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BOUQUET DECOMPOSITION FOR DETERMINANTAL MILNOR FIBERS

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ABSTRACT. We provide a bouquet decomposition for the determinantal Milnor fiber of an Essentially Isolated Determinantal Singularity (EIDS) of arbitrary type. The building blocks of the decomposition are (suspensions of) hyperplane sections in general position off the origin of the generic determinantal varieties. For the special case of $2 \times n$-matrices we give a full description of the homotopy types of the determinantal Milnor fibers as a wedge of spheres.

1. Results

In this note we will apply a general Bouquet Decomposition Theorem by M. Tibăr [13] in the case of an Essentially Isolated Determinantal Singularity (EIDS, see [4]) to prove the following:

Theorem 1.1. *Let $(X_0, 0) = (A^{-1}(M^t_{m,n}), 0) \subset (\mathbb{C}^N, 0)$ be an EIDS of type (m, n, t) and dimension $d = \dim(X_0, 0) = N - (m - t + 1)(n - t + 1) > 0$ given by a holomorphic map germ*

$$A \colon (\mathbb{C}^N, 0) \to (\operatorname{Mat}(m, n; \mathbb{C}), 0).$$

Suppose A_u is a stabilization of A and $\overline{X}_u = A_u^{-1}(M^t_{m,n})$ the determinantal Milnor fiber. Define

$$s_0 := \min\{s \in \mathbb{N} : (m - s + 1)(n - s + 1) \leq N\}.$$

Then $\overline{X}_u$ is homotopy equivalent to the bouquet

$$L^{t,N}_{m,n} \vee \bigvee_{s_0 \leq s \leq t} \bigvee_{i=1}^{r(s)} S^{N-(m-s+1)(n-s+1)+1}(L^{t-s+1,(m-s+1)(n-s+1)-1}_{m-s+1,n-s+1}) \tag{1}$$

for some numbers $r(s)$ with $s_0 \leq s \leq t$.

The spaces $M^t_{m,n}$ and $L^{t,k}_{m,n}$ appearing in this theorem are defined as follows. For any triple (m, n, t) of non-negative integers we set

$$M^t_{m,n} := \{M \in \operatorname{Mat}(m, n; \mathbb{C}) : \operatorname{rank} M < t\},$$

the *generic determinantal variety*. We define the space $L^{t,k}_{m,n}$ to be the interior of the determinantal Milnor fiber of a *linear* EIDS of type (m, n, t), i.e. the singularity obtained from a generic linear map germ

$$\Phi \colon (\mathbb{C}^k, 0) \to (\operatorname{Mat}(m, n; \mathbb{C}), 0).$$

Note that for the particular case $k = m \cdot n - 1$ the space $L^{t,k}_{m,n}$ is the *complex link* of $M^t_{m,n}$.

In Formula (1) we denote by $S^r(X)$ the r-fold repeated suspension of a topological space X. We use the same convention as in [13] and set $S^1(\emptyset) = S^0$, the sphere of dimension 0, and $S^0(X) = X$ for any X.

Theorem 1.1 is a major reduction step in the understanding of the vanishing topology of essentially isolated determinantal singularities. In particular it implies the known results for the Milnor fiber of an isolated complete intersection singularity $(X_0, 0) = (f^{-1}(\{0\}), 0) \subset (\mathbb{C}^N, 0)$ given by a holomorphic map germ

$$f \colon (\mathbb{C}^N, 0) \to (\mathbb{C}^d, 0)$$

which can naturally be regarded as EIDS of type $(d, 1, 1)$. In this particular case one has $L^{t,N}_{m,n} \cong_{ht} \{pt\}$, $s_0 = t = 1$, and $r(s) = \mu$ is the classical Milnor number of $(X_0, 0)$. Formula (1) therefore reads

$$\begin{aligned} \overline{X}_u &\cong_{ht} \{pt\} \vee \bigvee_{i=1}^{r(1)} S^{N-d+1}(\emptyset) \\ &\cong_{ht} \bigvee_{i=1}^{\mu} S^{N-d} \end{aligned}$$

In fact, it has already been shown in [13, Corollary 4.2] how to apply the Handlebody Theorem to reprove the known results [8] on isolated complete intersection singularities and we will follow the ideas presented there to obtain our generalization for EIDS.

While for ICIS the Milnor fiber is always homotopy equivalent to a bouquet of spheres of the same dimension, this is no longer the case for determinantal Milnor fibers of EIDS, see e.g. [3], [5], and Section 4. Several groups have studied the vanishing Euler characteristic for EIDS, see e.g. [4], [6], and [12]. One approach is to study the behavior of a generic hyperplane equation h in a determinantal deformation of a given EIDS $(X_0, 0)$. The determinantal Milnor fiber $\overline{X}_u$ is then obtained from its hyperplane section $\overline{X}_u \cap \{h = 0\}$ by attaching cells, or, more generally in the context of stratified Morse theory, so-called "thimbles[1]", at Morse critical points of h on $\overline{X}_u$. This way, one obtains nice formulas for the vanishing Euler characteristic in terms of the polar multiplicities of the singularity $(X_0, 0)$. However, it is hardly possible to describe the loci in the hyperplane section $\overline{X}_u \cap \{h = 0\}$ at which the attachments take place. This fact destroys any hope to arrive at a precise description of the homotopy type of $\overline{X}_u$.

It is the Carrousel by Lê which sits at the heart of the proof of the Handlebody Theorem (stated as Theorem 2.4 below) from [13] and which allows us to understand the attachments of the thimbles. As we will see, however, the setup for the application of the Handlebody Theorem is quite different from the viewpoint of EIDS. We will describe the transformation of any EIDS $(X_0, 0) = (A^{-1}(M^t_{m,n}) \subset (\mathbb{C}^N, 0)$ to an isolated *relative* complete intersection singularity (IRCIS, see Definition 3.2)

$$(X_0, 0) = (\{f_{1,1} = \cdots = f_{m,n} = 0\}, 0) \subset (Z, 0)$$

on a controlled Whitney stratified ambient space

$$(Z, 0) \cong (\mathbb{C}^N, 0) \times (M^t_{m,n}, 0)$$

in Section 3.1. Then, rather than doing an induction argument by cutting down with generic hyperplanes, we proceed by an inductive argument where we always trade one equation $f_{i,j}$ defining $(X_0, 0)$ in $(Z, 0)$ for a generic hyperplane equation and eventually end up with the space $L^{t,N}_{m,n}$ – a generic linear section of $M^t_{m,n}$ off the origin. During this process, the Handlebody Theorem allows us to really keep track of the involved attachment processes.

