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ALGEBRAIC DIFFERENTIAL EQUATIONS OF PERIOD-INTEGRALS

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ABSTRACT. We explain that in the study of the asymptotic expansion at the origin of a period-integral or of a hermitian period the computation of the Bernstein polynomial of the "fresco" (filtered differential equation) associated to the pair of germs of a holomorphic function with a holomorphic volume form gives a better control than the computation of the Bernstein polynomial of the full Brieskorn module of the germ of f at the origin. Moreover, it is easier to compute as it has a better functoriality and smaller degree. We illustrate this in the case where the polynomial f in (n+1) variables has (n+2) monomials and is not quasi-homogeneous, by giving an explicit simple algorithm to produce a multiple of this Bernstein polynomial in the case of a monomial holomorphic volume form. Several concrete examples are given.

1. INTRODUCTION

This article simplifies and improves two unpublished papers, see [3] and [2], on the computation of the Bernstein polynomial associated to a period-integral or to a hermitian period.
The main result of this paper gives a numerical necessary condition in order that the asymptotic expansion at $s = 0$ of a hermitian period

$$\frac{1}{(2i\pi)^n}\int_{f=s}\rho\frac{\omega}{df}\wedge\overline{\frac{\omega'}{df}}$$

associated to a pair ω, ω' of holomorphic volume forms (where $\rho \in \mathscr{C}^\infty$ is identically 1 near 0) has a non-zero singular term of the type $|s|^{2\xi}s^m\bar{s}^{m'}(Log\,|s|)^p$, where $\xi \in]-1, 0] \cap \mathbb{Q}, m, m' \in \mathbb{N}$, (we define $\mathbb{N}$ as the set of non-negative integers) and $p \in \mathbb{N}$ ($p \geq 1$ when $\xi = 0$), assuming that such a term does not exist when ρ is identically 0 near the origin.

Let us be more explicit about this goal. We consider a holomorphic function $f : U \to \mathbb{C}$ on an open neighborhood U of the origin in $\mathbb{C}^{n+1}$ which has a critical point at 0. The singularity is not assumed to be isolated, but we choose U small enough in order that $f(0) = 0$ is the only critical value of f on U[1]. We are interested, for instance, in the meromorphic extension of the holomorphic function for $\Re(\lambda) \gg |h|$:

$$\frac{1}{\Gamma(\lambda)}\int_U |f|^{2\lambda}\bar{f}^h\rho\omega\wedge\bar{\omega}'$$

where ω and ω' are given holomorphic $(n+1)-$forms on U, h is in $\mathbb{Z}$ and ρ is a $\mathscr{C}^\infty_c(U)$ function identically 1 near 0.

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[1] Recall that for any holomorphic function f with a critical point at the origin, such a U always exists.

This is the complex Mellin transform of the hermitian period (see [13]):

$$s \mapsto \frac{1}{(2i\pi)^n}\int_{f=s} \rho\frac{\omega}{df} \wedge \overline{\frac{\omega'}{df}}.$$

We shall assume that, for a given integer $q \geq 1$ and for a given $\xi \in \mathbb{Q}$, the meromorphic extension of

$$(H(\xi, q)) \qquad \frac{1}{\Gamma(\lambda)}\int_U |f|^{2\lambda}\varphi$$

has poles at points in $\xi + \mathbb{Z}$ which are of order at most $q-1$, for any $(n+1, n+1)$ differential form $\varphi \in \mathscr{C}^\infty_c(U \setminus \{0\})$ (so $\varphi \equiv 0$ near 0).
For instance, if the singularity of f is isolated at the origin, this hypothesis will always be true for $q = 1$ and any $\xi \in \mathbb{Q}$.
When the singularity of f is not isolated, this condition will be satisfied for $q = 1$ if and only if (thanks to [5]; see also [15]) the local monodromy of f acting on the reduced cohomology of the Milnor fibre at each point near 0, but distinct of 0, does not present the eigenvalue $\exp(2i\pi\xi)$.

In general, this assumption means that any pole of order $\geq q$ at a point in $\xi + \mathbb{Z}$ comes from the germ of our situation at the origin. But note that even if the eigenvalue $\exp(2i\pi\xi)$ of the local monodromy of f acting on the reduced cohomology of the Milnor fibre at any point near 0 (including 0) is simple, we may find a pole of order 2 for such an integral because the phenomenon of "entanglement of consecutive strata" may appear (see [7] for a topological description and [10] for a description in term of "Brieskorn modules" of this phenomenon).
We shall give in Theorem 3.2.1 and in Corollary 3.2.2 some necessary numerical conditions which control the order of poles at points in $\xi+\mathbb{Z}$ for a given holomorphic $(n+1)-$form ω which is much more precise than the "classical condition" asking that the Bernstein polynomial of f at 0 has at most $(q-1)$ roots (counting multiplicities) in the set $\xi + \mathbb{Z}$ (condition which in fact gives the result for such an integral when we replace $\rho\omega \wedge \omega'$ by any differential form $\varphi \in \mathscr{C}^\infty_c(U)^{n+1,n+1}$).
The precise result for the Mellin transforms of hermitian periods is given in Theorem 3.2.1 (the remark following the proof of Corollary 3.2.2 indicates also a variant which can be obtained by the same method.)
The examples given at the end of this paper show not only that the Bernstein polynomial of the **fresco** associated to the pair (f, ω) is much easier to compute than the full Bernstein polynomial of f, but also that it has, in general, a much smaller number of roots.

The main tool around this kind of technique will be the following generalization of the use of a Bernstein identity to control the poles of the Mellin transform of a "hermitian period" of the form

$$(1) \qquad F(\lambda) := \frac{1}{\Gamma(\lambda)}\int_U |f|^{2\lambda}\bar{f}^h\omega \wedge \psi$$

where ω is a holomorphic form in $\Omega^{n+1}(U)$, $\psi \in \mathscr{C}^{\infty,(0,n+1)}_c(U)$ is $d-$**closed near** 0 and h is in $\mathbb{Z}$.

Theorem 1.0.1. *Let $f : U \to \mathbb{C}$ be a holomorphic function on a polydisc U with center 0 in $\mathbb{C}^{n+1}$ and assume that $f(0) = 0$ is the only critical value of f on U. For ξ given in $\mathbb{Q}$, assume that the hypothesis $H(\xi, 1)$ is satisfied (see above).*
Let ω be a $(n+1)$ holomorphic differential form on U. Assume that the class induced by ω in

E_0^{n+1} is annihilated by the element[2]

$$P := (a - \lambda_1 b)S_1(b) \dots (a - \lambda_k b)S_k(b)$$

in $\tilde{\mathcal{A}}$, where for each $j \in [1, k]$, $S_j \in \mathbb{C}[[b]]$ satisfies $S_j(0) = 1$. Note that this assumption depends only on the germs at the origin of f and of ω.
*Now fix $\psi \in \mathscr{C}_c^{\infty,(0,n+1)}(U)$ which is $d-$***closed near** 0 *and assume that for some $h \in \mathbb{Z}$ the meromorphic extension of*

$$F_h^{\psi}(\lambda)[\omega] := \frac{1}{\Gamma(\lambda)} \int_U |f|^{2\lambda} \bar{f}^h \omega \wedge \psi$$

has a pole of order $d \geq 1$ at some point in $\xi + \mathbb{Z}$. Then there exist at least d values of $j \in [1, k]$ such that λ_j is in $\xi + \mathbb{Z}$.

The algebra $\tilde{\mathcal{A}}$ is defined in section 2 (see formula (6)) and for the definition of the geometric (a,b)-module E_0^{n+1} and the notion of the Bernstein polynomial of a fresco, see section 2 below.

Note that the hypothesis of the existence of such a P is always true. But in practice (see section 4) we may have such a P but we do not know that its initial form in (a,b) corresponds to the Bernstein polynomial of the fresco associated to ω. It is only a member of the (principal) left ideal which annihilates the class of ω in E_0^{n+1}.
So, under the hypothesis of this theorem, the Bernstein polynomial of the fresco F_ω associated to the pair (f, ω) (which is the geometric (a,b)-module generated by $[\omega]$ in E_0^{n+1}) divides the polynomial

$$B(\lambda) := \prod_{j=1}^{k} (\lambda + \lambda_j + j - k).$$

In the more precise statement given in section 3 (see Theorem 3.1.2) we make precise the values of these roots of the Bernstein polynomial of F_ω from the "jumps" of the orders of poles in $\xi + \mathbb{Z}$.

We shall give a more precise result in Theorem 3.1.2 and some interesting variants using the hypothesis $H(\xi, q)$ in Theorem 3.2.1 and Corollary 3.2.2 in section 3.
Remark. Notice that here we use in fact only a "one variable" differential equation (in fact multiplication by the variable in $\mathbb{C}$ and integration relative to this variable) instead of partial differential operators on $\mathbb{C}^{n+1}$ as in the Bernstein identity for f at the origin. This is precisely one of the points of interest of using an (a,b)-module structure in this setting.

So we begin this article by a short overview on geometric (a,b)-modules and frescos intended for the reader not familiar with the use of Brieskorn modules in the study of the singularities of a holomorphic function on a complex manifold.
In opposition with the preprint [2] cited above, we let aside the global point of view, that is to say the study of the global fresco associated to a period-integral in the case of a proper holomorphic function on a complex manifold, because it uses more heavy tools and very often the local study presented here would be enough to obtain good information, using a partition of unity.
The reader interested in this global setting may consult the preprint [2] mentioned above and also the preprint [1].
It is important to notice that we are dealing here with general singularities of a holomorphic

[2] Compare this hypothesis with Theorem 2.3.1 which is recalled in section 2.

function (not only the isolated singularity case as in the classical use of the Brieskorn module) and our illustration in the case of a (not quasi-homogeneous) polynomial in $\mathbb{C}[x_0, \dots, x_n]$ with $n+2$ monomials does not assume also that the singularity is isolated.

