\mathfrak{h}_b^\varepsilon$ the *horospherical geodesic curvature* of the cuspidal edge respectively. Since $\boldsymbol{b}=\boldsymbol{\gamma}\wedge\boldsymbol{t}\wedge\boldsymbol{\nu}$, the horospherical geodesic curvature corresponds to the singular curvature [34].

We denote $I=U\cap S(f)$ and introduce the following functions on $H^3\times I$:

$$\begin{array}{rcl}H_{l\nu}^\varepsilon(\boldsymbol{x},u)&=&\langle\boldsymbol{x},\boldsymbol{l}_\nu^\varepsilon(u)\rangle+1,\\ H_{lb}^\varepsilon(\boldsymbol{x},u)&=&\langle\boldsymbol{x},\boldsymbol{l}_b^\varepsilon(u)\rangle+1.\end{array} \tag{2.6}$$

One can also consider $H_\nu(\boldsymbol{x},u)=\langle\boldsymbol{x},\boldsymbol{\nu}(u)\rangle$ and $H_b(\boldsymbol{x},u)=\langle\boldsymbol{x},\boldsymbol{b}(u)\rangle$. Considering these functions is analogous notion in the Euclidean space [23]. See Appendix A for these cases.

We can take $\boldsymbol{x}$ as a parameter and regard these functions as parameter families of functions of u, then we can look at their corresponding discriminant set.

Let $g:(\mathbb{R},0)\to(\mathbb{R},0)$ be a function. For a manifold N and $p\in N$, a function

$$G:(N\times\mathbb{R},(p,0))\to(\mathbb{R},0)$$

is called an *unfolding* of g if $G(p,u)=g(u)$ holds. In this setting, we regard G as a parameter family of a function g. We assume that $g'(0)=0$ and define the set Σ_G and the *discriminant set* $\mathcal{D}_G$ of G as

$$\Sigma_G=\{(q,u)\in N\times\mathbb{R}\mid G(q,u)=G_u(q,u)=0\},$$
$$\mathcal{D}_G=\{q\in N\mid \text{ there exists } u\in\mathbb{R} \text{ such that } G(q,u)=G_u(q,u)=0\}.$$

If the map (G,G_u) is submersion at $(p,0)$, then Σ_G is a manifold. By definition, the discriminant set is the envelope of the family $\{q\in N|G(q,u)=0\}_{u\in\mathbb{R}}$ (see [3, Section 7] or [21, Section 5] for the general theory of unfoldings and their discriminant sets).

Now apply $(N,p)=(H^3,p)$ for $p\in H^3$ and $G=H_{l\nu}^\varepsilon,H_{lb}^\varepsilon$. Since $\boldsymbol{l}_\nu^\varepsilon$ and $\boldsymbol{l}_b^\varepsilon$ are lightlike, the discriminant sets $\mathcal{D}_{H_{l\nu}^\varepsilon}$ and $\mathcal{D}_{H_{lb}^\varepsilon}$ are the envelopes of families of horospheres. For a fixed

u, $\{\boldsymbol{x} \in H^3(-1) \mid H^{\varepsilon}_{l\nu}(\boldsymbol{x}, u) = 0\}$ are two horospheres tangent to the cuspidal edge at $\boldsymbol{\gamma}(u)$ and $\{\boldsymbol{x} \in H^3(-1) \mid H^{\varepsilon}_{lb}(\boldsymbol{x}, u) = 0\}$ are two horospheres normal to the cuspidal edge at $\boldsymbol{\gamma}(u)$, respectively. We investigate these functions and discriminant sets in Sections 4 and 5.

In what follows, we shall use the following abbreviation:

$$\kappa_\nu = \kappa^h_\nu, \quad \kappa_b = \kappa^h_b, \quad \kappa_t = \kappa^h_t.$$

3. Horocyclic surfaces

In this section, we give a quick review of general treatment of horocyclic surfaces. See [22] for detail. Let $\boldsymbol{g} : I \to H^3(-1)$ be a regular curve. Since $H^3(-1)$ is a Riemannian manifold, we can reparametrize $\boldsymbol{g}$ by the arc-length. Hence, we may assume that $\boldsymbol{g}(s)$ is a unit speed curve. Then the hyperbolic curvature κ_h and the hyperbolic torsion τ_h is defined by $\kappa_h(s) = |\boldsymbol{g}''(s) - \boldsymbol{g}(s)|$ and

$$\tau_h(s) = -\frac{\det(\boldsymbol{g}(s), \boldsymbol{g}'(s), \boldsymbol{g}''(s), \boldsymbol{g}'''(s))}{(\kappa_h(s))^2},$$

where $|\boldsymbol{v}| = \sqrt{|\langle \boldsymbol{v}, \boldsymbol{v}\rangle|}$ for $\boldsymbol{v} \in \mathbb{R}^4_1$. It can be shown that the curve $\boldsymbol{g}(s)$ satisfies the condition $\kappa_h(s) \equiv 0$ if and only if there exists a lightlike vector $\boldsymbol{c}$ such that $\boldsymbol{g}(s) - \boldsymbol{c}$ is a geodesic, where $\equiv$ stands for the equality holds identically. Such a curve is called an *equidistant curve*. Moreover $\boldsymbol{g}$ is called a *horocycle* if $\kappa_h(s) \equiv 1$ and $\tau_h(s) \equiv 0$. Let $\{\boldsymbol{\gamma}, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ be a pseudo-orthonormal basis of $\mathbb{R}^4_1$ which satisfies $\langle \boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle = -1$ and $\langle \boldsymbol{a}_i, \boldsymbol{a}_i\rangle = 1$ $(i = 1, 2, 3)$. Setting

$$\boldsymbol{g}(s) = \boldsymbol{\gamma} + s\boldsymbol{a}_1 + \frac{s^2}{2}(\boldsymbol{\gamma} + \boldsymbol{a}_2),$$

we see that $\kappa_h(s) \equiv 1$ and $\tau_h(s) \equiv 0$. Thus $s \mapsto \boldsymbol{g}(s)$ is a horocycle. Furthermore, let $\{\boldsymbol{\gamma}(u), \boldsymbol{a}_1(u), \boldsymbol{a}_2(u), \boldsymbol{a}_3(u)\}$ be a pseudo-orthonormal frame on an open interval I which satisfies $\langle \boldsymbol{\gamma}(u), \boldsymbol{\gamma}(u)\rangle = -1$ and $\langle \boldsymbol{a}_i(u), \boldsymbol{a}_i(u)\rangle = 1$ $(i = 1, 2, 3)$. Then the surface

$$F : (u, s) \mapsto \boldsymbol{\gamma}(u) + s\boldsymbol{a}_1(u) + \frac{s^2}{2}(\boldsymbol{\gamma}(u) + \boldsymbol{a}_2(u)) \tag{3.1}$$

is a one-parameter family of horocycles, namely, a horocyclic surface. We define fundamental invariants of horocyclic surfaces. Since a horocyclic surface (3.1) is determined by the frame $\{\boldsymbol{\gamma}(u), \boldsymbol{a}_1(u), \boldsymbol{a}_2(u), \boldsymbol{a}_3(u)\}$, the six functions $c_1(u), \dots, c_6(u)$ is defined by the following Frenet-Serre type equations:

$$\begin{pmatrix} \boldsymbol{\gamma}'(u) \\ \boldsymbol{a}_1'(u) \\ \boldsymbol{a}_2'(u) \\ \boldsymbol{a}_3'(u) \end{pmatrix} = \begin{pmatrix} 0 & c_1(u) & c_2(u) & c_3(u) \\ c_1(u) & 0 & c_4(u) & c_5(u) \\ c_2(u) & -c_4(u) & 0 & c_6(u) \\ c_3(u) & -c_5(u) & -c_6(u) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}(u) \\ \boldsymbol{a}_1(u) \\ \boldsymbol{a}_2(u) \\ \boldsymbol{a}_3(u) \end{pmatrix}. \tag{3.2}$$

Let α be a function of u, and set $\bar{F}(u, s) = F(u, s - \alpha(u))$. Then the images $\bar{F}(\mathbb{R} \times I)$ and $F(\mathbb{R} \times I)$ coincide. We set $\bar{c}_1, \dots, \bar{c}_6$ be the invariants defined by (3.2) of $\bar{F}(u, s)$. Then we have the equation

$$
\text{(3.3)}\qquad \begin{cases}
\bar{c}_1(u) &=& c_1(u)+\dfrac{\alpha(u)^2}{2}(c_4(u)-c_1(u))+\alpha(u)c_2(u)+\alpha'(u),\\
\bar{c}_2(u) &=& c_2(u)+\alpha(u)(c_4(u)-c_1(u)),\\
\bar{c}_3(u) &=& \left(1+\dfrac{\alpha(u)^2}{2}\right)c_3(u)+\alpha(u)c_5(u)+\dfrac{\alpha(u)^2}{2}c_6(u),\\
\bar{c}_4(u) &=& c_4(u)+\dfrac{\alpha(u)^2}{2}(c_4(u)-c_1(u))+\alpha(u)c_2(u)+\alpha'(u),\\
\bar{c}_5(u) &=& c_5(u)+\alpha(u)(c_3(u)+c_6(u)),\\
\bar{c}_6(u) &=& \left(1-\dfrac{\alpha(u)^2}{2}\right)c_6(u)-\alpha(u)c_5(u)-\dfrac{\alpha(u)^2}{2}c_3(u).
\end{cases}
$$

Then we see that $\bar{c}_1(u)-\bar{c}_4(u)=c_1(u)-c_4(u)$ and

$$
\text{(3.4)}\qquad \bar{c}_1(u)-\bar{c}_4(u)=\bar{c}_2(u)=0 \text{ if and only if } c_1(u)-c_4(u)=c_2(u)=0.
$$

Furthermore, the following proposition holds (see [22, Proposition 5.3]).