The homotopy type of the spaces $L^{t,k}_{m,n}$ has been studied in a few particular cases, see e.g. [5]. The Euler obstructions of the generic determinantal varieties $M^t_{m,n}$, which are closely related to their hyperplane sections $L^{t,m\cdot n-1}_{m,n}$, can be found in [6] and the Chern-Schwartz-MacPherson classes of their projectivizations $\mathbb{P}(M^t_{m,n})$ have been studied in [16]. However, there is – at least to the knowledge of the author – no complete understanding of the homotopy and homology groups of $L^{t,k}_{m,n}$ for arbitrary values of m, n, t, and k.

[1] By a thimble we mean the pair of topological spaces given by the product of the tangential and the normal Morse data at a given critical point. This might differ from the cell $(D^\lambda, \partial D^\lambda)$ occurring in classical Morse theory, see [7].

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2. Preliminaries

2.1. **Notations and Background.** In this article we will make use of the common terms of stratified Morse theory. The reader may consult the standard textbook reference [7]. Suppose we are given a manifold N and a closed subspace $Z \subset N$ with a Whitney stratification $\Sigma = (S_\alpha)_{\alpha \in A}$. For any point $p \in Z$ we will write

$$T_p Z := T_p S_\alpha$$

for the tangent space of the stratum S_α containing p. Furthermore, we say that a smooth map

$$f \colon M \to N \supset Z$$

from a manifold M to N is transverse to Z if f is transverse to all the strata.

Consider the set $X = f^{-1}(Z)$. It naturally decomposes into the sets $\Sigma_\alpha = f^{-1}(S_\alpha)$ given by the preimages of the strata of Z. Whenever $f \colon M \to N \supset Z$ is transverse to Z in M, the Σ_α form a Whitney stratification for X and we also say that X *inherits* the stratification of Z. In particular, this applies to the case of a closed embedding such as for example the fiber of a stratified submersion on Z induced from a map on N.

Throughout this article we usually consider *closed* Milnor balls B for singularities. This convention always assures that one automatically keeps track of the boundary behavior in deformations which can be a particularly tricky task in the setting of non-isolated singularities. Moreover, the resulting Milnor fibers are always *compact* stratified spaces which simplifies their treatment by Morse theory.

Since this note is merely an application of methods which had been developed before, we will restrict ourselves to the description of how the techniques can be used on determinantal singularities. To this end, we will review the cornerstones of the proofs of e.g. the Handlebody Theorem by Tibăr and other ideas behind it. However, the reader who is unfamiliar with the mathematical rigor on singularity theory on Whitney stratified spaces is strongly encouraged to consult the articles [13], [11], the references given there, and the standard textbook on stratified Morse theory [7].

2.2. **Essentially Isolated Determinantal Singularities.** Let $(M^t_{m,n}, 0) \subset (\mathrm{Mat}(m,n;\mathbb{C}), 0)$ be the generic determinantal variety of type (m,n,t):

$$M^t_{m,n} = \{M \in \mathrm{Mat}(m,n;\mathbb{C}) : \mathrm{rank}\, M < t\}.$$

The canonical rank stratification by

$$S^s_{m,n} = M^s_{m,n} \setminus M^{s-1}_{m,n}$$

for $0 < s \leq \min\{m,n\}+1$ is a Whitney stratification of $\mathrm{Mat}(m,n;\mathbb{C})$ and $M^t_{m,n}$. This can easily be deduced by induction from the observation that at any point $p \in S^s_{m,n}$ one has a product

$$(M^t_{m,n}, p) \cong (M^{t-s+1}_{m-s+1,n-s+1}, 0) \times (\mathbb{C}^{(m+n)\cdot(s-1)-(s-1)^2}, 0) \tag{2}$$

of analytic spaces. Consequently, the complex link of $M^t_{m,n}$ along the stratum $S^s_{m,n}$ is

$$\mathcal{L}^{t-s+1,(m-s+1)(n-s+1)-1}_{m-s+1,n-s+1}.$$

The complex links play a central role in the stratified Morse theory on complex analytic varieties because they determine the *normal Morse data*, see [7]. In the case of the generic determinantal variety $M^t_{m,n}$ we find from (2) that the normal Morse data along the stratum $S^s_{m,n}$ for $s \leq t$ is given by the pair of spaces

$$\left(C(L^{t-s+1,(m-s+1)(n-s+1)-1}_{m-s+1,n-s+1}), L^{t-s+1,(m-s+1)(n-s+1)-1}_{m-s+1,n-s+1} \right), \tag{3}$$

where $C(X)$ denotes the real cone over a given topological space X. We adopt the convention that $C(\emptyset) = \{pt\}$ is just one point.

Definition 2.1 ([4])**.** A determinantal singularity of type (m, n, t) is given by a holomorphic map germ

$$A\colon (\mathbb{C}^N, 0) \to (\mathrm{Mat}(m, n; \mathbb{C}), 0)$$

such that the space

$$(X_0, 0) := (A^{-1}(M^t_{m,n}), 0) \subset (\mathbb{C}^N, 0)$$

has *expected codimension* $\mathrm{codim}(X_0, 0) = \mathrm{codim}\, M^t_{m,n} = (m - t + 1)(n - t + 1)$.

A determinantal singularity $(X_0, 0)$ given by a matrix A is called *essentially isolated*, if the map A is transverse to the rank stratification of $\mathrm{Mat}(m, n; \mathbb{C})$ in a punctured neighborhood of the origin.

It follows directly from this definition that, away from the origin, X_0 inherits a canonical stratification by the strata

$$\Sigma^s := A^{-1}(S^s_{m,n}).$$

Counting dimensions yields that these strata are nonempty if and only if

$$\min\{r \in \mathbb{N} : (m - r + 1)(n - r + 1) < N\} \leq s \leq t. \tag{4}$$

and that

$$\dim \Sigma^s = N - (m - s + 1)(n - s + 1) > 0.$$

We supplement this stratification with the one-point stratum $\{0\} \subset X_0$ at the origin.

An *essential smoothing* of $(X_0, 0)$ is a family

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[d, "u"'] \\ \{0\} \arrow[r] & \mathbb{C} \end{tikzcd}$$

coming from a stabilization

$$\mathbf{A} : (\mathbb{C}^N, 0) \times (\mathbb{C}, 0) \to (\mathrm{Mat}(m, n; \mathbb{C}), 0) \times (\mathbb{C}, 0)$$

of the map A. That is $\mathbf{A} = \mathbf{A}(x, u) = (A_u(x), u)$ with $A_0 = A$ and A_u transversal to $M^t_{m,n}$ for all $u \neq 0$ sufficiently small. Then, the total space of the family above appears as $X = \mathbf{A}^{-1}(M^t_{m,n} \times \mathbb{C})$ and u is the map given by the deformation parameter.