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2. A short overview on (a,b)-modules and frescos

2.1. **Why use an $(a,b)-$module structure instead of a differential system ?** Note first that in "(a,b)" a is the multiplication by the variable z and b is the primitive vanishing at $z=0$, so $b(f)(z) := \int_0^z f(t)dt$ where f is, for instance, a holomorphic multivalued function with a possible ramification point at $z=0$. So we are working with the non-commutative algebra $\mathcal{A}$ generated by a and b with the commutation relation $ab - ba = b^2$ as unique relation. This relation corresponds to the usual commutation relation $\partial_z z - z\partial_z = 1$ in the Weyl algebra $\mathbb{C}\langle z, \partial_z\rangle$.

Then why not use the usual Weyl algebra ?

The initial motivation comes from the study of germs of isolated singularities of holomorphic functions $(f,0) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ initiated at the end of the sixties by Milnor [20], Brieskorn [16], Deligne [17], Malgrange, [19], Varchenko [24], Kyoji Saito [22], Morihiko Saito [23], ... and many others.
To my knowledge the first who introduced the "operator" ∂_z^{-1} was Kyoji Saito in the beginning of the eighties (see [22]). The main reason comes from the fact that, looking at period-integrals of the type $z \mapsto \int_{\gamma_z} \omega/df$ where $(f,0) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a germ of a holomorphic function with an isolated singularity, $\omega \in \Omega^{n+1}_{\mathbb{C}^{n+1},0}$ is a germ of a holomorphic volume form and $(\gamma_z, z \in H)$ is a horizontal family of compact $n-$cycles in the fibres $\{f = z\}$ of f, the map $H \to D^*$ being the universal cover of a small punctured disc D^* with center 0, the derivation ∂_z of such an integral is given by the following formula

$$\partial_z\big(\int_{\gamma_z} u\big) = \int_{\gamma_z} du/df = \int_{\gamma_z} \omega/df = \int_{\gamma_z} v \tag{2}$$

where u and v are in $\Omega^n_{\mathbb{C}^{n+1},0}$ and satisfy $\omega = df \wedge v = du$. But in general it is not possible to find such a $v \in \Omega^n_{\mathbb{C}^{n+1},0}$ because writing $\omega = g(x)dx$, the holomorphic germ g is not in the Jacobian ideal of f. Nevertheless, as the coherent sheaf $\Omega^{n+1}\big/df \wedge \Omega^n$ has support inside $\{f = 0\}$ near 0, the Nullstellensatz gives a positive integer p such that f^p annihilates this sheaf near 0 and we may find $v \in f^{-p}\Omega^n_{\mathbb{C}^{n+1},0}$ such that $\omega = df \wedge v$ and (2) holds. But, of course, this implies that in formula (2) the derivation in z needs a denominator which is a power of z.
Thanks to the positivity theorem of Malgrange (see [19]) we may write the formula (2) as follows:

$$\int_0^z \big(\int_{\gamma_t} v\big)dt = \int_0^z \big(\int_{\gamma_t} \omega/df\big)dt = \int_{\gamma_z} u \tag{3}$$

If we begin with $\omega := du \in \Omega^{n+1}_{\mathbb{C}^{n+1},0}$ we see that formula (3) does not need any denominator in z. Moreover the surjectivity in top degree of the de Rham differential $d : \Omega^n \to \Omega^{n+1}$ shows that du may be any germ in $\Omega^{n+1}_{\mathbb{C}^{n+1},0}$ and we may write

$$b\big(\int_{\gamma_z} du/df\big) = \int_{\gamma_z} df \wedge u/df = \int_{\gamma_z} u \tag{4}$$

so that the action of b only needs to solve the equation $du = \omega$, and this is always possible with $u \in \Omega^n_{\mathbb{C}^{n+1},0}$ without introducing a denominator in f (so no denominator in z downstairs).

As the action of a is given by the formula :

$$a\Big(\int_{\gamma_z} du/df\Big) = z\Big(\int_{\gamma_z} du/df\Big) = \int_{\gamma_z} f du/df\Big) \tag{5}$$

because the $n-$cycle γ_z is inside the fibre $f^{-1}(z)$ we see that the $\mathcal{A}-$module structure on the quotient[3] $E_f := \Omega^{n+1}_{\mathbb{C}^{n+1},0}\big/d\big(Ker(df)^n_0\big)$ does not need to consider meromorphic $(n+1)-$differential forms with poles along the fibre $\{f = 0\}$.
Note that in the case of an isolated singularity for f we have the equality

$$Ker(df)^n_0 = df \wedge \Omega^{n-1}_0$$

because the partial derivatives of f define a regular sequence at the origin. Also in this case we find that $E_f\big/bE_f$ is equal to the finite dimensional vector space $\mathcal{O}_{\mathbb{C}^{n+1},0}\big/J(f)_0$ where $J(f)$ is the Jacobian ideal of f. So we find the classical "Brieskorn module".

But why is this presentation interesting if, at the end, we are compelled to introduce denominators in f (or in z working downstairs) to reach an ordinary differential system (or a differential equation) ?

The answer comes from the following considerations:
If you keep a module structure over the algebra $\mathcal{A}$ as a substitute for a differential system you have a richer structure (so more precise information) than a structure of module over the localized Weyl algebra $\mathbb{C}\langle z, z^{-1}, \partial_z\rangle$ associated to your differential system. This comes from the fact that the commutation relation $ab - ba = b^2$ is homogeneous of degree 2 in (a, b) and implies the existence of the decreasing sequence of two-sided ideals in $\mathcal{A}$ given by $b^m\mathcal{A} = \mathcal{A}b^m, \forall m \in \mathbb{N}$. So any $\mathcal{A}-$module E is endowed with a "natural filtration" $(b^mE)_{m\in\mathbb{N}}$ by sub$-\mathcal{A}-$modules. For instance Varchenko [24] proves that in the case of an isolated singularity this filtration defines the Hodge filtration of the mixed Hodge structure on the cohomology of the Milnor fibre of f.

EXERCISE. Show that $ab^m = b^ma + mb^{m+1}, \forall m \in \mathbb{N}$ is a consequence of the commutation relation corresponding to $m = 1$.
Note that this relation implies that a and b^m commute modulo $b^{m+1}\mathcal{A}$.

Also, looking at the "natural action" of $\mathcal{A}$ on $\mathbb{C}[[z]]$ which is given by $a(z^m) = z^{m+1}$ and $b(z^m) = z^{m+1}/(m + 1)$, you will see that $b^m\mathbb{C}[[z]] = z^m\mathbb{C}[[z]], m \in \mathbb{N}$ so the $b-$filtration is the filtration defined by the valuation in z.

Another simple remark may also help to convince the reader that a module structure over $\mathcal{A}$ is interesting:

Lemma 2.1.1. *Let $E := \oplus_{j=1}^k \mathbb{C}[b]e_j$ be a free $\mathbb{C}[b]-$module with basis $e_1, \dots, e_k$ and let $x_1, \dots, x_k$ be any given collection of elements in E. Then there exists a unique $\mathcal{A}-$module structure on E such that*

a) The action of a is defined by $ae_j = x_j$ for each $j \in [1, k]$.
b) The action of b is given by the $\mathbb{C}[b]-$structure of E.

The proof of this lemma is easily deduced from the following formula which is an easy consequence of the exercise above:

$$a(S(b)e_j) = S(b)x_j + b^2S'(b)e_j \quad \forall j \in [1, k]$$

[3]This quotient allows us to define $b[\omega] = [df \wedge u]$ independently of the choice of $u \in \Omega^n_0$ such that $\omega = du$ because when ω is in $d(Ker(df)^n_0)$ the period-integral is identically 0 near $z = 0$.

where $S'(b)$ is the "usual" derivative of the polynomial $S \in \mathbb{C}[b]$.

In fact, the presence of the filtration by the two-sided ideals $b^m\mathcal{A}$ of the algebra $\mathcal{A}$ and the lemma above lead to the following considerations

- The "fundamental" operation[4] in the action of $\mathcal{A}$ is b !
- It seems convenient, as we are interested in the asymptotic expansions of the period-integrals $\int_{\gamma_z} \omega/df$ when $z \to 0$, to complete the algebra $\mathcal{A}$ for the uniform structure defined by the filtration $b^m\mathcal{A}, m \in \mathbb{N}$.
  Note that for the "obvious" action of $\mathcal{A}$ on formal power series in z this filtration is associated to the valuation in z (see the remark following the exercise above).

This means that we shall work with the algebra

$$\tilde{\mathcal{A}} := \Big\{ \sum_{\nu \geq 0} P_\nu(a) b^\nu,\ P_\nu \in \mathbb{C}[a] \quad \forall \nu \in \mathbb{N} \Big\}. \tag{6}$$

The initial idea of Kyoji Saito was to add some convergence conditions in order that such series act on convergent (multivalued) series likes

$$\sum_{r \in R, j \in [0,N]} \mathbb{C}\{z\} z^r (Log\, z)^j$$

where R is a finite subset in $\mathbb{Q}$ and N is a non-negative integer, which are the kind of asymptotic expansions which are valid for our period-integrals.
But thanks to the regularity of the Gauss-Manin connection, we do not lose any information by staying at the formal series level and this avoids a lot of painful (standard) estimates !
Remark also that the construction given in the lemma above is also valid for the algebra $\tilde{\mathcal{A}}$ and, moreover, that a module E over $\tilde{\mathcal{A}}$ without $b-$torsion is of finite type over $\mathbb{C}[[b]]$ if and only if the complex vector space E/bE is finite dimensional.
So, our definition of an (a, b)-module is:

- **An (a,b)-module is a left $\tilde{\mathcal{A}}-$module which is a free and finite type module over the (commutative) sub-algebra $\mathbb{C}[[b]] \subset \tilde{\mathcal{A}}$.**

Examples.