Proposition 3.1. *The horocyclic surface F is horo-flat if and only if $c_1(u)-c_4(u)=c_2(u)=0$ for any $u\in I$.*

4. Osculating horo-flat surfaces

In this section, we construct a parametrization of the discriminant set of $H^\varepsilon_{l\nu}$.

Let $f:(U,p)\to H^3$ be a cuspidal edge. As in Section 2, we assume $I=S(f)\cap U=\{(u,0)\}\cap U$ and set $\boldsymbol{\gamma}(u)=f(u,0)$. Then we have vector fields along $\boldsymbol{\gamma}$ as in (2.1). We consider invariants defined in (2.5). We assume $(\kappa_t,\mathfrak{h}^\varepsilon_\nu)(u)\neq(0,0)$ for any $u\in I$ unless otherwise stated.

4.1. **The discriminant set of $H^\varepsilon_{l\nu}$.** By differentiating (2.3), we have

(4.1)

$$
\begin{array}{rcl}
(\boldsymbol{l}^\varepsilon_\nu)' &=& \mathfrak{h}^\varepsilon_\nu\boldsymbol{t}+\varepsilon\kappa_t\boldsymbol{b},\\
(\boldsymbol{l}^\varepsilon_\nu)'' &=& \mathfrak{h}^\varepsilon_\nu\boldsymbol{\gamma}+\varepsilon(-\kappa_b\kappa_t-\kappa'_\nu)\boldsymbol{t}+(\kappa_\nu-\varepsilon\kappa_\nu^2+\varepsilon\kappa_t^2)\boldsymbol{\nu}+(\kappa_b-\varepsilon\kappa_\nu\kappa_b+\varepsilon\kappa'_t)\boldsymbol{b},\\
(\boldsymbol{l}^\varepsilon_\nu)''' &=& \varepsilon(-\kappa_b\kappa_t-2\kappa'_\nu)\boldsymbol{\gamma}\\
&& +\big(1-\kappa_b^2-\varepsilon\kappa_t\kappa'_b-2\varepsilon\kappa_b\kappa'_t-\varepsilon\kappa''_\nu+\varepsilon(-1+\kappa_b^2+\kappa_t^2)\kappa_\nu-\kappa_\nu^2+\varepsilon\kappa_\nu^3\big)\boldsymbol{t}\\
&& +(-\kappa_b\kappa_t+\kappa'_\nu-3\varepsilon\kappa_t\kappa'_t-3\varepsilon\kappa'_\nu\kappa_\nu)\boldsymbol{\nu}\\
&& +\big(-\varepsilon\kappa_b^2\kappa_t-\varepsilon\kappa_t^3-2\varepsilon\kappa_b\kappa'_\nu+\kappa'_b+\varepsilon\kappa''_t+(\kappa_t-\varepsilon\kappa'_b)\kappa_\nu-\varepsilon\kappa_t\kappa_\nu^2\big)\boldsymbol{b}.
\end{array}
$$

Since $\{\boldsymbol{\gamma},\boldsymbol{t},\boldsymbol{\nu},\boldsymbol{b}\}$ is a basis of $\mathbb{R}^4_1$, we can set $\boldsymbol{x}=x_\gamma\boldsymbol{\gamma}+x_t\boldsymbol{t}+x_\nu\boldsymbol{\nu}+x_b\boldsymbol{b}$. Then $H^\varepsilon_{l\nu}(\boldsymbol{x},u)=0$ if and only if $x_\gamma=\varepsilon x_\nu+1$. Moreover, $H^\varepsilon_{l\nu}(\boldsymbol{x},u)=(H^\varepsilon_{l\nu})_u(\boldsymbol{x},u)=0$ if and only if the equalities

$$
x_\gamma=\varepsilon x_\nu+1,\quad x_t=-\varepsilon\kappa_t s,\quad x_b=\mathfrak{h}^\varepsilon_\nu s
$$

hold for some $s\in\mathbb{R}$, under the assumption $(\kappa_t,\mathfrak{h}^\varepsilon_\nu)\neq(0,0)$. Since $\boldsymbol{x}\in H^3$, we have that $x_\nu=\varepsilon s^2(\kappa_t^2+(\mathfrak{h}^\varepsilon_\nu)^2)/2$. Thus $H^\varepsilon_{l\nu}(\boldsymbol{x},u)=(H^\varepsilon_{l\nu})_u(\boldsymbol{x},u)=0$ if and only if

$$
\boldsymbol{x}=\left(\frac{s^2}{2}(\kappa_t^2+(\mathfrak{h}^\varepsilon_\nu)^2)+1\right)\boldsymbol{\gamma}-\varepsilon\kappa_t s\boldsymbol{t}+\frac{\varepsilon s^2}{2}(\kappa_t^2+(\mathfrak{h}^\varepsilon_\nu)^2)\boldsymbol{\nu}+\mathfrak{h}^\varepsilon_\nu s\boldsymbol{b}
$$

for some $s\in\mathbb{R}$. Thus $\mathcal{D}_{H^\varepsilon_{l\nu}}$ is parameterized by

$$
(u,s)\mapsto\boldsymbol{x}=\boldsymbol{\gamma}+\Big(-\varepsilon\kappa_t\boldsymbol{t}+\mathfrak{h}^\varepsilon_\nu\boldsymbol{b}\Big)s+\frac{s^2}{2}\Big(\kappa_t^2+(\mathfrak{h}^\varepsilon_\nu)^2\Big)\boldsymbol{l}^\varepsilon_\nu.
$$

We set

$$
\overline{D^\varepsilon_l}=\frac{-\varepsilon\kappa_t\boldsymbol{t}+\mathfrak{h}^\varepsilon_\nu\boldsymbol{b}}{\sqrt{\kappa_t^2+(\mathfrak{h}^\varepsilon_\nu)^2}},
$$

and call it the *normalized* $\boldsymbol{b}$*-Darboux vector field*. By applying a parameter change

$$\tilde{s} = s\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2},$$

we obtain the following parameterization of $\mathcal{D}_{H_{l\nu}^\varepsilon}$

$$(u,s) \mapsto \boldsymbol{x} = \boldsymbol{\gamma} + \overline{D_l^\varepsilon} s + \frac{s^2}{2}\boldsymbol{l}_\nu^\varepsilon.$$

Since $|\overline{D_l^\varepsilon}| = 1$, for a fixed u, $s \mapsto \boldsymbol{\gamma} + \overline{D_l^\varepsilon} s + s^2 \boldsymbol{l}_\nu^\varepsilon/2$ is a horocycle, see §3. We also see that $\{\boldsymbol{\gamma}, \overline{D_l^\varepsilon}, \varepsilon\boldsymbol{\nu}, (\boldsymbol{l}_\nu^\varepsilon)'/|(\boldsymbol{l}_\nu^\varepsilon)'|\}$ is a pseudo-orthonormal frame of $\mathbb{R}^4_1$. Following §3, we set

$$\{\boldsymbol{\gamma}, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\} = \{\boldsymbol{\gamma}, \overline{D_l^\varepsilon}, \varepsilon\boldsymbol{\nu}, (\boldsymbol{l}_\nu^\varepsilon)'/|(\boldsymbol{l}_\nu^\varepsilon)'|\},$$

and

$$F_{l\nu}(u,s) = F_{l\nu}^\varepsilon(u,s) = \boldsymbol{\gamma} + \boldsymbol{a}_1 s + \frac{s^2}{2}(\boldsymbol{\gamma} + \boldsymbol{a}_2). \tag{4.2}$$

By definition, $F_{l\nu}$ is a Δ_2-dual of $\boldsymbol{l}_\nu^\varepsilon$. An example of the osculating horo-flat surface $F_{l\nu}$ of

$$f(u,v) = \Big(f_1(u,v), f_2(u,v), f_3(u,v), \sqrt{f_1(u,v)^2 + f_2(u,v)^2 + f_3(u,v)^2 - 1}\Big), \tag{4.3}$$

where $f_1(u,v) = 3 + u$, $f_2(u,v) = u^2/2 + v^2/2$, $f_3(u,v) = u^2/2 + uv^2/2 + v^3/2$ near $(0,0)$ is provided by Figure 1. We can now define invariants $c_{\nu,1}, \dots, c_{\nu,6}$ as in (3.2), namely,