From a stabilization we can construct the *determinantal Milnor fiber* as follows. Choose a representative

$$\mathbf{A} : W \times U \to \mathrm{Mat}(m, n; \mathbb{C}) \times U$$

of the stabilization $\mathbf{A}$ for some open sets $W \subset \mathbb{C}^N$ and $U \subset \mathbb{C}$ and let $B \subset \mathbb{C}^N$ be a Milnor ball for $(X_0, 0)$ in W. By this we mean a closed ball around the origin such that $\overline{X}_0 := X_0 \cap B$ is closed, the boundary ∂B intersects X_0 transversally, and

$$\overline{X}_0 \cong C(\partial \overline{X}_0)$$

is homeomorphic to the real cone over its boundary $\partial\overline{X}_0 = \partial B \cap X_0$. We can then consider the family $u : X \cap (B \times U) \to U$. It may be deduced from Thom's first Isotopy Lemma that u is a trivial topological fibration along the boundary $X \cap (\partial B \times U)$ over U and that

$$u : (X \cap (B \times U)) \setminus \overline{X}_0 \to U \setminus \{0\}$$

is a topological fiber bundle for U small enough.

Definition 2.2. It is the fiber of this bundle

$$\overline{X}_u \cong A_u^{-1}(M_{m,n}^t) \cap B$$

that we call the determinantal Milnor fiber.

Using the theory of versal unfoldings, one can show that in fact for any given EIDS $(X_0, 0)$ the determinantal Milnor fiber is unique up to homeomorphism, see [2] or [15].

Example 2.3. Consider the EIDS $(X_0, 0) \subset (\mathbb{C}^5, 0)$ of type $(2, 3, 2)$ given by the matrix

$$A = \begin{pmatrix} x & y & z \\ v & w & x \end{pmatrix}$$

together with the essential smoothing induced by the perturbation with

$$\begin{pmatrix} u & 0 & 0 \\ 0 & 0 & -u \end{pmatrix}.$$

It is easily seen that indeed the total space $(X, 0) \subset (\mathbb{C}^{5+1}, 0)$ is isomorphic to the generic determinantal variety $M_{2,3}^2 \subset \mathrm{Mat}(2, 3; \mathbb{C}) \cong \mathbb{C}^6$ and the map u is a generic linear form on it. Hence, the determinantal Milnor fiber of $(X_0, 0)$ is nothing but the (closure of the) complex link $L_{2,3}^{2,5}$ of $(M_{2,3}^2, 0)$. It is known that $L_{2,3}^{2,5}$ is homotopy equivalent to the 2-sphere S^2, see [5].

2.3. **The Handlebody Theorem.** In [13], M. Tibăr proofs the following theorem for the Milnor fiber F of an isolated hypersurface singularity

$$f : (Z, 0) \to (\mathbb{C}, 0)$$

on a complex analytic, Whitney stratified space $(Z, 0)$ of dimension $\dim(Z, 0) \geq 2$ and the complex link L of $(Z, 0)$:

Theorem 2.4 ([13], Handlebody Theorem)**.** *The Milnor fiber F is obtained from the complex link L to which one attaches cones over local Milnor fibers of stratified Morse singularities. The image of each such attaching map retracts within L to a point.*

We give a rough outline of the idea of the proof. We may assume $(Z, 0) \subset (\mathbb{C}^N, 0)$ to be embedded in some smooth ambient space. Let h be the linear equation on $\mathbb{C}^N$ defining the link L of $(Z, 0)$ and consider

$$\Phi = (h, f) : B \cap Z \cap \Phi^{-1}(D \times D') \to D \times D' \tag{5}$$

for a sufficiently small, closed ball B and discs $D, D' \subset \mathbb{C}$ around the origin. In [11], Lê has shown the following. There exists a Zariski open set $\Omega \subset \left(\mathbb{C}^N\right)^\vee$ of linear forms on the ambient space such that for $h \in \Omega$ the polar variety

$$\Gamma(h, f) := \overline{\{z \in Z \setminus f^{-1}(\{0\}) : \exists a \in \mathbb{C} : \mathrm{d}h(z)|_{T_z Z} = a \cdot \mathrm{d}f(z)|_{T_z Z}\}},$$

i.e. the critical locus of h on Z relative to f, is a curve which is branched over its image

$$\Delta = \Delta(h, f) = \Phi(\Gamma(h, f)) \subset D \times D',$$

the so-called Cerf-diagram. The proof for the set Ω of admissible hyperplane equations to be Zariski open can be found in [9]. Moreover, one can choose D' small enough such that the intersection $\Delta \cap (\partial D \times D')$ is empty. Then Φ is a topological fibration away from Δ and one has homeomorphisms

$$F \cong \Phi^{-1}(D \times \{\delta\})$$

and

$$L \cong \Phi^{-1}(\{\eta\} \times D')$$

for $0 \neq \delta$, resp. $0 \neq \eta$, sufficiently small. It is also shown in [9] that Ω can be chosen such that the restriction of $h \in \Omega$ to any fixed fiber $\Phi^{-1}(D \times \{\delta\})$ has only Morse singularities over the intersection points $\Delta \cap D \times \{\delta\}$ for $0 \neq \delta \in D'$.

At this point the so-called "carrousel" is furnished by the geometric monodromy of F along the boundary of D', i.e. by the variation of the value δ of f. But contrary to the classical viewpoint on monodromy one does not only construct a lifting of the unit tangent vector field along $\partial D'$ to $\Phi^{-1}(D \times \partial D')$, but one also keeps track of the monodromy induced on the disc $D \times \{\delta\}$, the intersection points $C = \Delta(h, f) \cap D \times \{\delta\}$, and the corresponding critical points of h on the Milnor fiber $\Phi^{-1}(D \times \{\delta\})$ over them.

Let $F' = \Phi^{-1}(\{(\eta, \delta)\})$. Then up to homotopy the Milnor fiber F is obtained from F' by attaching thimbles along suitably chosen paths in $D \times \{\delta\}$ from (η, δ) to the critical values of the stratified Morse points of h on F. The topology of each of these attachments is governed by the Morse data. In the situations we will encounter in the context of EIDS, the Morse data will always be of the following form:

Proposition 2.5. *Let $(X, p) \cong (M^s_{m,n}, 0) \times (\mathbb{C}^k, 0)$ and $h : (X, p) \to (\mathbb{C}, 0)$ a holomorphic map germ with a stratified Morse singularity at p. Then the thimble corresponding to this critical point is*

$$(C(S^k(L^{s,m\cdot n-1}_{m,n})), S^k(L^{s,m\cdot n-1}_{m,n})),$$

i.e. one attaches the real cone $C(S^k(L^{s,m\cdot n-1}_{m,n}))$ along its boundary $S^k(L^{s,m\cdot n-1}_{m,n})$.