(1) Let $(f,0) : (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of a holomorphic function with an isolated singularity. Let $\hat{\Omega}_0^p$ be the formal completion at the origin of $\Omega^p_{\mathbb{C}^{n+1},0}$. The quotient $\hat{E}_f := \hat{\Omega}_0^{n+1}\big/\big(df \wedge d(\hat{\Omega}_0^{n-1})\big)$ endowed with the actions of $a := \times f$ and $b := df \wedge d^{-1}$ is an (a,b)-module (note that the absence of $b-$torsion is a theorem; see [21] or [9]).

(2) Let $E := \mathbb{C}[[b]]e_0 \oplus \mathbb{C}[[b]]e_1$ be the $\tilde{\mathcal{A}}-$module defined by $ae_0 := be_0$ and $ae_1 := be_1 + be_0$. Then it is an easy exercice to show that E is isomorphic to

$$\mathbb{C}[[z]] \oplus \mathbb{C}[[z]] Log\, z$$

where $a := \times z$ and $b := \int_0^z$.
Determine the filtration $(b^m E)_{m\in\mathbb{N}}$ in this example and compare it with the filtration by the $(a^m E)_{m\in\mathbb{N}}$.
Compute the module over the Weyl algebra generated by $Log\, z$ and compare with E.

[4] This is psychologically the most difficult fact to accept after a standard education in maths.

2.2. **Geometric (a,b)-modules.** The (a,b)-modules which appear in singularity theory of a function are special. They correspond to **regular** differential systems and the notion of regularity is easy to define for an (a,b)-module:
First we say that the (a,b)-module E has a **simple pole** when $aE \subset bE$. When it is the case, $-b^{-1}a$ acts on the (finite dimensional) vector space E/bE and its **minimal polynomial** is called the **Bernstein polynomial** of E.
For a general (a,b)-module the saturation $E^\sharp$ of E by the action of $b^{-1}a$ is not always a finite type $\mathbb{C}[[b]]-$module. When $E^\sharp$ is of finite type over $\mathbb{C}[[b]]$, $E^\sharp$ is an (a,b)-module (with simple pole) with the same rank over $\mathbb{C}[[b]]$ as the rank of E.
We say in this case that E is **regular**. This is equivalent to the fact that E can be embedded in an (a,b)-module having a simple pole.
Then we defined the **Bernstein polynomial** of a regular (a,b)-module E as the Bernstein polynomial of its saturation $E^\sharp$ by $b^{-1}a$.

There is one more specific property for the (regular) (a,b)-modules coming from the singularity of a function f, which reflects the fact that the monodromy of f is quasi-unipotent and the positivity theorem of Malgrange: the fact that the roots of the Bernstein polynomial are negative rational numbers (compare with the famous theorem of Kashiwara [18]). So we call **geometric** a regular (a,b)-module whose Bernstein polynomial has **negative rational roots**.
Example. In the previous example 2 the (a,b)-module has a simple pole and its Bernstein polynomial is, by definition, the minimal polynomial of the matrix

$$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

so its Bernstein polynomial is $(\lambda+1)^2$. Compare with the Bernstein type identity

$$\big((2\lambda+1)\partial_z - z\partial_z^2\big)(z^{\lambda+1} Log\, z) = (\lambda+1)^2 z^\lambda Log\, z.$$

The following easy proposition will be needed in the sequel. Although it is rather standard, we shall sketch the proof for the convenience of the reader.

Recall that a sub-(a,b)-module F of a (a,b)-module E is **normal** when for each $x \in E$ such that bx is in F, then x is in F. This is the necessary and sufficient condition for the quotient E/F to be without $b-$torsion. It is a necessary and sufficient condition for E/F to be an (a,b)-module.

Proposition 2.2.1. *Let E be a geometric (a,b)-module and F any sub$-\tilde{\mathcal{A}}-$module in E. Then F is a geometric (a,b)-module. When F is normal, the quotient E/F is again a geometric (a,b)-module.*

proof. To prove the first point, thanks to the regularity of E, we may assume that E is a simple pole module (i.e. $aE \subset bE$). Then the Bernstein polynomial of E is the **minimal polynomial** of the action of $-b^{-1}a$ on the finite dimensional vector space E/bE. As F is a $\mathbb{C}[[b]]$ sub-module of E which is free and of finite rank on $\mathbb{C}[[b]]$, F is also free and of finite rank on $\mathbb{C}[[b]]$ and stable by a. So F is an (a,b)-module. Its saturation by $b^{-1}a$ is again contained in E and so it is also free of finite type on $\mathbb{C}[[b]]$. This gives the regularity of F. The last point to prove is the fact that the Bernstein polynomial of F has negative rational roots (i.e. F is geometric) and the fact that when F is normal E/F is also geometric. We shall argue by induction on the rank of F. In the rank 1 case let e be a generator of F over $\mathbb{C}[[b]]$ such that $ae = \lambda be$ (see the classification of rank 1 regular (a,b)-modules in [B.93], Lemma 2.4). Let ν in $\mathbb{N}$ be maximal such that $b^{-\nu}e$ lies in E. Then $\mathbb{C}[[b]]b^{-\nu}e = b^{-\nu}F$ is a normal sub-module of E and we have an exact sequence of simple poles (a,b)-modules

$$0 \to b^{-\nu}F \to E \to Q \to 0$$

and also an exact sequence of $(-b^{-1}a)$ finite dimensional vector spaces

$$0 \to \mathbb{C}b^{-\nu}e \to E/bE \to Q/bQ \to 0.$$

Then the minimal polynomial B_E of the action of $-b^{-1}a$ on E/bE is either equal to the minimal polynomial B_Q of the action of $-b^{-1}a$ on Q/bQ, and in this case $x + (\lambda - \nu)$ divides $B_Q = B_E$, or we have $B_E[x] = (x + (\lambda - \nu))B_Q[x]$.
In both cases, as E is geometric, we obtain that $-(\lambda - \nu)$ is a negative rational number, and so is $-\lambda$. Moreover, in both cases, Q is also geometric.
The induction step follows easily by considering a rank 1 normal sub-module G in F, using the following lemma and the fact which was already proved above that a quotient of a geometric (a,b)-module by a normal rank 1 sub-module is again a geometric (a,b)-module. ■

For a proof of the following lemma see for instance Remark 1.2 following Proposition 1.3 in [12].

Lemma 2.2.2. *Let E be a regular (a,b)-module and let $F \subset E$ be a sub-(a,b)-module. Assume that λ is a root of the Bernstein polynomial B_F of F. Then there exists $\lambda' \in \lambda + \mathbb{N}$ such that λ' is a root of the Bernstein polynomial B_E of E.* ■

As we want to consider, in the non-isolated singularity case, a sheaf of geometric (a,b)-modules along the singular set $\{df = 0\}$ of the zero set $\{f = 0\}$ of a holomorphic function on a complex manifold M, we have to replace the completion used in the classical case of an isolated singularity by an $f-$completion which is in fact the $z-$completion downstairs. This will not change seriously the considerations above, thanks to the following easy proposition which implies that any geometric (a,b)-module is in fact a module over the algebra $\hat{\mathcal{A}} := \{\sum_{p,q\geq 0} c_{p,q}a^pb^q\}$ which contains both $\mathbb{C}[[b]]$ and $\mathbb{C}[[a]]$.

Proposition 2.2.3. *Any geometric (a,b)-module is complete for the decreasing filtration by the $\mathbb{C}[a]-$sub-modules $(a^mE)_{m\in\mathbb{N}}$ (they are not stable by b in general).*

This result is an obvious consequence of the existence, for any regular (a,b)-module E, of a positive integer N such that $a^NE \subset bE$ (see [12]). Then $\mathbb{C}[[a]]$ acts on any regular (a,b)-module E. Note that the hypothesis "geometric" ensures that a is injective (this is not the case if we assume only the regularity).

2.3. **Frescos.** We have seen that the (a,b)-module structure may be an interesting way to study the differential system associated to period-integrals for a germ of a holomorphic function $(f, 0) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. It is given by the (a,b)-module $\hat{E}_f$ which is the completion of the Brieskorn module $\Omega_0^{n+1}\big/df \wedge d\Omega_0^{n-1}$ in the isolated singularity case. It gives in fact a filtered version of the differential system satisfied by **all the period-integrals** associated to the germ $(f, 0)$.
But if we are interested by the period-integrals corresponding to a specific holomorphic differential form, it is clear that such a differential system, that is to say the (a,b)-module $\hat{E}_f$, does not give very precise information. In terms of differential system, we would prefer to have a specific **differential equation** satisfied by the integral-periods $\int_{\gamma_z} \omega/df$ for our choice of ω rather than the differential system satisfied by all period-integrals, so associated to all choices of $\omega \in \Omega_0^{n+1}$.
The analogue of the differential equation in terms of (a,b)-modules is the notion of "fresco". A **fresco** is, by definition, a **geometric** (a,b)-module which is generated, as a $\tilde{\mathcal{A}}-$module, by **one generator**. For instance, in the previous situation, we shall consider the fresco given by

$\tilde{\mathcal{A}}[\omega] \subset \hat{E}_f$ and we shall call it the **fresco of the pair** (f, ω) **at the origin**. The following structure theorem describes in a very simple way such a fresco (see [8]).