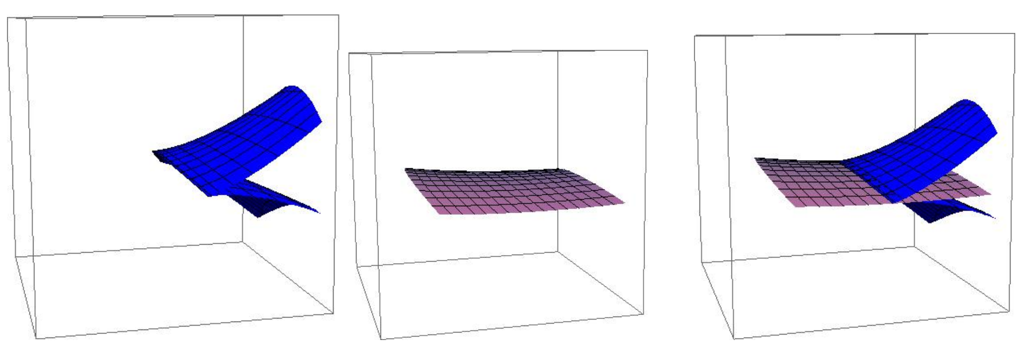

FIGURE 1. Cuspidal edge, $F_{l\nu}$ and the both surfaces together

$$\begin{pmatrix} \boldsymbol{\gamma}' \\ \boldsymbol{a}_1' \\ \boldsymbol{a}_2' \\ \boldsymbol{a}_3' \end{pmatrix} = \begin{pmatrix} 0 & c_{\nu,1} & c_{\nu,2} & c_{\nu,3} \\ c_{\nu,1} & 0 & c_{\nu,4} & c_{\nu,5} \\ c_{\nu,2} & -c_{\nu,4} & 0 & c_{\nu,6} \\ c_{\nu,3} & -c_{\nu,5} & -c_{\nu,6} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma} \\ \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix}. \tag{4.4}$$

It is not difficult to see that

$$
\begin{aligned}
c_{\nu,1} &= \frac{-\varepsilon\kappa_t}{\sqrt{\kappa_t^2+(\mathfrak{h}_\nu^\varepsilon)^2}},\\
c_{\nu,2} &= 0,\\
c_{\nu,3} &= \frac{\mathfrak{h}_\nu^\varepsilon}{\sqrt{\kappa_t^2+(\mathfrak{h}_\nu^\varepsilon)^2}},\\
c_{\nu,4} &= c_{\nu,1} = \frac{-\varepsilon\kappa_t}{\sqrt{\kappa_t^2+(\mathfrak{h}_\nu^\varepsilon)^2}},\\
c_{\nu,5} &= \frac{\delta_o^h}{\kappa_t^2+(\mathfrak{h}_\nu^\varepsilon)^2},\\
c_{\nu,6} &= \frac{-\varepsilon\kappa_\nu+\kappa_\nu^2+\kappa_t^2}{\sqrt{\kappa_t^2+(\mathfrak{h}_\nu^\varepsilon)^2}},\\
c_{\nu,3}+c_{\nu,6} &= \sqrt{\kappa_t^2+(\mathfrak{h}_\nu^\varepsilon)^2},
\end{aligned}
\tag{4.5}
$$

where we set

$$\delta_o^h = -\kappa_b\big((\mathfrak{h}_\nu^\varepsilon)^2+\kappa_t^2\big)+\varepsilon\kappa_t(\mathfrak{h}_\nu^\varepsilon)'-\varepsilon\mathfrak{h}_\nu^\varepsilon\kappa_t'.$$

By the condition $(\kappa_t,\mathfrak{h}_\nu^\varepsilon)\neq(0,0)$, we have $c_{\nu,3}+c_{\nu,6}\neq 0$. The invariant $c_{\nu,5}$ corresponds to the invariant δ of the Euclidean case (see [18, 23]). Note that if $(\kappa_t,\mathfrak{h}_\nu^\varepsilon)\equiv(0,0)$, then by (2.3), it holds that $(\boldsymbol{l}_\nu^\varepsilon)'\equiv 0$. This implies that $F_{l\nu}$ is a horosphere.

By (4.4) and (4.5), we see

$$
\begin{aligned}
F_{l\nu}' &= \boldsymbol{\gamma}'+\boldsymbol{a}_1's+\frac{s^2}{2}(\boldsymbol{\gamma}'+\boldsymbol{a}_2')\\
&= c_{\nu,1}s\boldsymbol{\gamma}+c_{\nu,1}\boldsymbol{a}_1+c_{\nu,1}s\boldsymbol{a}_2+\left(c_{\nu,3}+c_{\nu,5}s+\frac{s^2}{2}(c_{\nu,3}+c_{\nu,6})\right)\boldsymbol{a}_3
\end{aligned}
\tag{4.6}
$$

$$(F_{l\nu})_s = s\boldsymbol{\gamma}+\boldsymbol{a}_1+s\boldsymbol{a}_2, \tag{4.7}$$

where $' = \partial/\partial t$. We set

$$\lambda = (c_{\nu,3}+c_{\nu,6})s^2+2c_{\nu,5}s+2c_{\nu,3} \tag{4.8}$$

and

$$\eta = \partial u - c_{\nu,1}\partial s, \tag{4.9}$$

then we see $S(F_{l\nu})=\{(u,s)\in I\times\mathbb{R}\,|\,\lambda(u,s)=0\}$ by (4.6) and (4.7). We also see $\ker dF_{l\nu}=\langle\eta\rangle_{\mathbb{R}}$ on $S(F_{l\nu})$ holds. By (4.6), (4.7) and (4.5),

$$\boldsymbol{\nu}_l = \boldsymbol{a}_2 - s\boldsymbol{a}_1-\frac{s^2}{2}(\gamma+\boldsymbol{a}_2)\in S_1^3$$

is a Δ_1-dual of $F_{l\nu}$, and $F_{l\nu}+\boldsymbol{\nu}_l=\boldsymbol{\gamma}+\boldsymbol{a}_2$ is a Δ_2-dual of $F_{l\nu}$. Since the Δ_2-dual of $F_{l\nu}$ degenerates to a curve, $F_{l\nu}$ is a horo-flat surface. On the other hand, since $c_{\nu,1}-c_{\nu,4}\equiv c_{\nu,2}\equiv 0$, we also see that $F_{l\nu}$ is a horo-flat surface by Proposition 3.1. It follows that each of $F_{l\nu}$ is a horo-flat surface tangent to the cuspidal edge at any $\boldsymbol{\gamma}(u)$, so that we call it an *osculating horo-flat surface* (*along the cuspidal edge*).

4.2. **Singularities of osculating horo-flat surface.** We consider singularities of osculating horo-flat surface $F_{l\nu}$. A singular point p of the map-germ $f:(U,p)\to(\mathbb{R}^3,0)$ is a *swallowtail* if f is $\mathcal{A}$-equivalent to $(u,v)\mapsto(u,4v^3+2uv,3v^4+uv^2)$ at 0. A singular point p of f is a *cuspidal lip* (respectively, a *cuspidal beak*) if f is $\mathcal{A}$-equivalent to $(u,v)\mapsto(u,2v^3+\sigma u^2v,3v^4+\sigma u^2v^2)$ at 0 with $\sigma=+1$ (respectively, $\sigma=-1$). A singular point p of f is a *cuspidal cross cap* if f is $\mathcal{A}$-equivalent to $(u,v)\mapsto(u,v^2,uv^3)$ at 0.

Since $\boldsymbol{\nu}_l : U \to\in S^3_1$, the map $(F_{l\nu}, \boldsymbol{\nu}_l) : U \to \Delta_1$ is an immersion if and only if

$$\det(F'_{l\nu}, \nabla^F_\eta \boldsymbol{\nu}_l, \boldsymbol{\nu}_l, F_{l\nu})|_{S(F_{l\nu})} \neq 0, \tag{4.10}$$

where η is given by (4.9), and ∇^F_η be the canonical covariant derivative by η along F induced from the Levi-Civita connection on H^3. Since

$$\nabla^F_\eta \boldsymbol{\nu}_l = \alpha_0 \boldsymbol{\gamma} + \alpha_1 \boldsymbol{a}_1 + \alpha_2 \boldsymbol{a}_2 + (-c_{\nu,6} + sc_{\nu,5} + (s^2/2)(c_{\nu,3} + c_{\nu,6}))\boldsymbol{a}_3$$

($\alpha_0, \alpha_1, \alpha_2$ are some functions), the left hand side of (4.10) is $c_{\nu,1}(c_{\nu,3} + c_{\nu,6} - c_{\nu,1}\lambda)$. Thus by the assumption $c_{\nu,3} + c_{\nu,6} \neq 0$, the condition (4.10) is equivalent to $c_{\nu,1}(u) \neq 0$. Let Q be the discriminant of $\lambda = (c_{\nu,3} + c_{\nu,6})s^2 + 2c_{\nu,5}s + 2c_{\nu,3}$ (in (4.8)) regarding a quadratic equation of s:

$$Q(u) = c_{\nu,5}(u)^2 - 2c_{\nu,3}(u)(c_{\nu,3}(u) + c_{\nu,6}(u)) = c_{\nu,5}(u)^2 - 2\mathfrak{h}^\varepsilon_\nu(u).$$

If $Q < 0$, then there is no singular point. If $Q(u_0) = 0$, we set $s_0 = -c_{\nu,5}(u_0)/(c_{\nu,3}(u_0)+c_{\nu,6}(u_0))$. Then (u_0, s_0) is a singular point of $F_{l\nu}$.