The key observation from the Carrousel is that keeping track of the relative critical points of the hyperplane equation h on F allows one to determine exactly at which loci on F' these attachments take place.

As a final step, one constructs another homeomorphism $L \cong \Phi^{-1}(W) \subset F$ on a certain subspace $\Phi^{-1}(W)$ of F by "sliding along Δ". The space W is chosen such that $F' \subset \Phi^{-1}(W)$ and one can use the carrousel monodromy to show that for each thimble e one has to attach to $\Phi^{-1}(W)$ to complete it – up to homotopy – to F, there is already one thimble e' that had been attached to F' in the same spot as e to complete it to $\Phi^{-1}(W)$. This explains, why each attaching map in the statement of the Handlebody Theorem 2.4 retracts within L to a point.

3. Proof of the Main Theorem

3.1. **The Graph Transformation.** Let $(X_0, 0) \subset (\mathbb{C}^N, 0)$ be a determinantal singularity of type (m, n, t) given by a matrix A. In this section we will explain how to transform $(X_0, 0)$ into a *relative* complete intersection singularity on a canonical ambient space

$$(Z, 0) \cong (\mathbb{C}^N, 0) \times (M^t_{m,n}, 0),$$

see Definition 3.2.

Let $Y = \mathrm{Mat}(m, n; \mathbb{C}) \cong \mathbb{C}^{m\cdot n}$, $\mathbb{C}[\underline{y}] = \mathbb{C}[y_{i,j} | 1 \leq i \leq m, 1 \leq j \leq n]$ the associated coordinate ring and $\mathcal{O}_{m\cdot n} = \mathbb{C}\{\underline{y}\}$ the local ring of $(Y, 0)$. By abuse of notation, we will also write y for the tautological matrix $y \in \mathrm{Mat}(m,n;\mathbb{C})$ with entries $y_{i,j}$:

$$y = \begin{pmatrix} y_{1,1} & \cdots & y_{1,n} \\ \vdots & & \vdots \\ y_{m,1} & \cdots & y_{m,n} \end{pmatrix}.$$

Choose a representative $A : U \to Y$ of the matrix A defining $(X_0, 0)$ and let

$$\Gamma_A = \{(x,y) : y = A(x)\} \subset U \times Y$$

be the graph of A. Set $Z := U \times M^t_{m,n}$. Then, by construction, $X_0 \cong \Gamma_A \cap Z$.

We define two maps

$$\begin{aligned} p\colon U \times Y &\to Y, \qquad &(x,y) &\mapsto y - A(x), \\ q\colon U \times Y &\to U, \qquad &(x,y) &\mapsto x \end{aligned}$$

and form the commutative diagram

$$\begin{tikzcd} X_y \arrow[r, hook] \arrow[d] & Z \arrow[r, "q"] \arrow[d, "p"'] & U \\ \{y\} \arrow[r, hook] & Y. & \end{tikzcd} \tag{6}$$

While q is the projection to the first factor, the map p can be considered as the "projection to Y along the graph Γ_A". Clearly, for every point $y \in Y$ the space X_y is the determinantal variety

$$X_y = (A - y)^{-1}\left(M^t_{m,n}\right) = q\left(p^{-1}(\{y\})\right)$$

defined by the perturbation of A by the constant matrix y and we can consider X_y as a determinantal deformation of the EIDS $(X_0, 0)$.

Note that $(Z, 0)$ enjoys a canonical Whitney stratification by the strata

$$(\tilde{S}^s_{m,n}, 0) = (S^s_{m,n}, 0) \times (\mathbb{C}^N, 0)$$

inherited from the rank stratification on $M^t_{m,n}$. Whenever A is defining an EIDS, i.e. A is transverse to the rank stratification in a punctured neighborhood of the origin in $\mathbb{C}^N$, the above construction turns $(X_0, 0)$ into the fiber of a map $p|(Z, 0)$ which is a stratified submersion along $X_0 \subset Z$ on a punctured neighborhood of the origin in $\mathbb{C}^N \times \mathbb{C}^{m \cdot n}$:

Lemma 3.1. *Let $(x, A(x))$ be a point in the graph Γ_A of A. The restriction $p|Z$ is a stratified submersion on Z at $(x, A(x))$ if and only if the map $A : U \to Y$ is transverse to the rank stratification at $x \in U$.*

Proof. Let $(v_1, \dots, v_d)$ be local coordinates at $y = A(x)$ of the stratum $S^s_{m,n}$ containing y. Together with the standard coordinates of U, they form a coordinate system (x, v) of the stratum $\tilde{S}^s_{m,n}$ of Z at $(x, A(x))$. Now note that on the one hand the jacobian matrix of $p|Z$ at this point is of block form

$$\begin{pmatrix} \frac{\partial p(x,v)}{\partial x} & \frac{\partial p(x,v)}{\partial v} \end{pmatrix} = \begin{pmatrix} -\frac{\partial A(x)}{\partial x} & \frac{\partial y(v)}{\partial v} \end{pmatrix}$$

and p is a stratified submersion at $(x, A(x))$ if and only if this matrix has full rank $m \cdot n$. On the other hand, the map A is transverse to the rank stratification of Y at x, if and only if the tangent space $T_y Y$ of the ambient space Y at $y = A(x)$ can be generated by both the image of the differential of A – i.e. the span of the columns of the matrix $\frac{\partial A}{\partial x}$ – and the tangent space $T_y S^s_{m,n}$ of the stratum $S^s_{m,n}$. Since $T_y S^s_{m,n}$ is by definition the span of the second block $\frac{\partial y}{\partial v}$ in the jacobian matrix of p, the claim follows. $\square$

The components of the map p define the graph Γ_A via

$$p_{i,j}(x,y) = y_{i,j} - a_{i,j}(x) = 0$$

and clearly, Γ_A is a complete intersection in $U \times Y$. The determinantal singularity $X_0 \cong Z \cap \Gamma_A$ appears as the intersection of Γ_A with $Z = \mathbb{C}^N \times M^t_{m,n}$. While $M^t_{m,n}$ is not a complete intersection in general, it is nevertheless always a Cohen-Macaulay space, see [10]. Since $(X_0, 0)$ has expected dimension and

$$\mathcal{O}_{X_0,(x,y)} \cong \mathcal{O}_{Z,(x,y)}/\langle p_{1,1}, \dots, p_{m,n}\rangle,$$

the components $p_{i,j}(x,y)$ of p must also form a regular sequence on $\mathcal{O}_Z$, the structure sheaf of Z; cf. [1, Theorem 2.1.2 c)]. We give a general definition of the object we just encountered.