Theorem 2.3.1. *Any rank k fresco*[5] *F with generator e is isomorphic (as a left $\tilde{\mathcal{A}}-$module) to a quotient $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}\Pi$, the isomorphism $F \to \tilde{\mathcal{A}}/\tilde{\mathcal{A}}\Pi$ being defined by sending the generator e of F to the class of 1. We may choose Π having the following form*

$$\Pi := (a - \lambda_1 b)S_1^{-1}(a - \lambda_2 b)S_2^{-1} \dots S_{k-1}^{-1}(a - \lambda_k b)S_k^{-1} \tag{7}$$

where the numbers $-(\lambda_j + j - k)$ are the roots of the Bernstein polynomial of F and where S_j are in $\mathbb{C}[b]$ and satisfy $S_j(0) = 1$ (so each S_j is invertible in $\mathbb{C}[[b]]$).

Note that the initial form in (a,b) of Π is $P_F := (a - \lambda_1 b) \dots (a - \lambda_k b)$. It is called the **Bernstein element** of the fresco F. It does not depend on the choice of the generator of F over $\tilde{\mathcal{A}}$ (choice which determines Π) and is related to the **Bernstein polynomial** $B_F \in \mathbb{C}[\lambda]$ of F by the following relation in the ring $\mathcal{A}[b^{-1}]$:

$$(-b)^k B_F(-b^{-1}a) = P_F, \quad \text{where} \quad k := rk(F). \tag{8}$$

In the case of a fresco F the Bernstein polynomial B_F is equal to the **characteristic polynomial** of the action of $-b^{-1}a$ on $F^\sharp/bF^\sharp$ where $F^\sharp$ is the saturation of F by $b^{-1}a$ (see [8] Theorem 3.2.1). This makes the computation of the Bernstein polynomial of a fresco easier than for a general geometric (a,b)-module, for instance by the use of the following remark:
If $0 \to F \to G \to H \to 0$ is an exact sequence of frescos we have the relation $P_G = P_F P_H$ (product in $\mathcal{A}$) between the Bernstein elements and this gives the relation (see [8] Proposition 3.4.4):

$$B_G(x) = B_F\big(x + rk(H)\big)B_H(x)$$

between the Bernstein polynomials.

Recall that any element P in the algebra $\mathcal{A}$ which is homogeneous of degree k in (a,b) and monic in a may be written $P = (a - r_1 b) \dots (a - r_k b)$ where $r_1, \dots, r_k$ are complex numbers. This equality is not unique but the sequence $r_j + j - k, j \in [1, k]$ depends only on P (see [8] Proposition 2.0.2).

Our next proposition will be useful in our computations of examples.

Proposition 2.3.2. *Let F be a rank k fresco with generator e. Assume that $\mathcal{Q} \in \hat{\mathcal{A}}$ has the following properties:*

i) The initial form Q of $\mathcal{Q}$ in (a, b) has degree d and is monic in a.
ii) $\mathcal{Q}[e] = 0$ in F.

Then Q is a left multiple in $\mathcal{A}$ of P_F, the Bernstein element of F.
If moreover we have $d = k$, then Q is the Bernstein element of F.

Proof. Using the structure theorem of [8] recalled in Theorem 2.3.1 above, we have an isomorphism $F \simeq \tilde{\mathcal{A}}/\tilde{\mathcal{A}}\Pi$ where the initial form in (a,b) P_F of Π is the Bernstein element of F. As F is a $\hat{\mathcal{A}}-$module (see Proposition 2.2.3) we have also an isomorphism $F \simeq \hat{\mathcal{A}}/\hat{\mathcal{A}}\Pi$ of $\hat{\mathcal{A}}-$modules and our hypothesis ii) implies that there exists $Z \in \hat{\mathcal{A}}$ such that

$$\mathcal{Q} = Z\Pi.$$

This gives $Q = in(Z)P_F$ where $in(Z)$ is the initial form in (a,b) of Z. This already implies that $d \geq k$ and that $in(Z)$ is of degree $d - k$. In the case $d = k$ we have $in(Z) = 1$ and $Q = P_F$. ∎

[5] Recall that we consider the rank as a $\mathbb{C}[[b]]-$module, where $\mathbb{C}[[b]] \subset \tilde{\mathcal{A}}$.

2.4. **A general existence theorem.** Now consider a germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ such that $\{f = 0\}$ is reduced. Let $\hat{\Omega}^\bullet$ be the formal $f-$completion of the sheaf of holomorphic differential forms on $(\mathbb{C}^{n+1}, 0)$ and let $\hat{K}er\, df^\bullet$ be the kernel of the map

$$\wedge df : \hat{\Omega}^\bullet \longrightarrow \hat{\Omega}^{\bullet+1}.$$

Then for any $p \geq 2$ the $p-$th cohomology sheaf of the complex $(\hat{K}er\, df^\bullet, d^\bullet)$ has a natural structure of left $\hat{\mathcal{A}}-$module, where the action of a is given by multiplication by f and the action of b is (locally) given by $df \wedge d^{-1}$.

The following result is known (see [9], [14] and [10])

Theorem 2.4.1. *For each integer p the germ at* 0 *of the p−th cohomology sheaf of the complex* $(\hat{K}er\, df^\bullet, d^\bullet)$ *(modified for* $p = 1$*)*[6]*, denoted by* $\mathcal{E}_0^p$*, satisfies the following properties:*

i) We have in $\mathcal{E}_0^p$ *the commutation relation* $ab - ba = b^2$.

ii) $\mathcal{E}_0^p$ *is b-separated and b-complete (so also a-complete). Then it is a* $\tilde{\mathcal{A}}-$*module (and also a* $\hat{\mathcal{A}}-$*module).*

iii) There exists an integer $m \geq 1$ *such that* $a^m \mathcal{E}_0^p \subset b\mathcal{E}_0^p$.

iv) We have $B(\mathcal{E}_0^p) = A(\mathcal{E}_0^p) = \tilde{A}(\mathcal{E}_0^p)$ *and there exists an integer* $N \geq 1$ *such that* $a^N A(\mathcal{E}_0^p) = 0$ *and* $b^{2N} B(\mathcal{E}_0^p) = 0$.

v) The quotient $E_0^p := \mathcal{E}_0^p / B(\mathcal{E}_0^p)$ *is a geometric (a,b)-module.*

Recall that $B(\mathcal{E}_0)$ is the b-torsion in $\mathcal{E}_0$, $A(\mathcal{E}_0)$ the a-torsion of $\mathcal{E}_0$ and $\tilde{A}(\mathcal{E}_0)$ the $\mathbb{C}[b]-$module generated by $A(\mathcal{E}_0)$ in $\mathcal{E}_0$.

We shall mainly use this result in the case where ω is an $(n+1)-$holomorphic differential form in an open neighborhood U of the origin in $\mathbb{C}^{n+1}$; note that the condition $df \wedge \omega = d\omega = 0$ is automatic in this case. So it defines a class $[\omega]$ in E_0^{n+1} and generates the fresco

$$F_\omega := \tilde{\mathcal{A}}[\omega] \subset E_0^{n+1}$$

thanks to Proposition 2.2.1 and property v) of the previous theorem.

Definition 2.4.2. *We shall denote by* $B_\omega \in \mathbb{C}[x]$ *and by* $P_\omega \in \mathcal{A}$ *respectively the* **Bernstein polynomial** *and the* **Bernstein element** *of the fresco* F_ω.

We shall study several examples in section 4.

3. Mellin transform of hermitian periods

3.1. **The main result.** We consider now a holomorphic function on an open polydisc U centered at the origin in $\mathbb{C}^{n+1}$ such that $f(0) = 0$ is the only critical **value** of f on U. Let ω be a holomorphic $(n+1)-$differential form on U . Then let ψ be a $\mathscr{C}^\infty$ differential form with compact support in the polydisc U of type $(0, n+1)$ which satisfies $d\psi \equiv 0$ near 0. For any $h \in \mathbb{Z}$ define, at least for $\Re(\lambda)$ large enough, the holomorphic function

$$\lambda \mapsto F_h^\psi(\lambda)[\omega] = \frac{1}{\Gamma(\lambda)} \int_U |f|^{2\lambda} \bar{f}^h \omega \wedge \psi. \tag{9}$$

Note that the existence of a Bernstein identity for f in a neighborhood of 0 ensures that for U small enough and any differential form $\varphi \in \mathscr{C}_c^\infty(U)^{(n+1,n+1)}$ the holomorphic function defined by $\int_U |f|^{2\lambda}\varphi$ has a meromorphic extension to the whole complex plane with a finite series of poles of

[6]For $p = 1$ we have to replace $\hat{K}er\, df^1$ by a quotient ; see [10].

order at most $n+1$ at points of the form $\xi_j - \mathbb{N}$ where ξ_j are negative rational numbers (see [18]).

We shall make the following hypothesis, which we shall denote by $H(\xi, 1)$:

- For a given $\xi \in \mathbb{Q}$ the local monodromy of f acting on the reduced cohomology of the Milnor fibre of f at any point distinct from 0 does not admit the eigenvalue $\exp(2i\pi\xi)$.