Proposition 4.1. *Under the above notation, we have the following.*

(I) *If $Q(u_0) = 0$, the singular point (u_0, s_0) of $F_{l\nu}$ is a cuspidal edge if and only if*

$$c_{\nu,1}((c'_{\nu,3} + c'_{\nu,6})s^2 + 2c'_{\nu,5}s + 2c'_{\nu,3}) \neq 0$$

at u_0. Moreover, there are no swallowtails. The singular point (u_0, s_0) is a cuspidal lip if and only if $c_{\nu,1} \neq 0$, $(c'_{\nu,3} + c'_{\nu,6})s^2 + 2c'_{\nu,5}s + 2c'_{\nu,3} = 0$, and

$$\det\begin{pmatrix} (c''_{\nu,3} + c''_{\nu,6})s^2 + 2c''_{\nu,5}s + 2c''_{\nu,3} & 2(c'_{\nu,3} + c'_{\nu,6})s + 2c'_{\nu,5} \\ 2(c'_{\nu,3} + c'_{\nu,6})s + 2c'_{\nu,5} & 2(c_{\nu,3} + c_{\nu,6}) \end{pmatrix} > 0 \tag{4.11}$$

at (u_0, s_0). The singular point (u_0, s_0) is a cuspidal beak if and only if $c_{\nu,1} \neq 0$,

$$(c'_{\nu,3} + c'_{\nu,6})s^2 + 2c'_{\nu,5}s + 2c'_{\nu,3} = 0,$$

the left hand side of the determinant (4.11) is negative, and

$$\begin{aligned} &s^2(c''_{\nu,3} + c''_{\nu,6}) + 2sc''_{\nu,5} + c''_{\nu,3} - 2c'_{\nu,1}(s(c_{\nu,3} + c_{\nu,6}) + c_{\nu,5}) \\ &\qquad - 4c_{\nu,1}(s(c'_{\nu,3} + c'_{\nu,6}) + c'_{\nu,5}) + 2c^2_{\nu,1}(c_{\nu,3} + c_{\nu,6}) \neq 0 \end{aligned} \tag{4.12}$$

at (u_0, s_0). The singular point (u_0, s_0) is a cuspidal cross cap if and only if $c_{\nu,1} = 0$ and $c'_{\nu,1}((c'_{\nu,3} + c'_{\nu,6})s^2 + 2c'_{\nu,5}s + 2c'_{\nu,3}) \neq 0$ at u_0.

(II) *If $Q(u) > 0$, let s be the solution of $\lambda = 0$, namely,*

$$s = \frac{-c_{\nu,5} \pm \sqrt{c^2_{\nu,5} - 2c_{\nu,3}(c_{\nu,3} + c_{\nu,6})}}{c_{\nu,3} + c_{\nu,6}}. \tag{4.13}$$

Then (u, s) is a singular point. The singular point is a cuspidal edge if and only if $c_{\nu,1} \neq 0$ and

$$(c'_{\nu,3} + c'_{\nu,6})s^2 + 2c'_{\nu,5}s + 2c'_{\nu,3} - 2c_{\nu,1}((c_{\nu,3} + c_{\nu,6})s + c_{\nu,5}) \neq 0 \tag{4.14}$$

at (u, s). The singular point is a swallowtail if and only if $c_{\nu,1} \neq 0$ and the left hand side of (4.14) vanishes, $(c_{\nu,3} + c_{\nu,6})s + c_{\nu,5} \neq 0$, and (4.12) holds at (u, s). Moreover, there are no cuspidal lips and cuspidal beaks. The singular point (u, s) is a cuspidal cross cap if and only if $c_{\nu,1} = 0$ and $c'_{\nu,1}((c'_{\nu,3} + c'_{\nu,6})s^2 + 2c'_{\nu,5}s + 2c'_{\nu,3}) \neq 0$ at (u, s).

There are criteria for these singularities of horo-flat surfaces in [22, Theorem 6.2]. However, since the condition $c_{\nu,3} \equiv 0$ is assumed in [22, Theorem 6.2], we give a proof.

Proof. Since (4.10) is equivalent to $c_{\nu,1}(u) \neq 0$, $F_{l\nu}$ is a front at a singular point if and only if $c_{\nu,1} \neq 0$ when $Q \geq 0$. We show the proposition by using Proposition B.1. By (4.6) and (4.7), λ in (4.8) is an identifier of singularities which is defined just before Proposition B.1. If $Q(u_0) = 0$, then $\lambda_s(u_0, s_0) = 0$. Thus $\eta\lambda(u_0, s_0) \neq 0$ if and only if $\lambda_u(u_0, s_0) \neq 0$. This proves the assertion for a cuspidal edge. Furthermore, since $\eta\lambda(u_0, s_0) = 0$ implies $(\lambda_u, \lambda_s)(u_0, s_0) = (0, 0)$, this proves the assertion for a swallowtail. When $(\lambda_u, \lambda_s)(u_0, s_0) = (0, 0)$, calculating the Hesse matrix of λ and $\eta\eta\lambda$, we have the assertion of the case of $Q(u_0) = 0$ by (3) of Proposition B.1. If $Q(u) > 0$, by Proposition B.1 with the data $\lambda = (c_{\nu,3}+c_{\nu,6})s^2+2c_{\nu,5}s+2c_{\nu,3}$ and $\eta = \partial u - c_{\nu,1}\partial s$, we can show the assertion. □

By (4.8), if $c_{\nu,3} \equiv 0$, then $(u, 0)$ is a singular point of $F_{l\nu}$. This means that all generating horocycles are tangent to $F_{l\nu}|_{S(F_{l\nu})}$ at all the regular points of this curve. Thus $F_{l\nu}$ is said to be *horo-flat tangent* if $c_{\nu,2} \equiv c_{\nu,3} \equiv c_{\nu,1} - c_{\nu,4} \equiv 0$ holds (see [22, Section 5] for detail). See also Section 6.3. If $F_{l\nu}$ is horo-flat tangent, then we have the following corollary. In this case, since $c_{\nu,3} + c_{\nu,6} \neq 0$, it holds that $c_{\nu,6} \neq 0$, and $S(F_{l\nu}) = \{s(c_{\nu,6}s + 2c_{\nu,5}) = 0\}$.

Corollary 4.2. *Under the assumptions $c_{\nu,2} \equiv c_{\nu,3} \equiv c_{\nu,1} - c_{\nu,4} \equiv 0$ and $c_{\nu,6} \neq 0$ on the singularities of $F_{l\nu}$, the map $F_{l\nu}$ is a front, and the following assertions hold:*
(I) *If $c_{\nu,5}(u_0) = 0$, then $Q(u_0) = 0$ and $d\lambda(u_0, 0) = 0$ hold, in particular there are no cuspidal edge and swallowtail. The singular point $(u_0, 0)$ is a cuspidal beak if and only if*

$$c'_{\nu,5}(-2c'_{\nu,5} + c_{\nu,1}c_{\nu,6}) \neq 0$$

at u_0, Moreover, there are no cuspidal lips.
(II) *If $c_{\nu,5}(u) \neq 0$, then $Q(u_0) > 0$ and*

(1) *$d\lambda \neq 0$ at both $(u, 0)$ and $(u, -c_{\nu,5}/c_{\nu,6})$.*
(2) *A singular point $(u, 0)$ is a cuspidal edge. A singular point $(u, -c_{\nu,5}/c_{\nu,6})$ is a cuspidal edge if and only if $c_{\nu,5}c'_{\nu,6} - 2c'_{\nu,5}c_{\nu,6} \neq 0$ at u.*
(3) *A singular point $(u, 0)$ is not a swallowtail. A singular point $(u, -c_{\nu,5}/c_{\nu,6})$ is a swallowtail if and only if $c_{\nu,5}c'_{\nu,6} - 2c'_{\nu,5}c_{\nu,6} = 0$ and a formula (4.12) with $c_{\nu,3} \equiv 0$, $s = -c_{\nu,5}/c_{\nu,6}$ holds at u.*

Proof. Since $c_{\nu,3} \equiv 0$, we have $\mathfrak{h}_\nu^\varepsilon \equiv 0$ by (4.5). By the assumption $c_{\nu,6} \neq 0$, it holds that $\kappa_t \neq 0$. Again by (4.5), we get $c_{\nu,1} \neq 0$. By (4.10), this condition is equivalent to that $F_{l\nu}$ is a front, we have the first assertion. One can easily show the other assertions by applying Proposition 4.1. □

5. Normal horo-flat surfaces

In this section, we construct a parametrization of the discriminant set of H_{lb}^ε.