Definition 3.2. Let $(Z,0) \subset (\mathbb{C}^r, 0)$ be a germ of a complex analytic space and

$$f\colon (\mathbb{C}^r, 0) \to (\mathbb{C}^c, 0)$$

a holomorphic map.

We say that the restriction $f|(Z,0)$ is a complete intersection morphism, if the components $f_1, \dots, f_c$ form a regular sequence on $\mathcal{O}_{Z,0}$.

If, moreover, $(Z,0)$ is endowed with a Whitney stratification, we say that $f|(Z,0)$ has an isolated *relative* complete intersection singularity (IRCIS) on $(Z,0)$ whenever there exists a punctured neighborhood U of 0 in $\mathbb{C}^r$ such that at every point $z \in U \cap Z \cap f^{-1}(\{0\})$ in the central fiber, the restriction $f|(Z,0)$ is a stratified submersion at z.

We have just verified:

Proposition 3.3. *The restriction $p|(Z,0)$ is a complete intersection morphism which realizes $(X_0,0) = p^{-1}(\{0\}) \cap (Z,0)$ as an IRCIS of p on $(Z,0)$.*

We will refer to the above construction as the *graph transformation* of the EIDS $(X_0,0) = (A^{-1}(M^t_{m,n}), 0)$. This transformation allows us to study $(X_0,0)$ with the classical methods for complete intersections. To this end, we will fix some notation. Let

$$\mathbf{W} = (\{0\} = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_{m\cdot n-1} \subsetneq W_{m\cdot n} = \mathbb{C}^{m\cdot n})$$

be a maximal ascending flag in $Y = \mathrm{Mat}(m,n;\mathbb{C})$ and

$$\mathbf{V} = \left(\mathbb{C}^N \times \mathrm{Mat}(m,n;\mathbb{C}) = V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_{m\cdot n-1} \supsetneq V_{m\cdot n}\right)$$

a descending flag in $\mathbb{C}^N \times \mathrm{Mat}(m,n;\mathbb{C})$ with $\dim V_i/V_{i+1} = 1$ for each i.

For each $k > 0$ we set

$$Z_k := Z \cap p^{-1}(W_k) \cap V_{k-1}. \tag{7}$$

The two projections p and q induce natural maps

$$\begin{tikzcd} & Z_k \arrow[dl, "f_k"'] \arrow[dr, "h_k"] & \\ W_k/W_{k-1} & & V_{k-1}/V_k. \end{tikzcd} \tag{8}$$

Proposition 3.4. *If the flags $\mathbf{W}$ and $\mathbf{V}$ are in general position, then the following holds.*

1. *Each of the spaces Z_k inherits the canonical Whitney stratification from $(Z,0)$ outside the origin.*
2. *Each f_k defines an isolated hypersurface singularity on $(Z_k,0)$ relative to the given stratification.*

(3) *The function h_k is a linear equation on $(Z_k, 0)$, which can be used to define the complex link and the carrousel.*

Proof. We do induction on k. Let $k = 1$. By definition $V_{k-1} = V_0 = \mathbb{C}^N \times \operatorname{Mat}(m, n; \mathbb{C})$. Consider the map

$$\mathbb{P}p : Z \setminus X_0 \to \mathbb{P}^{m \cdot n - 1}, \quad (x, y) \mapsto (p_{1,1}(x, y) : \cdots : p_{m,n}(x, y))$$

and let $[W_1] \in \mathbb{P}^{m \cdot n - 1}$ be a regular value of this map. Choose a splitting $\mathbb{C}^{m \cdot n} \cong (\mathbb{C}^{m \cdot n}/W_1) \oplus W_1$ and write $p = (\tilde{p}, f_1)$ with

$$\tilde{p} \colon z \mapsto p(z) + W_1 \in \mathbb{C}^{m \cdot n}/W_1 \cong \mathbb{C}^{m \cdot n - 1}.$$

Then $Z_1 = p^{-1}(W_1) = \tilde{p}^{-1}(\{0\})$ does not have critical points of $\tilde{p}$ outside $X_0 = \{f_1 = 0\} \subset Z_1$. Suppose $(x, y) \in X_0$, $x \neq 0$ was a critical point of $\tilde{p}$ on Z_1 in X_0 and S the stratum of Z containing it. Then the differential $\mathrm{d}(\tilde{p}|S)(x, y)$ does not have full rank and, hence, also $\mathrm{d}(p|S)(x, y)$ can not have full rank – a contradiction to X_0 being an IRCIS. We conclude that $\tilde{p}$ is a stratified submersion on Z at all points of Z_1 except the origin. Therefore, Z_1 inherits the Whitney stratification from $(Z, 0)$ and $f_1 : (Z_1, 0) \to \mathbb{C}$ defines an IRCIS on $(Z_1, 0)$.

For a given isolated singularity $f_1 \colon (Z_1, 0) \to (\mathbb{C}, 0)$ the condition on a linear equation h_1 to be sufficiently general to define the carrousel is Zariski open; cf. [13]. We may choose h_1 accordingly and set $V_1 = \{h_1 = 0\}$.

For the induction step we start by projectivizing the map $\tilde{p}$:

$$\mathbb{P}\tilde{p} \colon Z \cap V_k \setminus p^{-1}(W_{k-1}) \to \mathbb{P}(\mathbb{C}^{m \cdot n}/W_{k-1}), \quad (x, y) \mapsto [p(x, y) + W_{k-1}].$$

Choose a subspace $W_k \subset \mathbb{C}^{m \cdot n}$ such that $[W_k/W_{k-1}]$ is a regular value of this map. The rest of the induction step is merely a repetition of the above said and left to the reader. ☐

In what follows, we will from now on assume that the flags $\mathbf{V}$ and $\mathbf{W}$ have been chosen to fulfill Proposition 3.4. For any $k > 0$ let

$$F_k = f_k^{-1}(\{\delta\}) \cap Z_k \cap B \tag{9}$$

be the Milnor fiber of f_k on Z_k for a suitable choice of a Milnor ball B and $\delta \in \mathbb{C} \setminus \{0\}$ small enough. We denote the complex link of Z_k by

$$L_k = h_k^{-1}(\{\eta\}) \cap Z_k \cap B, \tag{10}$$

$\eta \in \mathbb{C} \setminus \{0\}$ small enough.