Proposition 3.1.1. *We assume the hypothesis $H(\xi,1)$. Let ω be a holomorphic $(n+1)-$differential form on U and let ψ be a $\mathscr{C}^\infty$ differential form with compact support in U of type $(0, n+1)$ which satisfies $d\psi \equiv 0$ near 0. We have the following formulas in E_0^{n+1} (recall that the geometric (a,b)-module E_0^{n+1} is defined in the theorem 2.4.1 above and that the action of a and b are defined by: $a[\omega] = [f\omega]$, $b[\omega] = [df \wedge u]$ where $u \in \Omega^p(U)$ satisfies $du = \omega$[7]).*

i) If there exists $v \in \Omega^n(U)$ satisfying $df \wedge v \equiv 0$ and $dv = \omega$ on U, then $F_h^\psi[\omega]$ has no pole in $\xi + \mathbb{Z}$ for any h and any ψ.
ii) $F_h^\psi(\lambda)[a\omega] - (\lambda+1)F_{h-1}^\psi(\lambda+1)[\omega]$ has no pole in $\xi + \mathbb{Z}$.
iii) $F_h^\psi(\lambda)[b\omega] + F_{h-1}^\psi(\lambda+1)[\omega]$ has no pole in $\xi + \mathbb{Z}$.
iv) So for any $\mu \in \mathbb{C}$ the meromorphic function

$$F_h^\psi(\lambda)[(a-\mu b)\omega] - (\lambda+\mu+1)F_{h-1}^\psi(\lambda+1)[\omega]$$

has no pole in $\xi + \mathbb{Z}$, combining ii) and iii).

Of course, the simplest example of such a ψ is given by $\psi := \rho\bar{\omega}'$ where ρ is a $\mathscr{C}^\infty$ function with compact support such that $\rho \equiv 1$ in a neighborhood of 0 and where ω' is a holomorphic $(n+1)-$differential form in U.

PROOF. Write $\omega = du$ on U with $u \in \Omega^n(U)$. This is always possible as U is a polydisc so Stein and contractible. Then for $\Re(\lambda) > 1 + |h|$ the differential form $\alpha := |f|^{2\lambda}\bar{f}^h u \wedge \psi$ is $\mathscr{C}^1$ in U and has compact support. So we have:

$$d\alpha = |f|^{2\lambda}\bar{f}^h du \wedge \psi + \lambda |f|^{2(\lambda-1)}\bar{f}^{h+1} df \wedge u \wedge \psi + (-1)^n |f|^{2\lambda}\bar{f}^h u \wedge d\psi.$$

Stokes' formula gives, as $d\psi$ vanishes near 0,

$$\frac{1}{\Gamma(\lambda)}\int_U d\alpha = 0 = F_h^\psi(\lambda)[\omega] + F_{h+1}^\psi(\lambda-1)[b\omega] + G(\lambda)$$

where $G(\lambda)$ is a meromorphic function on $\mathbb{C}$ which has no pole in $\xi + \mathbb{Z}$ thanks to our hypothesis $H(\xi,1)$. This implies $i)$ and $iii)$ using $\Gamma(\lambda) = \lambda\Gamma(\lambda-1)$. The formula $ii)$ is easy and left to the reader. ∎

REMARK. The point $i)$ of the previous proposition shows that $F_h^\psi(\lambda)[\omega]$ has no pole in $\xi + \mathbb{Z}$ when ω induces the zero class in $\mathcal{E}_0^{n+1} = H^{n+1}(\hat{K}er\, df_0^\bullet, d^\bullet)$, and the point $iii)$ implies the same conclusion when the class of ω in $\mathcal{E}_0^{n+1}$ is of $b-$torsion. So the polar part of $F_h^\psi(\lambda)[\omega]$ in $\xi + \mathbb{Z}$ depends only on the class induced by ω in $E_0^{n+1} = \mathcal{E}_0^{n+1}/b-torsion$ for h and ψ fixed. This remark will be crucial in the sequel.

We shall give the following result which is more precise than Theorem 1.0.1 stated in the introduction. Let us recall the situation. Let f be a holomorphic function on a polydisc U with center 0 in $\mathbb{C}^{n+1}$ and assume that $f(0) = 0$ is the only critical value of f on U. Let ω be a

[7] Such a u always exists.

$(n+1)$ holomorphic differential form on U and let ψ be a $\mathscr{C}^\infty$ differential form with compact support in U of type $(0, n+1)$ which satisfies $d\psi \equiv 0$ near 0. Then define as above

$$F_h^\psi(\lambda)[\omega] := \frac{1}{\Gamma(\lambda)} \int_U |f|^{2\lambda} \bar{f}^h \omega \wedge \psi. \tag{9}$$

Theorem 3.1.2. *Fix a number $\xi \in \mathbb{Q}$ and an integer $d \geq 1$. Assume that the hypothesis $H(\xi, 1)$ is satisfied. Then assume that, for a given ψ, the meromorphic extension of $F_h^\psi(\lambda)[\omega]$ has no pole of order $\geq d+1$ at any point in $\xi + \mathbb{Z}$, for any choice of $h \in \mathbb{Z}$, but that there exists a point in $\xi + \mathbb{Z}$ where, for some h, this meromorphic extension has a pole of order d.*
For each integer $s \in [1, d]$ let ξ_s be the biggest element in $\xi + \mathbb{Z}$ for which there exists $h \in \mathbb{Z}$ such that $F_h^\psi(\lambda)[\omega]$ has a pole of order at least equal to s at ξ_s. Then each ξ_s for $s \in [1, d]$ is a root of the Bernstein polynomial of the fresco F_ω.
Moreover, if $\xi_s = \xi_{s+1} = \cdots = \xi_{s+p}$ then ξ_s is a root of the Bernstein polynomial of F_ω with multiplicity at least equal to $p+1$.

Remark that the theorem implies that the Bernstein polynomial of the fresco F_ω is a multiple of $\prod_{s=1}^d (\lambda - \xi_s)$ and that we have $\xi_d \leq \xi_{d-1} \leq \cdots \leq \xi_1 < 0$ by definition.

The proof of this theorem needs some lemmas.

Lemma 3.1.3. *In the situation of the theorem 3.1.2, let $S \in \mathbb{C}[[b]]$ which satisfies $S(0) = 1$ and let $\mu \in \mathbb{C}$ such that $\mu \neq -\xi_s$ for a given $s \in [1, d]$. Then*

1. *ξ_s is still the biggest pole of order $\geq s$ in $\xi + \mathbb{Z}$ for the meromorphic extension of $F_h^\psi(\lambda)[S(b)\omega]$ when h varies in $\mathbb{Z}$.*
2. *$\xi_s - 1$ is the biggest pole of order $\geq s$ in $\xi + \mathbb{Z}$ for the polar part of the meromorphic extension of $F_h^\psi(\lambda)[(a - \mu b)\omega]$ when h varies in $\mathbb{Z}$.*

PROOF. Write $S(b) := 1 + \sum_{m=1}^\infty s_m b^m$. As $F_h^\psi(\lambda)[b^m\omega] = (-1)^m F_{h-m}^\psi(\lambda + m)[\omega]$, for each $m \geq 1$, thanks to point $iii)$ in Proposition 3.1.1, the meromorphic extension of $F_h^\psi(\lambda)[b^m\omega]$ cannot have a pole of order $\geq s$ at the point ξ_s for any choice of h. So the pole of order $\geq s$ given by the initial term (i.e. $m = 0$) for a suitable value of h, stays maximal. This proves 1.
Because we assume $\mu + \xi_s \neq 0$, the point $iv)$ in Proposition 3.1.1 shows that the pole at $\xi_s - 1$ of $F_h^\psi(\lambda)[(a - \mu b)\omega]$ has the same order as the pole at ξ_s for $F_{h+1}^\psi(\lambda)[\omega]$. Also, the same formula shows that for any integer $p \geq 0$ the order of the pole $\xi_s + p$ for $F_h^\psi(\lambda)[(a - \mu b)\omega]$ is less than or equal to the order of the pole $\xi_s + p + 1 \geq \xi_s + 1$ for $F_{h-1}^\psi(\lambda)[\omega]$ which is at most $s - 1$ by definition of ξ_s. This allows us to conclude. ■

Lemma 3.1.4. *In the situation of the theorem 3.1.2, assume that $\xi_{s+1} = \xi_s$, for some $s \in [1, d-1]$. Then $\xi_s - 1$ is the biggest pole of order $\geq s$ in $\xi + \mathbb{Z}$ for the meromorphic extension of $F_h^\psi(\lambda)[(a + \xi_s b)\omega]$ when h varies in $\mathbb{Z}$.*

PROOF. Using the formula of point $iv)$ in Proposition 3.1.1 we obtain that, for some suitable choice of h, the meromorphic extension of $F_h^\psi(\lambda)[(a + \xi_{s+1} b)\omega]$ has a pole of order $\geq s$ at the point $\xi_{s+1} - 1 = \xi_s - 1$. Assume that for some integer $p \geq 0$ and some $h \in \mathbb{Z}$ the meromorphic extension of $F_h^\psi(\lambda)[(a + \xi_{s+1} b)\omega]$ has a pole of order $\geq s$ at $\xi_s + p$. Then using again the formula of the point $iv)$ in Proposition 3.1.1 we find that $(\lambda - \xi_s + 1)F_{h-1}^\psi(\lambda + 1)[\omega]$ has, for a suitable choice of $h \in \mathbb{Z}$, a pole of order $\geq s$ at $\lambda = \xi_s + p$. But $\xi_s + p - \xi_s + 1 = p + 1 \neq 0$ so we find a pole of order $\geq s$ at the point $\xi_s + p + 1$ for $F_{h-1}(\lambda)[\omega]$. As $p + 1 \geq 1$ this contradicts the definition of ξ_s, so $\xi_s - 1$ is the biggest pole of order $\geq s$ in $\xi + \mathbb{Z}$ when h varies in $\mathbb{Z}$, for the meromorphic extension of $F^\psi_h(\lambda)[(a+\xi_s b)\omega]$. ■

Lemma 3.1.5. *In the situation of Theorem 3.1.2, there exists a minimal integer $j_d \in [0, k-1]$ such that $\xi_d - j_d = -\lambda_{k-j_d}$ and for each $s \in [1, d-1]$ there exists a minimal integer $j_s \in [j_{s+1}+1, k-1]$ such that $\xi_s - j_s = -\lambda_{k-j_s}$. Moreover the meromorphic extension of $F^\psi_h(\lambda)[P_{k-j_s-1}\omega]$ has a pole of order $\geq s-1$ at the point $\xi_s - j_s - 1$, where we define*

$$P_r := (a-\lambda_1 b)S_1(b)\dots(a-\lambda_r b)S_r(b) \quad \text{for} \quad r \in [1, k]$$

where $P_k := \Pi$ is given by applying Theorem 2.3.1 to the fresco F_ω and satisfies moreover the condition that the sequence $(\lambda_j + j), j \in [1, k]$ is non-decreasing.