Let $f : (U, p) \to H^3$ be a cuspidal edge. Under the same notation as in Section 4, we assume $(\kappa_t, \mathfrak{h}_b^\varepsilon)(u) \neq (0, 0)$ for any $u \in I$ unless otherwise stated.

By using similar arguments to those of Section 4, we obtain the following. Since

$$(\boldsymbol{l}_b^\varepsilon)' = \mathfrak{h}_b^\varepsilon \boldsymbol{t} - \varepsilon\kappa_t \boldsymbol{\nu},$$

we have that $H_{lb}(\boldsymbol{x}, u) = (H_{lb})_u(\boldsymbol{x}, u) = 0$ if and only if

$$x_\gamma = \varepsilon x_b + 1, \quad x_t = \varepsilon\kappa_t s, \quad x_\nu = \mathfrak{h}_b^\varepsilon s, \quad \text{for some } s \in \mathbb{R},$$

where $\boldsymbol{x} = x_\gamma \boldsymbol{\gamma} + x_t \boldsymbol{t} + x_\nu \boldsymbol{\nu} + x_b \boldsymbol{b}$.

Since $\boldsymbol{x} \in H^3$, it holds that $x_b = \varepsilon \dfrac{s^2}{2}(\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2)$ and thus

$$x_\gamma = \frac{s^2}{2}(\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2) + 1, \quad x_t = \varepsilon\kappa_t s, \quad x_\nu = \mathfrak{h}_b^\varepsilon s, \quad x_b = \frac{\varepsilon s^2}{2}(\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2).$$

We set

$$\overline{D_{lb}^\varepsilon} = \frac{\varepsilon\kappa_t \boldsymbol{t} + \mathfrak{h}_b^\varepsilon \boldsymbol{\nu}}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}},$$

and call it the *normalized* $\boldsymbol{\nu}$*-Darboux vector field.* Now, by a parameter change,

$$\tilde{s} = s\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2},$$

and rewriting $\tilde{s}$ as s, we obtain the following parameterization of $\mathcal{D}_{H_{lb}^\varepsilon}$

$$(u, s) \mapsto \boldsymbol{x} = \boldsymbol{\gamma} + \overline{D_{lb}^\varepsilon} s + \frac{s^2}{2}\boldsymbol{l}_b^\varepsilon.$$

As seen in the case of $H_{l\nu}$, since $|\overline{D_{lb}^\varepsilon}| = 1$, for a fixed u, $s \mapsto \boldsymbol{\gamma} + \overline{D_{lb}^\varepsilon} s + \dfrac{s^2}{2}\boldsymbol{l}_b^\varepsilon$ is a parabola and thus a horocycle ([22, Section 4]).

We have that $\{\boldsymbol{\gamma}, \overline{D_{lb}^\varepsilon}, \varepsilon\boldsymbol{b}, (\boldsymbol{l}_b^\varepsilon)'/|(\boldsymbol{l}_b^\varepsilon)'|\}$ is a pseudo-orthonormal frame. Analogously to Section 4, we set

$$\{\boldsymbol{\gamma}_b, \boldsymbol{a}_{b,1}, \boldsymbol{a}_{b,2}, \boldsymbol{a}_{b,3}\} = \{\boldsymbol{\gamma}, \overline{D_{lb}^\varepsilon}, \varepsilon\boldsymbol{b}, (\boldsymbol{l}_b^\varepsilon)'/|(\boldsymbol{l}_b^\varepsilon)'|\}$$

and

$$F_{lb}(u, s) = F_{lb}^\varepsilon(u, s) = \boldsymbol{\gamma}_b + \boldsymbol{a}_{b,1} s + \frac{s^2}{2}(\boldsymbol{\gamma}_b + \boldsymbol{a}_{b,2}). \tag{5.1}$$

By definition, F_{lb} is a Δ_2-dual of $\boldsymbol{l}_b^\varepsilon$. Similarly to the case of $H_{l\nu}^\varepsilon$, the invariants $c_{b,1}, \dots, c_{b,6}$ are defined by the relation

$$\begin{pmatrix} \boldsymbol{\gamma}_b' \\ \boldsymbol{a}_{b,1}' \\ \boldsymbol{a}_{b,2}' \\ \boldsymbol{a}_{b,3}' \end{pmatrix} = \begin{pmatrix} 0 & c_{b,1} & c_{b,2} & c_{b,3} \\ c_{b,1} & 0 & c_{b,4} & c_{b,5} \\ c_{b,2} & -c_{b,4} & 0 & c_{b,6} \\ c_{b,3} & -c_{b,5} & -c_{b,6} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_b \\ \boldsymbol{a}_{b,1} \\ \boldsymbol{a}_{b,2} \\ \boldsymbol{a}_{b,3} \end{pmatrix}. \tag{5.2}$$

Then we have

$$\begin{array}{rcl} c_{b,1} & = & \dfrac{\varepsilon\kappa_t}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}}, \\ c_{b,2} & = & 0, \\ c_{b,3} & = & \dfrac{\mathfrak{h}_b^\varepsilon}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}}, \\ c_{b,4} & = & c_{b,1} = \dfrac{\varepsilon\kappa_t}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}}, \\ c_{b,5} & = & \dfrac{\delta_n^h}{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}, \\ c_{b,6} & = & \dfrac{-\varepsilon\kappa_b + \kappa_b^2 + \kappa_t^2}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}} \\ c_{b,3} + c_{b,6} & = & \sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}, \end{array} \tag{5.3}$$

where we set

$$\delta_n^h = -\kappa_\nu((\mathfrak{h}_b^\varepsilon)^2 + \kappa_t^2) - \varepsilon\kappa_t(\mathfrak{h}_b^\varepsilon)' + \varepsilon\mathfrak{h}_b^\varepsilon\kappa_t'.$$

By (5.3), $\boldsymbol{\nu}_{lb} = -(s^2/2)\boldsymbol{\gamma}_b - s\boldsymbol{a}_{b,1} + (1 - s^2/2)\boldsymbol{a}_{b,2}$ is a Δ_1-dual of F_{lb}, and $F_{lb} + \boldsymbol{\nu}_{lb} = \boldsymbol{\gamma}_b + \boldsymbol{a}_{b,2}$ is a Δ_2-dual of F_{lb}. Since the Δ_2-dual of F_{lb} degenerates to a curve, F_{lb} is a horo-flat surface.

It follows that each of F_{lb} is a horo-flat surface normal to the cuspidal edge at any $\boldsymbol{\gamma}(u)$, so that we call it a *normal horo-flat surface* (*along the cuspidal edge*). An example of the normal horo-flat surface F_{lb} of f as in (4.3) near $(0,0)$ is provided by Figure 2. Similar calculations

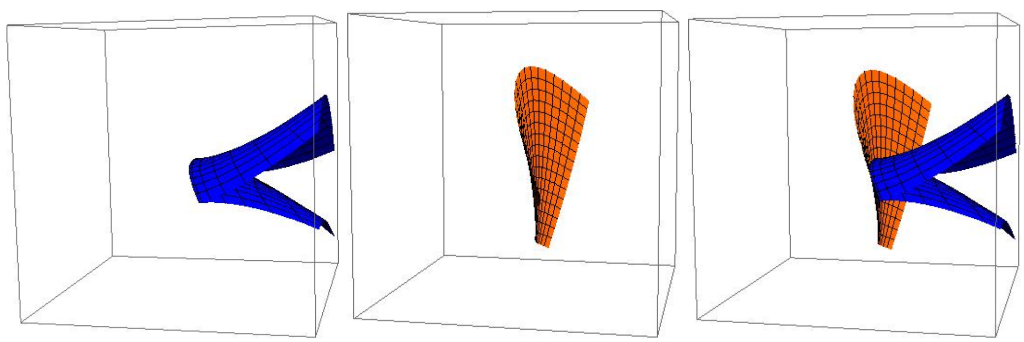

FIGURE 2. Cuspidal edge, F_{lb} and the both surfaces together

to those in Section 4, lead to the characterization of the singularities of F_{lb} (just substitute $c_{b,i}$ into $c_{\nu,i}$ $(i=1,\dots,6)$ in Proposition 4.1 and Corollary 4.2). By comparing (4.5) and (5.3), we see that changing κ_ν to κ_b and κ_t to $-\kappa_t$ in the formulae for $c_{\nu,i}$, leads to the formulae for $c_{b,i}$ $(i=1,\dots,6)$.

6. Special cuspidal edges

We consider a cuspidal edge f, where either $F_{l\nu}$ or F_{lb} has special properties. The special horo-flat surfaces which are one-parameter families of horocycles (horo-flat horocyclic surfaces) are classified in [22, pp815–818]. We consider here the cases of the horo-cylinder and the horocone. We review the special horo-flat surfaces given in [22].