3.2. **The induction argument.** We can apply the Handlebody Theorem of Tibăr at each step k in the setup of the previous section to obtain our Main Theorem. The key lemma for this induction can already be extracted from [13, Corollary 4.2]:

Lemma 3.5. *In the final setup of the standard transformation we have for each $0 < k < m \cdot n$ a (non-canonical) homeomorphism*

$$L_k \cong F_{k+1}. \tag{11}$$

Proof. One has homeomorphisms

$$\begin{array}{rcl} F_{k+1} & = & Z_{k+1} \cap f_{k+1}^{-1}(\{\delta\}) \cap B \\ & = & Z \cap V_k \cap p^{-1}(W_{k+1}) \cap f_{k+1}^{-1}(\{\delta\}) \cap B \\ & = & Z \cap V_{k-1} \cap p^{-1}(W_{k+1}) \cap h_k^{-1}(\{0\}) \cap f_{k+1}^{-1}(\{\delta\}) \cap B \\ & \cong & Z \cap V_{k-1} \cap p^{-1}(W_{k+1}) \cap h_k^{-1}(\{\eta\}) \cap f_{k+1}^{-1}(\{\delta\}) \cap B \\ & \cong & Z \cap V_{k-1} \cap p^{-1}(W_{k+1}) \cap h_k^{-1}(\{\eta\}) \cap f_{k+1}^{-1}(\{0\}) \cap B \\ & \cong & Z \cap V_{k-1} \cap p^{-1}(W_k) \cap h_k^{-1}(\{\eta\}) \cap B \\ & \cong & L_k \end{array}$$

for a Milnor ball B and sufficiently small values for δ and η. The homeomorphisms are induced from the parallel transport in the fibration given by

$$\Phi = (h_k, f_{k+1})\colon Z \cap V_{k-1} \cap p^{-1}(W_{k+1}) \cap B \to \mathbb{C} \times \mathbb{C}$$

as in (5) over suitably chosen paths connecting $(0, \delta)$, (η, δ), and $(\eta, 0)$. □

Proof. (of Theorem 1.1) After applying the graph transformation we obtain for $k = 1$:

$$\overline{X}_u = f_1^{-1}(\{\delta\}) \cap Z_1 \cap B = F_1,$$

because W_1 was in general position. This space is naturally stratified by the strata Σ^s of dimension

$$\dim \Sigma^s = N - (m - s + 1)(n - s + 1)$$

for $s_0 \leq s \leq t$ with $s_0 = \min\{r \in \mathbb{N}_0 : (m - r + 1)(n - r + 1) \leq N\}$ and the complex link along Σ^s is $L_{m-s+1,n-s+1}^{t-s+1,(m-s+1)(n-s+1)-1}$. We may apply Proposition 2.5 to determine the thimbles associated to Morse critical points on the strata. It is the pair of spaces consisting of

$$S^{N-(m-s+1)(n-s+1)}(L_{m-s+1,n-s+1}^{t-s+1,(m-s+1)(n-s+1)-1})$$

and the cone over it. According to the Handlebody Theorem [13], the space F_1 then has a bouquet decomposition

$$F_1 \cong_{ht} L_1 \vee \bigvee_{s_0 \leq s \leq t} \bigvee_{i=1}^{r_1(s)} S^{N-(m-s+1)(n-s+1)+1}(L_{m-s+1,n-s+1}^{t-s+1,(m-s+1)(n-s+1)-1}).$$

Note that, since the image of the attaching maps in L_1 retract to a point, we obtain one more suspension compared to the formula for the thimble.

We may now proceed inductively and replace L_k by F_{k+1} in this formula according to Lemma 3.5. At each step we attach a certain number $r_k(s)$ of thimbles and we may add them up to $r(s) = \sum_{k=1}^{m \cdot n-1} r_k(s)$. This finishes the proof. □

Corollary 3.6. *If the singularity $(X_0, 0)$ in the setting of Theorem 1.1 is smoothable (i.e. if $N < (m - t + 2)(n - t + 2)$), then*

$$\overline{X}_u \cong_{ht} L_{m,n}^{t,N} \vee \bigvee_{i=1}^{r} S^d, \tag{12}$$

with $d = N - (m - t + 1)(n - t + 1) = \dim(X_0, 0)$.

4. EIDS of type $(2, n, 2)$

In this section we will be concerned with arbitrary EIDS $(X_0, 0) \subset (\mathbb{C}^N, 0)$ of type $(2, n, 2)$. The requirement on $(X_0, 0)$ to have expected dimension relates N and the dimension $d = \dim(X_0, 0)$ via

$$d = N - (2 - 2 + 1)(n - 2 + 1) = N - n + 1.$$

In particular, we always have $n - 1 \leq N$. Note that Theorem 1.1 is only applicable if $n \leq N$.

If we require $(X_0, 0)$ to be smoothable, we also obtain an upper bound on N given by

$$N < (2 - 2 + 2)(n - 2 + 2) = 2n.$$

4.1. **The homotopy type of $L^{2,N}_{2,n}$.** We shall first determine the homotopy type of all the spaces $L^{2,N}_{2,n}$, see (13), (15), (17), and (18).

Whenever $N \geq 2n$, any generic linear map

$$\Phi \colon (\mathbb{C}^N, 0) \to (\mathrm{Mat}(2, n; \mathbb{C}), 0)$$

is a submersion and in particular stable. The interior of the determinantal Milnor fiber of Φ is therefore given by

$$L^{2,N}_{2,n} \cong \mathbb{C}^{N-2n} \times M^2_{2,n} \cong_{ht} \{pt\}. \tag{13}$$

Suppose $N < 2n$. Let $M = M^2_{2,n}$ be the generic determinantal variety and

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, "\pi"'] & \mathrm{Mat}(2, n; \mathbb{C}) \times \mathbb{P}^1 \arrow[r] \arrow[d] & \mathbb{P}^1 \\ M \arrow[r, hook] & \mathrm{Mat}(2, n; \mathbb{C}) & \end{tikzcd}$$

its Tjurina transform (see e.g. [14], or [15]) resulting from the blowup of the rational map

$$\Psi : M \dashrightarrow \mathbb{P}^1, \quad y \mapsto [\ker(y)].$$

If we let $y_{i,j}$ be the canonical coordinates of $\mathrm{Mat}(2, n; \mathbb{C})$ and $(s_1 : s_2)$ the homogeneous coordinates of $\mathbb{P}^1$ then the equations for W are

$$s_1 \cdot y_{2,j} - s_2 \cdot y_{1,j} = 0 \text{ for } j = 1, \ldots, n. \tag{14}$$

We may consider $y_{1,j}$ and $y_{2,j}$ as linear fiber coordinates in local trivializations of the tautological bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ for every j. Thus, W is a smooth complex manifold isomorphic to the total space of the vector bundle $(\mathcal{O}_{\mathbb{P}^1}(-1))^n$.