PROOF. The fact that we may assume that the sequence $(\lambda_j + j), j \in [1, k]$ is non-decreasing is a consequence of the existence of a **principal Jordan-Hölder sequence** for a fresco (see [4] Theorem 1.2.5 or [8] Proposition 3.5.2). Note also that the roots of the Bernstein polynomial of F_ω are given by the numbers $-(\lambda_j + j - k)$ for $j \in [1, k]$.
Now the first point is to prove that there exists an integer $j \in [0, k-1]$ such that $\xi_d - j = -\lambda_{k-j}$, because we may then define j_d as the minimal such integer. So assume that no such j exists. Then applying Lemma 3.1.3 we will obtain that the meromorphic extension of $F^\psi_h(\lambda)(\Pi\omega)$ has a pole of order at least equal to $d \geq 1$ at the point $\xi_d - k$. But points $i)$ and $iii)$ in Proposition 3.1.1 imply that there is no pole in $\xi + \mathbb{Z}$ in the meromorphic extension of $F^\psi_h(\lambda)[\Pi\omega]$ as $[\Pi\omega] = 0$ in E_0^{n+1}. Contradiction. So such a j exists and j_d is well defined.
The same argument as for $s = d$ shows that for each $s \in [1, d-1]$ there exists at least one $j \in [1, k]$ such that $\xi_s - j = -\lambda_{k-j}$. Now we have

$$\xi_{s+1} = -\lambda_{k-j_{s+1}} + j_{s+1} \leq \xi_s = -\lambda_{k-j} + j.$$

The non-decreasing property of the sequence $\lambda_j + j$ implies then that $j \geq j_{s+1}$. If the inequality is strict, then we obtain $j_s := j$ and the point $iv)$ in Proposition 3.1.1 implies that $F^\psi_h(\lambda)[P_{k-j_s-1}\omega]$ has a pole of order $\geq s-1$ at the point $\xi_s - j_s - 1$.
If we have $\xi_{s+1} = \xi_s$ then Lemma 3.1.4 shows that $\xi_s - j_{s+1} - 1$ is still in this case the biggest pole of order $\geq s$ in $\xi + \mathbb{Z}$ for the meromorphic extension of $F^\psi_h(\lambda)[(a+\xi_s b)P_{k-j_{s+1}-1}\omega]$ when h varies in $\mathbb{Z}$. So we can continue to apply $S_{k-j_{s+1}-2}$ and then $a - \lambda_{k-j_{s+1}-2}b$, *etc...* until we reach another $j < j_{s+1}$ such that $\xi_s - j = -\lambda_j$, and then we conclude using the same argument as above. ■

REMARKS.

(1) The sequence $j_s, s \in [1, d]$ is strictly decreasing, so the sequence $k - j_s$ is strictly increasing and there are exactly d rational numbers

$$\xi_s = -(\lambda_{k-j_s} - j_s) = -(\lambda_{k-j_s} + k - j_s - k)$$

counting multiplicities (we remark that the multiplicities correspond to equalities $\lambda_{k-j_s} - \lambda_{k-j_{s+1}} = j_s - j_{s+1}$).

(2) As long as $r \leq j_s$ the rational number ξ_s remains the biggest pole of order $\geq s$ in $\xi + \mathbb{Z}$ for the meromorphic extension of $F^\psi_h(\lambda)[Q_r\omega]$ when h describes $\mathbb{Z}$, where

(@) $$Q_r := S_r(a - \lambda_{r+1}b)S_{r+1}...(a - \lambda_k b)S_k$$

(3) Note that if we have several ψ which are d−closed near 0 and for which the meromorphic extension of $F^\psi_h[\omega]$ presents poles in $\xi + \mathbb{Z}$, we may obtain more roots in $\xi + \mathbb{Z}$ for the Bernstein polynomial of F_ω from this result.

PROOF OF THE THEOREM 3.1.2. We shall prove first, by induction on $d \geq 1$, that there exist at least d values of $j \in [1, k]$ such that $-\lambda_j$ belongs to $\xi + \mathbb{Z}$.
So assume that either $d = 1$ or that $d \geq 2$ and that our claim is proved for $d - 1$. Then consider the poles of the meromorphic extension of $F_h(\lambda)[Q_{k-j_d}\omega]$ (where Q_r is defined in the formula (@) above) and where the integer j_d is defined in Lemma 3.1.5. Using Lemma 3.1.3 applied to ξ_d, we obtain that it has a maximal pole of order d at the point $\xi_d - j_d$ for a suitable choice of h and, applying Lemma 3.1.5 we conclude that the meromorphic extension of $F_h(\lambda)[(a-\lambda_{k-j_d}b)Q_{k-j_d}\omega]$ has a pole of order at least equal to $d-1$ at the point $\xi_d - j_d - 1$. But the form $\omega' := (a - \lambda_{k-j_d}b)Q_{k-j_d}\omega$ is killed in E_0^{n+1} by P_{k-j_d-1}, so the induction hypothesis gives at least $d-1$ values of $j \in [1, k - j_d - 1]$ such that $-\lambda_j$ is in $\xi + \mathbb{Z}$. As $-\lambda_{k-j_d} = \xi_d - j_d$ belongs to $\xi + \mathbb{Z}$ this completes the proof of our induction.
So we obtain that at least d roots (counting multiplicities) of the Bernstein polynomial of F_ω are among the ξ_s for $s \in [1, d]$. ■

3.2. **Some variants.** The following variant of the previous result is also useful.

Assume now that we fix a rational number ξ and an integer $q \geq 1$ and that we make the following hypothesis:

- For any differential form $\varphi \in \mathscr{C}_c^{\infty,(n+1,n+1)}(U \setminus \{0\})$ the meromorphic extension of

$$\frac{1}{\Gamma(\lambda)}\int_U |f|^{2\lambda}\varphi \qquad (H(\xi, q))$$

  has no pole of order at least equal to q at some point in $\xi + \mathbb{Z}$.
  Note that we require that $\varphi \equiv 0$ near 0.

Theorem 3.2.1. *Let $\omega \in \Omega^{n+1}(U)$ and fix a differential form $\psi \in \mathscr{C}_c^{\infty,(0,n+1)}(U)$ which is* $d-$**closed near** 0.
Assume that the condition $H(\xi, q)$ is satisfied and that for some integer $h \in \mathbb{Z}$ the meromorphic extension of

$$F_h^\psi(\lambda)[\omega] := \frac{1}{\Gamma(\lambda)}\int_U |f|^{2\lambda}\bar{f}^h\omega \wedge \psi \qquad (10)$$

has a pole of order $q+d-1$ at some point in $\xi+\mathbb{Z}$ with $d \geq 1$, maximal in $\mathbb{N}^*$. Let ξ_0 be maximal in $\xi+\mathbb{Z}$ such that there exists $h \in \mathbb{Z}$ with the property that $F_h^\psi(\lambda)[\omega]$ has a pole of order $q+d-1$ at ξ_0. Then ξ_0 is a root of the Bernstein polynomial B_ω of the fresco F_ω associated to the germs of f and ω at the origin, and B_ω admits at least d roots in $\xi_0 + \mathbb{N}$ (counting multiplicities).

PROOF. The argument is analogous to the one given in the proof of Theorem 3.1.2. The change to make in the proof is that we must take into account here only the polar parts of order at least equal to q of the poles at points in $\xi + \mathbb{Z}$. So in Proposition 3.1.1 and in Lemmas 3.1.3, 3.1.4 and 3.1.5 we have to replace "no poles in $\xi + \mathbb{Z}$" by "no pole of order $\geq q$ in $\xi + \mathbb{Z}$" under the hypothesis $H(\xi, q)$. Also we define ξ_s as the maximal pole of order $\geq q + s - 1$ for $s \in [1, d]$.
The other difference in the argument lies in the fact that in the Stokes' formula the extra term given by the differential of ψ equal to:

$$G(\lambda) := \frac{(-1)^n}{\Gamma(\lambda)}\int_U |f|^{2\lambda}\bar{f}^h u \wedge d\psi$$

has no pole of order $\geq q$ at a point in $\xi + \mathbb{Z}$ because we assumed $d\psi \equiv 0$ near 0 and we may apply our hypothesis $H(\xi, q)$ with $\varphi := \bar{f}^h u \wedge d\psi$ for $h \geq 0$ or with $\varphi := f^{-h} u \wedge d\psi$ for $h \leq 0$. ■

Let us specialize now the form $\psi \in \mathscr{C}_c^{\infty,(0,n+1)}(U)$ such that $d\psi \equiv 0$ near the origin by defining $\psi := \rho\bar{\omega}'$ where ω' is a fixed holomorphic $(n+1)-$differential form on U and where ρ is a function in $\mathscr{C}_c^{\infty}(U)$ which is identically equal to 1 near the origin (so $d(\rho\bar{\omega}') = d'\rho \wedge \bar{\omega}'$ vanishes identically near the origin). Then we consider the Mellin transform of the hermitian period

$$z \mapsto \frac{1}{(2i\pi)^n} \int_{f=z} \rho(\omega/df) \wedge (\overline{\omega'/df}).$$

In the following corollary we use the hermitian symmetry between ω and ω' in order to obtain a better control of the poles of the Mellin transform using the Bernstein polynomials of the frescos associated to (f, ω) and (f, ω') at the origin.