Definition 6.1. A horocyclic surface with the invariants $c_1,\dots,c_6$ is called a *regular horocylindrical surface* if $c_1\equiv c_2\equiv c_4\equiv c_5\equiv 0$, and $c_3(c_3+c_6)>0$. A horocyclic surface is called a *secondary regular horocylindrical surface* if $c_1\equiv c_2\equiv c_4\equiv c_6\equiv 0$, and $c_5^2-2c_3^2<0$.

Definition 6.2. A horocyclic surface with the invariants $c_1,\dots,c_6$ is called a *generalized horocone* if $c_1\equiv c_2\equiv c_3\equiv c_4\equiv 0$. A generalized horocone is called a *horocone with a single vertex* if $c_5\equiv 0$ and there is no subinterval $J\subset I$ such that $c_6|_J=0$. A *horocone with two vertices* is a generalized horocone with the property that there is no subinterval $J\subset I$ such that $c_5|_J=0$, and there exists $\lambda\in\mathbb{R}$ such that $c_6=\lambda c_5$. A generalized horocone is called a *semi-horocone* if the following holds for $(i,j)=(5,6)$ or $(i,j)=(6,5)$: There is no subinterval $J\subset I$ such that $c_i|_J=0$ and c_j/c_i is not constant on $\{t\in I\,|\,c_i(u)\neq 0\}$. If the condition $c_1\equiv c_2\equiv c_3\equiv c_4\equiv c_6\equiv 0$ holds and there is no subinterval $J\subset I$ such that $c_5|_J=0$, then the image of the horocyclic surface is a horosphere. We call this a *conical horosphere*.

Let α be a function. By (4.5) and substituting $c_{\nu,1} - c_{\nu,4} \equiv 0, c_{\nu,2} \equiv 0$ in (3.3), we get

$$
\left\{
\begin{array}{rcl}
\bar{c}_{\nu,1} &=& \bar{c}_{\nu,4} = \dfrac{-\varepsilon\kappa_t}{\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2}} + \alpha', \\
\bar{c}_{\nu,2} &=& 0, \\
\bar{c}_{\nu,3} &=& \dfrac{\alpha^2}{2}\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2} + \alpha\dfrac{\delta_o^h}{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2} + \dfrac{\mathfrak{h}_\nu^\varepsilon}{\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2}}, \\
\bar{c}_{\nu,5} &=& \dfrac{\delta_o^h}{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2} + \alpha\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2}, \\
\bar{c}_{\nu,6} &=& -\dfrac{\alpha^2}{2}\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2} - \alpha\dfrac{\delta_o^h}{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2} + \dfrac{-\varepsilon\kappa_\nu + \kappa_\nu^2 + \kappa_t^2}{\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2}}, \\
\bar{c}_{\nu,3} + \bar{c}_{\nu,6} &=& \sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2},
\end{array}
\right. \tag{6.1}
$$

and similarly, for a function β, we get

$$
\left\{
\begin{array}{rcl}
\bar{c}_{b,1} &=& \bar{c}_{b,4} = \dfrac{\varepsilon\kappa_t}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}} + \beta', \\
\bar{c}_{b,2} &=& 0, \\
\bar{c}_{b,3} &=& \dfrac{\beta^2}{2}\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2} + \beta\dfrac{\delta_n^h}{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2} + \dfrac{\mathfrak{h}_b^\varepsilon}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}}, \\
\bar{c}_{b,5} &=& \dfrac{\delta_n^h}{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2} + \beta\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}, \\
\bar{c}_{b,6} &=& -\dfrac{\beta(u)^2}{2}\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2} - \beta\dfrac{\delta_n^h}{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2} + \dfrac{-\varepsilon\kappa_b + \kappa_b^2 + \kappa_t^2}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}}, \\
\bar{c}_{b,3} + \bar{c}_{b,6} &=& \sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}.
\end{array}
\right. \tag{6.2}
$$

We remark that one can obtain the formula for F_{lb} by interchanging κ_ν to κ_b and κ_t to $-\kappa_t$ in the formula for $F_{l\nu}$.

6.1. **Horocylinders as osculating and normal horo-flat surfaces.** We consider the condition for $F_{l\nu}$ and F_{lb} to be horocylinders. By (6.1) and (6.2), setting

$$
\alpha_c = \frac{-\delta_o^h}{(\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2)\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2}}, \tag{6.3}
$$

and

$$
\beta_c = \frac{-\delta_n^h}{(\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2)\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}}, \tag{6.4}
$$

we see that $c_{\nu,5} \equiv 0$, $c_{b,5} \equiv 0$. Thus, $\bar{c}_{\nu,1} = \bar{c}_{\nu,4} \equiv 0$ if and only if

$$
\frac{-\varepsilon\kappa_t}{\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2}} + \alpha_c' \equiv 0, \tag{6.5}
$$

and $\bar{c}_{b,1} = \bar{c}_{b,4} \equiv 0$ if and only if

$$
\frac{\varepsilon\kappa_t}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}} + \beta_c' \equiv 0. \tag{6.6}
$$

Set

$$
C_o^h = -2\varepsilon\kappa_t(\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2)^2 - 2(\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2)(\delta_o^h)' + 3\delta_o^h(\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2)'
$$

and

$$
C_n^h = 2\varepsilon\kappa_t(\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2)^2 - 2(\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2)(\delta_n^h)' + 3\delta_n^h(\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2)'.
$$

Then the condition (6.5) is equivalent to $C_o^h \equiv 0$, and (6.6) is equivalent to $C_n^h \equiv 0$. Moreover, if α_c satisfies (6.3), then $\bar{c}_{\nu,3}$ is equal to a positive functional multiplication of

$$\frac{-(\delta_o^h)^2}{2(\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2)^2} + \mathfrak{h}_\nu^\varepsilon,$$

and if β_c satisfies (6.4), then $\bar{c}_{b,3}$ is equal to a positive functional multiplication of

$$\frac{-(\delta_n^h)^2}{2(\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2)^2} + \mathfrak{h}_b^\varepsilon.$$

Thus we obtain the following proposition.

Proposition 6.3. *The horocyclic surface $F_{l\nu}$ is a regular horocylindrical surface if and only if $C_o^h \equiv 0$ and*

$$\frac{-(\delta_o^h)^2}{2(\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2)^2} + \mathfrak{h}_\nu^\varepsilon > 0.$$

The horocyclic surface F_{lb} is a regular horocylindrical surface if and only if $C_n^h \equiv 0$ and

$$\frac{-(\delta_n^h)^2}{2(\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2)^2} + \mathfrak{h}_b^\varepsilon > 0.$$

We see that if $\kappa_t \equiv 0$. Then $\delta_o^h = -\kappa_b(\mathfrak{h}_\nu^\varepsilon)^2$ and $\delta_n^h = -\kappa_\nu(\mathfrak{h}_b^\varepsilon)^2$ hold, and also $\bar{c}_{\nu,1} = \bar{c}_{\nu,4} = \alpha_c'$ and $\bar{c}_{b,1} = \bar{c}_{b,4} = \beta_c'$. We give examples of cuspidal edge whose osculating and normal horo-flat surfaces are horocylinders.

Example 6.4. (regular horocylindrical surface) We set $\kappa_t \equiv \kappa_b \equiv 0$ and κ_ν satisfies $\mathfrak{h}_\nu^\varepsilon > 0$. Setting $\alpha_c = 0$, then we see that $\bar{c}_{\nu,1} = \bar{c}_{\nu,2} = \bar{c}_{\nu,4} = \bar{c}_{\nu,5} = 0$, and $\bar{c}_{\nu,3}(\bar{c}_{\nu,3} + \bar{c}_{\nu,6}) > 0$. Then by definition, $F_{l\nu}$ is a regular horocylindrical surface. Similarly, we set $\kappa_t \equiv \kappa_\nu \equiv 0$ and κ_b satisfies $\mathfrak{h}_b^\varepsilon > 0$. Setting $\beta_c = 0$, then we see that $\bar{c}_{b,1} = \bar{c}_{b,2} = \bar{c}_{b,4} = \bar{c}_{b,5} = 0$, and $\bar{c}_{b,3}(\bar{c}_{b,3} + \bar{c}_{b,6}) > 0$. Then by definition, F_{lb} is a regular horocylindrical surface.

Example 6.5. (secondary regular horocylindrical surface) We set $\kappa_t \equiv \kappa_\nu \equiv 0$ and $\kappa_b = 1$. Setting $\alpha_c = 0$, then we see that $\bar{c}_{\nu,1} = \bar{c}_{\nu,2} = \bar{c}_{\nu,4} = \bar{c}_{\nu,6} = 0$, and $\bar{c}_{\nu,5}^2 - 2\bar{c}_{\nu,3}^2 < 0$. Then by definition, $F_{l\nu}$ is a secondary regular horocylindrical surface. We set $\kappa_t \equiv \kappa_b \equiv 0$ and $\kappa_\nu = 1$. Setting $\beta_c = 0$, then we see that $\bar{c}_{b,1} = \bar{c}_{b,2} = \bar{c}_{b,4} = \bar{c}_{b,6} = 0$, and $\bar{c}_{b,5}^2 - 2\bar{c}_{b,3}^2 < 0$. Then by definition, F_{lb} is a secondary regular horocylindrical surface.