Instead of describing an embedding

$$\Phi \colon \mathbb{C}^N \to \mathrm{Mat}(2, n; \mathbb{C})$$

of a linear subspace defining an EIDS $(X_0, 0) = (\Phi^{-1}(M^2_{2,n}), 0)$, we may also choose a linear form

$$l = (l^1, \ldots, l^{2n-N}) \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{Mat}(2, n; \mathbb{C}), \mathbb{C}^{2n-N})$$

such that $\Phi(\mathbb{C}^N) = \ker(l)$. Since all equations involved in this process are either linear or homogeneous, we may neglect the choice of Milnor balls. We obtain an extension of the above diagram to the left:

$$\begin{tikzcd}
\tilde{X}_0 \arrow[r, hook] \arrow[d] & W \arrow[r, hook] \arrow[d, "\pi"'] & \mathrm{Mat}(2,n;\mathbb{C}) \times \mathbb{P}^1 \arrow[r] \arrow[d] & \mathbb{P}^1 \\
X_0 \arrow[r, hook] \arrow[d] & M \arrow[r, hook] \arrow[d, "l"'] & \mathrm{Mat}(2,n;\mathbb{C}) & \\
\{0\} \arrow[r, hook] & \mathbb{C}^{2n-N}, & &
\end{tikzcd}$$

with $\tilde{X}_0 := \pi^{-1}(X_0)$. The interior of the determinantal Milnor fiber $L^{2,N}_{2,n}$ of Φ is then given by

$$L^{2,N}_{2,n} = M \cap l^{-1}(\{u\})$$

for some regular value u of l on M.

Utilizing the trace pairing (see e.g. [4])

$$\mathrm{Mat}(2,n;\mathbb{C}) \times \mathrm{Mat}(2,n;\mathbb{C}) \to \mathbb{C}, \quad (A,B) \mapsto \mathrm{trace}(A^T \cdot B),$$

we may write the components of l in the form

$$l^k = \begin{pmatrix} l^k_{1,1} & l^k_{1,2} & \cdots & l^k_{1,n} \\ l^k_{2,1} & l^k_{2,2} & \cdots & l^k_{2,n} \end{pmatrix}$$

for constant entries $l^k_{i,j} \in \mathbb{C}$. We leave it to the reader to verify that in the range $n \leq N < 2n$, a sufficiently general choice for l is given by choosing the $2n-N$ components l^k from the following n matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}, \quad \cdots$$
$$\cdots, \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Fix one value $n \leq N < 2n$ and the linear form $l \colon \mathrm{Mat}(2,n;\mathbb{C}) \to \mathbb{C}^{2n-N}$ as above and consider the algebraic sets

$$W \supset \pi^{-1}(\{l=0\}) = \tilde{X}_0 \xrightarrow{\pi} X_0 = M \cap l^{-1}(\{0\}).$$

Using the above equations (14) for W and $\pi^* l^k, k = 1, \dots, 2n-N$ we see that $\tilde{X}_0$ is a local complete intersection in $\mathrm{Mat}(2,n;\mathbb{C}) \times \mathbb{P}^1$.

Moreover, whenever $N > n$ – i.e. whenever $d = \dim(X_0, 0) > 1$ – $\tilde{X}_0$ is isomorphic to the total space of the vector bundle

$$\mathcal{O}_{\mathbb{P}^1}(-(2n-N+1)) \oplus (\mathcal{O}_{\mathbb{P}^1}(-1))^{N-n-1}$$

and in particular smooth of dimension $d = N-n+1$. Passing from $l = 0$ to a regular value $l = u$ therefore results in a flat deformation of $\tilde{X}_0$ which is topologically trivial due to Ehresmann's theorem. Since the set $X_u = M \cap \{l = u\}$ does not meet the locus $M^1_{2,n} = \{0\}$ where Ψ is not defined, the projection $\pi \colon \tilde{X}_u \to X_u$ is an isomorphism and we obtain homotopy equivalences

$$S^2 \cong \mathbb{P}^1 \cong_{ht} \tilde{X}_0 \cong_{ht} \tilde{X}_u \cong_{ht} X_u \cong_{ht} L^{2,N}_{2,n} \text{ for } n < N < 2n. \tag{15}$$

In the particular case where $N = n$ – i.e. when X_0 is a curve and the components of l comprise all of the above listed linear forms – we find the following system of equations for $\tilde{X}_0$ in the chart

$\{s_1 \neq 0\}$:

$$
\begin{aligned}
y_{2,j} &= \frac{s_2}{s_1} y_{1,j}, && j = 1, \dots, n, \\
y_{1,j} &= \left(-\frac{s_2}{s_1}\right) y_{1,j+1}, && j = 1, \dots, n-1, \\
y_{1,n} &+ y_{2,1} = 0.
\end{aligned}
$$

We may eliminate the variables $y_{2,j}$ for all j and express all $y_{1,j}$ in terms of $y_{1,n}$ for $j < n$. Substituting this into the last equation yields

$$
y_{1,n} \left(1 - \left(-\frac{s_2}{s_1}\right)^n\right) = 0.
$$

Thus,

$$
\tilde{X}_0 = \tilde{L}_1 \cup \tilde{L}_2 \cup \cdots \cup \tilde{L}_n \cup E \quad \xrightarrow{\pi} \quad L_1 \cup L_2 \cup \cdots \cup L_n = X_0 \subset \mathbb{C}^N \tag{16}
$$

consists of exactly n lines $\tilde{L}_1, \dots, \tilde{L}_n$ meeting the exceptional set $E = \{0\} \times \mathbb{P}^1$ of π transversally in the points $(s_1 : s_2) = (1 : -\zeta_n^k)$, $\quad k = 0, \dots, n$ with ζ_n a primitive n-th root of unity. Since the projection π is an isomorphism outside E, the $\tilde{L}_i$ are taken to a set of lines $L_i \subset \mathbb{C}^N$, which meet pairwise at the origin.

The situation is depicted in Figure 1 for the case $n = 3$. Note that X_0 is drawn as three cones touching each other at their vertices. This is intrinsically homeomorphic to three complex lines meeting at the origin, but drawn as embedded in real 3-space. In fact, all the pictures really capture the described objects up to homeomorphism.