Corollary 3.2.2. *Let $\tilde{f} : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant holomorphic germ. Fix $\xi \in \mathbb{Q}$ and a positive integer q. Assume that the hypothesis $H(\xi, q)$ holds for $\tilde{f}$ and consider ω and ω' two germs of $(n+1)-$holomorphic forms. Let $q + d - 1, d \geq 1$, be the maximal order of pole*[8] *for*

$$(2i\pi)^{n+1} F^h_{\omega,\omega'}(\lambda) := \frac{1}{\Gamma(\lambda)} \int_U |f|^{2\lambda} \bar{f}^h \rho\omega \wedge \bar{\omega}'$$

for any choice of a point in $\xi + \mathbb{Z}$ and for any choice of $h \in \mathbb{Z}$.
Let ξ_0 be maximal in $\xi + \mathbb{Z}$ such that there exists some $h \in \mathbb{Z}$ and a pole of order $q + d - 1$ at $\xi_0 \in \xi + \mathbb{Z}$ for $F^h_{\omega,\omega'}(\lambda)$. Then ξ_0 is a root of the Bernstein polynomial of the fresco F_ω and there exist at least d roots of the Bernstein polynomial of F_ω in $(\xi_0 + \mathbb{N}) \cap [\xi_0, 0[$ counting multiplicities.
Moreover, under our hypothesis, there exists $\xi_1 \in \xi + \mathbb{Z}$ such that for some $h \in \mathbb{Z}$, $F^h_{\omega',\omega}(\lambda)$ has a pole of order $q + d - 1$ at ξ_1. Let ξ_1 be maximal in $\xi + \mathbb{Z}$ such that this happens. Then ξ_1 is a root of the Bernstein polynomial of the fresco $F_{\omega'}$ and there exist at least d roots of the Bernstein polynomial of $F_{\omega'}$ in $(\xi_1 + \mathbb{N}) \cap [\xi_1, 0[$ counting multiplicities.

PROOF. The first statement is a special case of the previous theorem.
We shall deduce the second statement by using complex conjugation. Let $\xi_0 \in \xi + \mathbb{Z}$ and $h_0 \in \mathbb{Z}$ be such that $F(\lambda) := (2i\pi)^{n+1} F^{h_0}_{\omega,\omega'}(\lambda)$ has a pole of order $q + d - 1$ at ξ_0. As $F(\lambda)$ has only real poles, the poles of $\overline{F(\bar{\lambda})}$ are the same as the poles of $F(\lambda)$ with the same orders. Moreover we may assume that the function ρ is real, so $\overline{F(\bar{\lambda})}$ is given by

$$\frac{(2i\pi)^{-(n+1)}}{\Gamma(\lambda)} \int_U |f|^{2(\lambda+h_0)} \bar{f}^{-h_0} \rho\omega' \wedge \bar{\omega} = \frac{(2i\pi)^{-(n+1)}}{\Gamma(\mu - h_0)} \int_U |f|^{2\mu} \bar{f}^{-h_0} \rho\omega' \wedge \bar{\omega}.$$

where $\mu = \lambda + h_0$. But $F(\lambda)$ is holomorphic when $\Re(\lambda) \geq 0$ and $\Re(\lambda + h_0) \geq 0$ so we may replace $\Gamma(\mu - h_0)$ by $\Gamma(\mu)$ in the right hand-side without changing the poles and their orders. We conclude that $F^{-h_0}_{\omega',\omega}$ has a pole of order $q + d - 1$ at $\xi_0 + h_0$ and applying the first statement gives the conclusion. ■

REMARK. Of course, with the same method, we can obtain a result analogous to that in Theorem 3.2.1 for the asymptotic expansion at the origin of a period-integral of the type

$$s \mapsto \int_{\gamma_s} \omega/df$$

[8] Note that the polar parts of order $\geq q$ of the poles in $\xi + \mathbb{Z}$ are independent of the choice of ρ because of our hypothesis $H(\xi, q)$.

where ω is a holomorphic $(n+1)-$form on U and where $(\gamma_s)_{s\in H}$ is a horizontal family of compact $n-$cycles in the fibers of f:
Assuming the hypothesis $H(\xi, q)$ for $q \geq 1$ and the existence of a non-zero term like $s^{m-\xi}(Log\, s)^{q+d-2}$ with $d \geq 1$ (or $s^m(Log\, s)^{q+d-1}$ for $\xi = 0$), in such an expansion will imply that the Bernstein polynomial of the fresco F_ω will have at least d roots (counting multiplicities) in the set $\xi + \mathbb{Z}$.
This gives a numerical criterion to ensure that such a term will not appear in the expansion we are interested in.

4. The case of a polynomial with $(n+1)$ variables and $(n+2)$ monomials

The purpose of this section is to give a general algorithm in order to obtain an "estimate" of the Bernstein polynomial of the fresco associated to (f, ω) for any polynomial $f \in \mathbb{C}[x_0, \dots, x_n]$ with $(n+2)$ monomials and for any monomial holomorphic differential form $\omega = x^\beta dx$ of degree $n+1$, where β is in $\mathbb{N}^{n+1}$ (we exclude the quasi-homogeneous case for f which is classical, see [16]). Using the results of the previous sections we obtain rather precise information on the exponents of the asymptotic expansions of the period integrals $\int_{\gamma_z} \omega/df$ where $(\gamma_z)_{z\in H}$ is a horizontal family of $n-$cycles in the fibres of f. This gives also rather precise information on the poles of the meromorphic extensions of the Mellin transform of the hermitian periods $z \mapsto \int_{f=z} \rho(\omega/df) \wedge \overline{(\omega'/df)}$:

$$\frac{1}{\Gamma(\lambda)} \int_{\mathbb{C}^{n+1}} |f|^{2\lambda} \bar{f}^h \rho\omega \wedge \bar{\omega}'$$

where $\rho \in \mathscr{C}_c^\infty(\mathbb{C}^{n+1})$ satisfies $\rho \equiv 1$ near 0, where ω, ω' are monomial holomorphic differential forms of degree $n+1$ and where h is in $\mathbb{Z}$. We shall illustrate the result by several examples.

4.1. **Our setting.** We consider a polynomial $f \in \mathbb{C}[x_0, \dots, x_n]$ which is the sum of $n+2$ monomials

$$f = \sum_{j=1}^{n+2} m_j$$

where $m_j := \sigma_j x^{\alpha_j}$, with $\sigma_j \in \mathbb{C}^*$ and $\alpha_j \in \mathbb{N}^{n+1}$ are not 0. Define the matrix with $(n+1)$ lines and $(n+2)$ columns $M = (\alpha_{i,j})$ and let $\tilde{M}$ be the square $(n+2, n+2)$ matrix obtained from M by adding a first line equal to $(1, \dots, 1)$. We shall assume the following conditions:
(C1) $\alpha_1, \dots, \alpha_{n+1}$ is a $\mathbb{Q}-$basis of $\mathbb{Q}^{n+1}$.
(C2) The rank of $\tilde{M}$ is $n+2$.

Remarks.

1. Only the condition (C2) is restrictive on f: when (C2) is fulfilled the condition (C1) may always be satisfied without changing f by a suitable ordering of the $n+2$ monomials.
2. The condition (C2) is equivalent to the fact that f is **not quasi-homogeneous**.

A diagonal linear change of variables allows us to reduce the study to the case where

$$f(x) = \sum_{j=1}^{n+1} x^{\alpha_j} + \lambda x^{\alpha_{n+2}} \tag{a}$$

for some $\lambda \in \mathbb{C}^*$. So in what follows, we shall assume that $m_j = x^{\alpha_j}$ for $j \in [1, n+1]$ and $m_{n+2} = \lambda x^{\alpha_{n+2}}$ where $\lambda \in \mathbb{C}^*$ is a parameter.

Then we shall write, using our hypothesis (C1):

$$\alpha_{n+2} = \sum_{j=1}^{n+1} \rho_j \alpha_j \tag{b}$$

where ρ_j are rational numbers. We define

$$\begin{aligned} H &:= \{j \in [1, n+1] \ / \ \rho_j = 0\}; \\ J_+ &:= \{j \in [1, n+1] \ / \ \rho_j > 0\}; \\ J_- &:= \{j \in [1, n+1] \ / \ \rho_j < 0\}. \end{aligned}$$

Let $|r|$ be the smallest positive integer such that $|r|\rho_j := p_j$ is an integer for each $j \in [1, n+1]$. Write now the relation above as

$$|r|\alpha_{n+2} + \sum_{j \in J_-} (-p_j)\alpha_j = \sum_{j \in J_+} p_j \alpha_j. \tag{c}$$

Now define $d + h$ and d as respectively the supremum and infimum of the two numbers $|r| + \sum_{j \in J_-} (-p_j)$ and $\sum_{j \in J_+} p_j$.
Then d and h are positive :
- - the non-vanishing of d is a consequence of the fact that $|r| \geq 1$ and that at least one p_j is positive.
- - the non-vanishing of h is a consequence of the fact that the equality of these two integers would imply that the first line in $\tilde{M}$ satisfies the same linear relation (b) as all the other lines in $\tilde{M}$, contradicting our hypothesis (C2).
The relation (e) above gives the following equality between the monomials $(m_j)_{j \in [1, n+2]}$:

$$m_{n+2}^{|r|} \prod_{j \in J_-} m_j^{-p_j} = \lambda^{|r|} \prod_{j \in J_+} m_j^{p_j} \tag{d}$$

and we shall write it

$$m^{\Delta} = \lambda^r m^{\delta} \tag{e}$$

where Δ and δ are in $\mathbb{N}^{n+2}$ of respective lengths $d + h$ and d.
Remark that Δ_j and δ_j are zero for each $j \in H$.
Note that the relation (e) defines the sign of r which is well defined in $\mathbb{Z}^*$ by $|r|$ and its sign.