6.2. **Horocones as osculating and normal horo-flat surfaces.** If the discriminant $Q_{l\nu}$ (respectively, Q_{lb}) of

$$\bar{c}_{\nu,3} = \frac{\alpha^2}{2}\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2} + \alpha\frac{\delta_o^h}{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2} + \frac{\mathfrak{h}_\nu^\varepsilon}{\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2}} = 0$$

$$\left(\text{respectively}, \bar{c}_{b,3} = \frac{\beta^2}{2}\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2} + \beta\frac{\delta_n^h}{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2} + \frac{\mathfrak{h}_b^\varepsilon}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}} = 0\right)$$

as an equation of α (respectively, β) is non-negative, then we have a solution α (respectively, β). We set

$$\sigma_o^h = \frac{-\varepsilon\kappa_t}{\sqrt{\kappa_t^2 + (\mathfrak{h}_\nu^\varepsilon)^2}} + \alpha'$$

and

$$\sigma_n^h = \frac{\varepsilon\kappa_t}{\sqrt{\kappa_t^2 + (\mathfrak{h}_b^\varepsilon)^2}} + \beta'.$$

Then if $\sigma_o^h \equiv 0$, (respectively, $\sigma_n^h \equiv 0$,) $F_{l\nu}$ (respectively, F_{lb}) is a generalized horocone. Thus we can state the following:

Proposition 6.6. *The horocyclic surface $F_{l\nu}$ is a generalized horocone if and only if $\sigma_o^h \equiv 0$ and $Q_{l\nu} \geq 0$. The horocyclic surface F_{lb} is a generalized horocone if and only if $\sigma_n^h \equiv 0$ and $Q_{lb} \geq 0$.*

We give examples of cuspidal edge whose osculating and normal horo-flat surfaces are horocones.

Example 6.7. (horocone with single and two vertices) We take $\kappa_t \equiv 0$, a non-zero constant κ_b and a constant κ_ν satisfying $\mathfrak{h}_\nu^\varepsilon > 0$ and $\kappa_b^2 - 2\mathfrak{h}_\nu^\varepsilon \geq 0$. We also take a constant α which is a solution that $\bar{c}_{\nu,3} = 0$. Then $\bar{c}_{\nu,1} \equiv \bar{c}_{\nu,2} \equiv \bar{c}_{\nu,3} \equiv \bar{c}_{\nu,4} \equiv 0$ holds. Moreover, we see

$$\bar{c}_{\nu,5} = -\kappa_b + \alpha\mathfrak{h}_\nu^\varepsilon, \quad \bar{c}_{\nu,6} = \mathfrak{h}_\nu^\varepsilon.$$

Thus setting κ_b and κ_ν satisfying $-\kappa_b + \alpha\mathfrak{h}_\nu^\varepsilon = 0$, then we obtain a horocone with a single vertex. On the other hand, $-\kappa_b + \alpha\mathfrak{h}_\nu^\varepsilon \neq 0$, then we obtain a horocone with two vertices.

Similarly, we take $\kappa_t \equiv 0$, a non-zero constant κ_ν and a constant κ_b satisfying $\mathfrak{h}_b^\varepsilon > 0$ and $\kappa_\nu^2 - 2\mathfrak{h}_b^\varepsilon \geq 0$. We also take a constant β which is a solution that $\bar{c}_{b,3} = 0$. Then

$$\bar{c}_{b,1} \equiv \bar{c}_{b,2} \equiv \bar{c}_{b,3} \equiv \bar{c}_{b,4} \equiv 0$$

holds. Moreover, we see

$$\bar{c}_{b,5} = -\kappa_\nu + \beta\mathfrak{h}_b^\varepsilon, \quad \bar{c}_{b,6} = \mathfrak{h}_b^\varepsilon.$$

Thus setting that κ_ν and κ_b satisfy $-\kappa_\nu + \beta\mathfrak{h}_b^\varepsilon = 0$, then we obtain a horocone with a single vertex. On the other hand, if $-\kappa_\nu + \beta\mathfrak{h}_b^\varepsilon \neq 0$, then we obtain a horocone with two vertices.

Example 6.8. (semi-horocone) We set $\varepsilon\kappa_\nu \equiv 1$ and $\varepsilon\kappa_t < 0$. By (6.1),

$$\bar{c}_{\nu,1} = \bar{c}_{\nu,4} = -\varepsilon\kappa_t / \sqrt{\kappa_t^2} + \alpha'.$$

Let α be a solution of $1 + \alpha' = 0$, i.e., $\alpha = -u + A$, where A is a sufficiently large positive constant such that $-\varepsilon\kappa_t$ is positive around $u = 0$. We take $\kappa_t = -2\varepsilon(u + A)$ and $\kappa_b = -u^2 + A^2$. Then $\bar{c}_{\nu,1} \equiv \bar{c}_{\nu,2} \equiv \bar{c}_{\nu,3} \equiv \bar{c}_{\nu,4} \equiv 0$. Moreover, $\bar{c}_{\nu,5} = -u^2 + A^2$ and $\bar{c}_{\nu,6} = 2(u + A)$. Thus we get a semi-horocone $F_{l\nu}$.

Similarly, set $\varepsilon\kappa_b \equiv 1$ and $\varepsilon\kappa_t > 0$. Let β be a solution of $1 + \beta' = 0$ i.e., $\beta = -u + B$, where B is a sufficiently small negative constant such that $\varepsilon\kappa_t$ is positive around $u = 0$. We take $\kappa_t = -2\varepsilon(u + B)$ and $\kappa_\nu = u^2 - B^2$. Then, we see that $\bar{c}_{b,1} \equiv \bar{c}_{b,2} \equiv \bar{c}_{b,3} \equiv \bar{c}_{b,4} \equiv 0$, $\bar{c}_{b,5} = u^2 - B^2$ and $\bar{c}_{b,6} = -2(u + B)$. Thus we get a semi-horocone F_{lb}.

6.3. **Special cases.** If $\kappa_b \equiv 0$, then $\boldsymbol{\nu}$ is the principal normal direction of $\boldsymbol{\gamma}$, or equivalently, $\boldsymbol{b}$ is the bi-normal direction of $\boldsymbol{\gamma}$. If $\kappa_\nu \equiv 0$, then $\boldsymbol{\nu}$ is the bi-normal direction of $\boldsymbol{\gamma}$, and which to say that $\boldsymbol{b}$ is the principal normal direction of $\boldsymbol{\gamma}$.

Important particular cases are:

(i) $\kappa_\nu \equiv \varepsilon$ (i.e., $\mathfrak{h}_\nu^\varepsilon \equiv 0$) in $H_{l\nu}^\varepsilon$,
(ii) $\kappa_b \equiv \varepsilon$ (i.e., $\mathfrak{h}_b^\varepsilon \equiv 0$) in H_{lb}^ε.

If (i) is satisfied, then $c_{\nu,3} \equiv 0$ holds, and if (ii) is satisfied, then $c_{b,3} \equiv 0$ holds. Namely, the singular set of the original cuspidal edge and the singular set of the osculating and the normal horo-flat surfaces coincide respectively. By Proposition 4.1, we have the conditions of singularities of the osculating and normal horo-flat surfaces in terms of the information of the singular locus of the cuspidal edge.

7. Duals of the singular set of the cuspidal edge

Since the curve $\boldsymbol{\gamma}$ of the parameterization (4.2) takes values in H^3, we can consider the Δ_1 and Δ_2 duals of $\boldsymbol{\gamma}$. We set $H^s_\gamma : S^3_1 \times I \to \mathbb{R}$ (respectively, $H^l_\gamma : LC^* \times I \to \mathbb{R}$) by

$$H^s_\gamma(\boldsymbol{x}, u) = \langle \boldsymbol{x}, \boldsymbol{\gamma}(u)\rangle \Big(\text{respectively, } H^l_\gamma(\boldsymbol{x}, u) = \langle \boldsymbol{x}, \boldsymbol{\gamma}(u)\rangle + 1\Big).$$

Then we have a parameterization of the discriminant set of H^s_γ given by

$$DD_l(\phi, u) = \cos\phi\boldsymbol{\nu}(u) + \sin\phi\boldsymbol{b}(u).$$

The corresponding singular set $S(DD_l)$ is

$$S(DD_l) = \left\{(\phi, u) \middle| \cos\phi = \pm\kappa_b/\sqrt{\kappa_\nu^2 + \kappa_b^2},\ \sin\phi = \mp\kappa_\nu/\sqrt{\kappa_\nu^2 + \kappa_b^2}\right\},$$

with

$$\begin{pmatrix}\boldsymbol{\gamma}'\\ \boldsymbol{t}'\\ \boldsymbol{n}'\\ \boldsymbol{e}'\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & \kappa_h & 0\\ 0 & -\kappa_h & 0 & \tau_h\\ 0 & 0 & -\tau_h & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}\\ \boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{e}\end{pmatrix},$$

where $\{\boldsymbol{\gamma}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{e}\}$ is the hyperbolic Frenet frame along $\boldsymbol{\gamma}$ and $\kappa_h = |\boldsymbol{t}' - \gamma|$ (see Section 3). Since $DD_l|_{S(DD_l)} = \pm\boldsymbol{e}$, it follows that $\pm\boldsymbol{e} = \pm(\kappa_b\boldsymbol{\nu} - \kappa_\nu\boldsymbol{b})/\sqrt{\kappa_b^2 + \kappa_\nu^2}$, and $\tau_h = -\kappa_t + \dfrac{\kappa_\nu'\kappa_b - \kappa_b'\kappa_\nu}{\kappa_b^2 + \kappa_\nu^2}$ ([20, p109]).