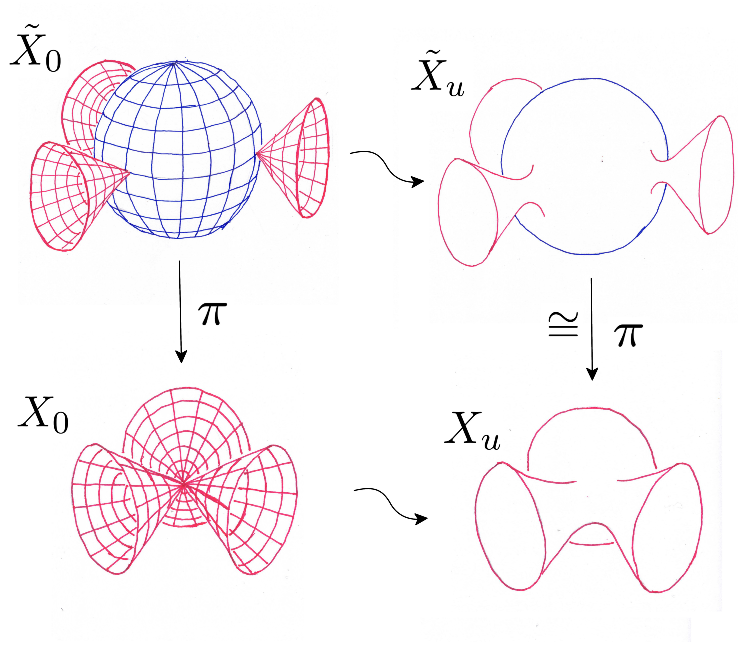

FIGURE 1. Deformation of a space curve and its Tjurina transform for $n = 3$

It is not clear a priori how the topology of X_u changes compared to X_0 when passing to a regular value u of l. For $\tilde{X}_0$, however, the induced deformation must be a smoothing of the n

distinct singularities of $\tilde{X}$ at the points $(1 : \zeta_n^k)$, because again $\pi\colon \tilde{X}_u \to X_u$ is an isomorphism and X_u is smooth. Locally and up to homotopy, the smoothing replaces a neighborhood $D_k \cup \tilde{L}_k$ of the line $\tilde{L}_k$ in X_0 by a punctured disc D_k^* at every such point. Thus $\tilde{X}_u$ has the homotopy type of a punctured 2-sphere with n points missing:

$$\tilde{X}_u \cong_{ht} L_{2,n}^{2,n} \cong_{ht} S^2 \setminus \{n \text{ points}\} \cong_{ht} \bigvee_{i=1}^{n-1} S^1. \tag{17}$$

For the last admissible value $N = n-1$ of N observe that the space $L_{2,n}^{2,n-1}$ is given by the intersection of

$$X_0 = L_1 \cup L_2 \cup \cdots \cup L_n$$

in (16) from the previous considerations with a further codimension one hyperplane in general position off the origin. Clearly, this intersection consists of precisely n points and therefore

$$L_{2,n}^{2,n-1} = \{n \text{ points}\}. \tag{18}$$

4.2. **Arbitrary EIDS of type** $(2, n, 2)$**.** Suppose that

$$A\colon (\mathbb{C}^N, 0) \to (\mathrm{Mat}(2, n; \mathbb{C}), 0)$$

defines an arbitrary EIDS $(X_0, 0) = (A^{-1}(M_{2,n}^2), 0) \subset (\mathbb{C}^N, 0)$ of type $(2, n, 2)$. We will describe the homotopy type of its determinantal Milnor fiber in all cases (19), (20), (21), and (22).

Whenever $N = n-1$, i.e. if $\dim(X_0, 0) = 0$ and $(X_0, 0)$ is a fat point, the determinantal Milnor fiber will consist of a finite number of distinct, regular points

$$\overline{X}_u = \{k \text{ points}\} \quad \text{if } N = n-1. \tag{19}$$

Since $(X_0, 0)$ is Cohen-Macaulay, we may use the principle of conservation of number and compute this number k directly from the local algebra:

$$k = \dim_{\mathbb{C}} \mathcal{O}_{X_0,0}.$$

Now let $(X_0, 0) = (A^{-1}(M_{2,n}^2), 0)$ be a curve, i.e. $d = 1 \Leftrightarrow N = n$. Theorem 1.1 is applicable and we have $s_0 = t = 2$. Hence, there is only one number $r = r(2)$ which is relevant in the bouquet decomposition (1). The homotopy type of the determinantal Milnor fiber $\overline{X}_u$ is

$$\overline{X}_u \cong_{ht} \left(\bigvee_{i=1}^{n-1} S^1\right) \vee \left(\bigvee_{i=1}^{r} S^1\right) \quad \text{if } N = n. \tag{20}$$

Suppose $d = \dim(X_0, 0) > 1$ and $(X_0, 0)$ is smoothable. This allows a range $n < N < 2n$ for N and according to the computations in the previous section we find

$$\overline{X}_u \cong_{ht} S^2 \vee \bigvee_{i=1}^{r} S^{N-n+1} \quad \text{if } n < N < 2n. \tag{21}$$

Note that whenever $d \geq 3$, there is still a 2-sphere in the decomposition! This is a striking difference to any behavior which can be observed for ICIS.

Finally, for values $N \geq 2n$, a determinantal singularity $(X_0, 0)$ of type $(2, n, 2)$ does not admit a determinantal smoothing. Nevertheless, the determinantal Milnor fiber $\tilde{X}_u$ is defined up to homeomorphism. In this case we find $s_0 = 1 \leq s \leq t = 2$ and we have different contributions in the bouquet decomposition. The complex link $L_{2,n}^{2,N}$ is homotopically trivial. But the thimble which is being attached to $L_{2,n}^{2,N}$ at a Morse critical point for $s = 1$ has a nontrivial normal Morse datum

$$\left(C(L_{2,n}^{2,2n-1}), L_{2,n}^{2,2n-1}\right).$$

Thus, according to (15) we find

$$\overline{X}_u \cong_{ht} \{pt\} \vee \left(\bigvee_{i=1}^{r(1)} S^3\right) \vee \left(\bigvee_{i=1}^{r(2)} S^{N-n+1}\right) \qquad \text{for } N \geq 2n. \tag{22}$$

Remark 4.1. The decomposition (1) in Theorem 1.1 reduces the question about the homotopy type of a determinantal Milnor fiber to the question about the topology of the spaces $L_{m,n}^{t,k}$ appearing in the formula. In those cases, where all these $L_{m,n}^{t,k}$ themselves are homotopy equivalent to a bouquet of spheres, the same holds for the determinantal Milnor fiber.

Moreover, the generalized Milnor numbers $r(s)$ measuring the contributions from critical points on the different strata are invariants of the singularity. Using computer algebra systems like Singular, one can compute these numbers for any given singularity from the Cerf-diagrams Δ in the carrousel [13, Section 1.4] at each induction step in the proof of Theorem 1.1. However, these computations involve random choices of linear equations and it would be appealing to have a concise formula relating the numbers $r(s)$ to analytic invariants of the singularity itself.

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