On the other hand, we have a parameterization of the discriminant set of H^l_γ given by

$$HS_l(\phi, u) = \boldsymbol{\gamma}(u) + \cos\phi\boldsymbol{\nu}(u) + \sin\phi\boldsymbol{b}(u),$$

where $\phi \in [0, 2\pi)$. We also have

$$S(HS_l) = \left\{(\phi, u) \middle| \cos\phi = \frac{\kappa_\nu \pm \sqrt{\kappa_\nu + \kappa_b^2 - 1}}{\kappa_\nu^2 + \kappa_b^2}\right\}.$$

Thus DD_l and HS_l are Δ_3-dual each other. Here, $\Delta_3 = \{(\boldsymbol{v}, \boldsymbol{w}) \in LC^* \times S^3_1 \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 1\}$ and as in Section 2, the phrase "DD_l and HS_l are Δ_3-dual" amounts to say that the map $(DD_l, HS_l) : U \to \Delta_3$ is isotropic with respect to the contact structure defined by the restrictions of the 1-forms

$$\theta_{31} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|_{\Delta_3}, \quad \theta_{32} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle|_{\Delta_3}.$$

See [15] for details.

Now, we give a global property of a curve in the hyperbolic space. There is a relation

$$\begin{pmatrix}\boldsymbol{\gamma}'\\ \boldsymbol{t}'\\ \boldsymbol{n}'\\ \boldsymbol{e}'\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \pm\dfrac{\kappa_\nu}{\sqrt{\kappa_\nu^2 + \kappa_b^2}} & \pm\dfrac{\kappa_b}{\sqrt{\kappa_\nu^2 + \kappa_b^2}}\\ 0 & 0 & \pm\dfrac{\kappa_b}{\sqrt{\kappa_\nu^2 + \kappa_b^2}} & \mp\dfrac{\kappa_\nu}{\sqrt{\kappa_\nu^2 + \kappa_b^2}}\end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}\\ \boldsymbol{t}\\ \boldsymbol{\nu}\\ \boldsymbol{b}\end{pmatrix} \tag{7.1}$$

between $\{\boldsymbol{\gamma}, \boldsymbol{t}, \boldsymbol{n}, \boldsymbol{e}\}$ and $\{\boldsymbol{\gamma}, \boldsymbol{t}, \boldsymbol{\nu}, \boldsymbol{b}\}$. If we define θ by

$$\cos\theta = \pm\frac{\kappa_\nu}{\sqrt{\kappa_\nu^2 + \kappa_b^2}}, \quad \sin\theta = \pm\frac{\kappa_b}{\sqrt{\kappa_\nu^2 + \kappa_b^2}},$$

then we get that $\kappa_t = \theta' - \tau_h$. And in the case that the singular set forms a circle $C = \mathbb{R}/\mathbb{Z}$, we obtain

$$\int_C (\tau_h + \kappa_t)\, du = \theta(1) - \theta(0) = 2n\pi \qquad (n \in \mathbb{Z}).$$

Observe that the integer n is the *linking number* of $\{\boldsymbol{n}, \boldsymbol{e}\}$ around $\{\boldsymbol{\gamma}, \boldsymbol{t}\}$ along $\boldsymbol{\gamma}(C)$, with respect to $\{\boldsymbol{\nu}, \boldsymbol{b}\}$.

Appendix A. Osculating and normal extrinsic flat surfaces

We consider the following smooth functions on $H^3(-1) \times I$:

$$\begin{aligned} H_\nu(\boldsymbol{x}, u) &= \langle \boldsymbol{x}, \boldsymbol{\nu}(u) \rangle, \\ H_b(\boldsymbol{x}, u) &= \langle \boldsymbol{x}, \boldsymbol{b}(u) \rangle. \end{aligned}$$

Then by using the functions H_ν and H_b, we can obtain analogous results. The discriminant set of these functions are envelopes of the osculating or the rectifying hyperbolic planes. In the Euclidean case, the discriminant set of the functions corresponding to them are envelopes of the osculating or the rectifying planes. The results and the geometric meaning of them for these cases are quite similar to those of the case in the Euclidean space [18, 23]. Thus we only give here the parameterizations for the discriminant sets of H_ν and H_b.

The discriminant set $\mathcal{D}_{H_\nu}$ of the function H_ν can be parameterized by

$$(u, \phi) \mapsto \cosh \phi \boldsymbol{\gamma}(u) + \sinh \phi \overline{D}_\nu(u), \quad \overline{D}_\nu(u) = \frac{\kappa_t \boldsymbol{t} + \kappa_\nu \boldsymbol{b}}{\kappa_t^2 + \kappa_\nu^2}(u),$$

where we assume $(\kappa_t, \kappa_\nu) \neq (0,0)$. This is a one-parameter family of geodesics tangent to the cuspidal edge. Therefore, $\mathcal{D}_{H_\nu}$ is called an *osculating extrinsic flat surface along the cuspidal edge*.

The discriminant set $\mathcal{D}_{H_b}$ of the function H_b can be parameterized by

$$(u, \phi) \mapsto \cosh \phi \boldsymbol{\gamma}(u) + \sinh \phi \overline{D}_b(u), \quad \overline{D}_b(u) = \frac{-\kappa_t \boldsymbol{t} + \kappa_b \boldsymbol{\nu}}{\kappa_t^2 + \kappa_b^2}(u),$$

where we assume $(\kappa_t, \kappa_b) \neq (0,0)$. This is a one-parameter family of geodesics normal to the cuspidal edge, so that $\mathcal{D}_{H_b}$ is called a *normal extrinsic flat surface along the cuspidal edge*.

Appendix B. Criteria for singularities

We state the some criteria to characterize the singularities used in Sections 4 and 5. Let $f : U \to H^3$ be a frontal with a Δ_1-dual $g : U \to S^3_1$. A function Λ is called an *identifier of singularities* if it is a non-zero functional multiplication of the function $\det(f_u, f_v, g, f)$ for a coordinate system (u, v) on U. If $p \in U$ satisfies $\operatorname{rank} df_p = 1$, then there exists a vector field η such that $\langle \eta_q \rangle_{\mathbb{R}} = \ker df_q$ for all $q \in S(f)$. We call η a *null vector field*. Let $p \in U$ be a singular point satisfying $d\Lambda(p) \neq 0$. Then there exists a parametrization $c : ((-z, z), 0) \to (U, p)$ of $S(f)$ near p, where $z > 0$. Let ∇^f_η be the canonical covariant derivative by η along a map f induced from the Levi-Civita connection on H^3. We set

$$\psi(u) = \det \left(\frac{df(\gamma(u))}{dt}, \ \frac{d(\nabla^f_\eta g)(\gamma(u))}{dt}, \ g(\gamma(u)), \ f(\gamma(u)) \right).$$

Then we have the following criteria for singularities:

Proposition B.1. *Let $p \in U$ be a singular point of f satisfying $\operatorname{rank} df_p = 1$. Then p is*

(1) *a cuspidal edge if and only if f is a front at p, and $\eta\Lambda(p) \neq 0$.*
(2) *a swallowtail if and only if f is a front at p, $d\Lambda(p) \neq 0$, $\eta\Lambda(p) = 0$ and $\eta\eta\Lambda(p) \neq 0$.*
(3) *a cuspidal beak (respectively, cuspidal lip) if and only if f is a front at p, $d\Lambda(p) = 0$, $\det \operatorname{Hess} \Lambda(p) < 0$ and $\eta\eta\Lambda(p) \neq 0$ (respectively, $\det \operatorname{Hess} \Lambda(p) > 0$).*
(4) *a cuspidal cross cap if and only if $\eta\Lambda(p) \neq 0$, $\psi(0) = 0$ and $\psi'(0) \neq 0$.*

These criteria for singularities in H^3 can be easily shown by well-known criteria in [35, Corollary 2.5] (see also [25, Proposition 1.3]) for (1) and (2), in [22, Theorem A.1] for (3), and in [8, Corollary 1.5] for (4).

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