We shall also use the following observation later on :

Lemma 4.1.1. *The $j-$th element of the first column of the matrix $\tilde{M}^{-1}$ is zero if and only if j is in H.*

Proof. The co-factor of the element $(1, j)$ in $\tilde{M}$ is the $(n+1, n+1)$ determinant of the matrix with columns $\alpha_1, \dots, \hat{\alpha}_j, \dots, \alpha_{n+2}$. This matrix has rank at most n if and only if α_{n+2} is a linear combination of $\alpha_1, \dots, \hat{\alpha}_j, \dots, \alpha_{n+1}$. This is the case if and only if $\rho_j = 0$, thanks to our hypothesis (C1). ■

4.2. **The result.** Let Ω^p be the $\mathbb{C}[x_0, \dots, x_n]-$module of polynomial $p-$differential forms on $\mathbb{C}^{n+1}$ and fix a polynomial $f \in \mathbb{C}[x_0, \dots, x_n]$ with $(n+2)$ monomials

$$f := \sum_{j=1}^{n+2} m_j \tag{11}$$

where $m_j := x^{\alpha_j}$ for $j \in [1, n+1]$ and $m_{n+2} := \lambda x^{\alpha_{n+2}}$, with $\lambda \in \mathbb{C}^*$, satisfying the conditions (C1) and (C2) above.

We define $df^p : \Omega^p \to \Omega^{p+1}$ the $\mathbb{C}[x_0, \dots, x_n]-$linear map given by exterior product by df and we note $Ker(df)^p$ its kernel.
Let $E_f := \Omega^{n+1}\big/d(Ker(df)^n\big)$ endowed with its natural structure of module over the $\mathbb{C}-$algebra $\mathcal{A} := \mathbb{C}\langle a, b\rangle$ where the variables a and b satisfy the commutation relation $ab - ba = b^2$. Recall that on E_f the action of a is the multiplication by f and the action of b is given by $df \wedge d^{-1}$ where d is the de Rham differential (which is surjective on Ω^{n+1}).
We extend this structure of (left) $\mathcal{A}-$module to $E_f[\lambda] := E_f \otimes_{\mathbb{C}} \mathbb{C}[\lambda]$ by asking that a and b are $\mathbb{C}[\lambda]-$linear.

Lemma 4.2.1. *For any monomial x^β the image of $\mathbb{C}[m_1, \dots, m_{n+2}].x^\beta$ via the map defined by*

$$m^\eta \mapsto \lambda^{\eta_{n+2}} \prod_{i=0}^{n} x_i^{\sum_{j=1}^{n+2} \alpha_{i,j}\eta_j} \quad \text{where} \quad \eta \in \mathbb{N}^{n+2},$$

is a sub-$\mathcal{A}$-module of $E_f[\lambda]$.

Proof. We want to prove that this image is stable by the action of a and b.
Let $\gamma \in \mathbb{N}^{n+1}$ and compute b using a primitive in x_i, for any $i \in [0, n]$:

$$b(x^{\gamma+\beta}dx) = \frac{1}{\gamma_i + \beta_i + 1} x_i x^{\gamma+\beta} \frac{\partial f}{\partial x_i} dx.$$

Remark now that $x_i \frac{\partial f}{\partial x_i} = \sum_{j=1}^{n+2} \alpha_{i,j} m_j$ so the previous computation gives, with $\gamma := M\eta$ for some $\eta \in \mathbb{N}^{n+2}$ and then $\gamma_i := \sum_{j=1}^{n+2} \alpha_{i,j}\eta_j$,

$$\Gamma_i(\eta, \beta) b(m^\eta x^\beta dx) = \sum_{j=1}^{n+2} \alpha_{i,j} m_j m^\eta x^\beta dx \quad \forall i \in [0, n], \tag{$@_i$}$$

where $\Gamma_i(\eta, \beta) := 1 + \beta_i + \sum_{j=1}^{n+2} \alpha_{i,j}\eta_j$.
Note that we have also

$$a(m^\eta x^\beta dx) = \sum_{j=1}^{n+2} m_j m^\eta x^\beta dx \tag{$@_{n+1}$}$$

The formulas $(@_i)$ for $i \in [0, n+1]$ are enough to conclude the proof. ■

Corollary 4.2.2. *Fix $\beta \in \mathbb{N}^{n+1}$. For each $\eta \in \mathbb{N}^{n+2}$ there exists an element $P_{\beta,\eta}(a, b)$ in $\mathcal{A}$, homogeneous of degree $q := |\eta| := \sum_{j=1}^{n+2} \eta_j$ in (a,b) such that:*

(1) There exists $c(\beta, \eta) \in \mathbb{Q}^*$ such that $P_{\beta,\eta}(a, b)[x^\beta dx] = c(\beta, \eta) m^\eta x^\beta dx$ in E_f.
(2) Assuming that η satisfies $\eta_j = 0$ for each $j \in H$, there exist rational numbers (depending on β and η) $r_1, \dots, r_q$ such that $P_{\beta,\eta}(a, b) = \prod_{h=1}^{q}(a - r_h b)$ in $\mathcal{A}$.

Proof. Let us first show by induction on $q \geq 0$ that such a $P_{\beta,\eta}(a, b)$ satisfying condition 1 exists and that it satisfies condition 2 when η has no component on H. As for $q = 0$ the assertion is clear with $P \equiv 1$, assume that our assertion is proved for any η with $|\eta| = q - 1$ with $q \geq 1$.
Then it is enough to prove the assertion for $m_j m^\eta$ for each $j \in [1, n+2]$ and each η with $|\eta| = q - 1$.
Then consider the equations $(@_i)$ for $i \in [0, n+1]$ as a square $\mathbb{Q}-$linear system of size $(n+2, n+2)$ with unknown the elements $m_j m^\eta x^\beta dx$ in E_f. The matrix of this system is in $Gl(\mathbb{Q}, n+2)$ thanks

to our hypothesis (C2), and so there exist rationals numbers u_j and v_j such that we have, for each $j \in [1, n+2]$,

$$m_j m^\eta x^\beta dx = (u_j a + v_j b)[m^\eta x^\beta dx].$$

With our induction hypothesis this gives that the homogeneous degree q element in $\mathcal{A}$ defined by $P_{\beta,\eta+1_j}(a,b) := (u_j a + v_j b)P_{\beta,\eta}(a,b)$ satisfies 1.
Assuming that $\eta + 1_j$ has no component on H we obtain that $P_{\beta,\eta}(a,b)$ is monic in a (up to a non-zero rational number) and then it is enough to show that u_j is not zero to conclude the induction. This is given by Lemma 4.1.1. ■

Theorem 4.2.3. *Assume that $f \in \mathbb{C}[x_0, \dots, x_n]$ has $(n+2)$ monomials and satisfies the conditions (C1) and (C2) described above. Let d and h be the positive integers defined after (c) and $r \in \mathbb{Z}^*$ defined in (c) and (e). For each $\beta \in \mathbb{N}^{n+1}$ there exist homogeneous elements P_{d+h} and P_d of respective degrees $d+h$ and d which are products of homogeneous factors of degree 1 of the form $a - \xi b$ where $\xi \in \mathbb{Q}$ such that*

$$\Big(P_{d+h}(a,b) - c\lambda^r P_d(a,b)\Big)[x^\beta dx] = 0 \quad \text{in} \quad E_f[\lambda] \quad \text{where} \quad c \in \mathbb{Q}^* \tag{12}$$

Proof. The previous corollary applied to both sides of the equality in $E_f[\lambda]$
$m^\Delta x^\beta dx = \lambda^r m^\delta x^\beta dx$ deduced from (e) and Corollary 4.2.2 allow us to conclude because we know that $\Delta_j = 0$ and $\delta_j = 0$ for each $j \in H$. ■

This theorem has the following corollary.

Corollary 4.2.4. *Let $f \in \mathbb{C}[x_0, \dots, x_n]$ be as in the previous theorem and choose any monomial x^β. Define for any horizontal family of $n-$cycles $(\gamma_s)_{s\in S}$ over a simply connected open set S in $\mathbb{C}^*$ avoiding the critical values of f and having 0 in its boundary, the integral-period:*

$$\varphi_\beta(s) := \int_{\gamma_s} x^\beta dx/df \tag{13}$$

Then φ_β is a solution on S of the differential equation (which is regular singular at 0) obtained from (12) by letting $a = \times s$ and $b := \int_0^s$.

Proof. Thanks to Proposition 2.3.2 the annihilator in $\hat{\mathcal{A}}$ of the element $[x^\beta dx]$ in $E_f \otimes_{\mathcal{A}} \hat{\mathcal{A}}$ is a left multiple of $P_d(a,b)$ as c and λ are not 0. This is enough to conclude. ■

This shows that the computation of $P_d(a,b)$ gives rather precise information on the asymptotic expansion at 0 of such a period-integral.
We leave the corresponding statement for the poles of the Mellin transform of the hermitian period-integrals corresponding to such an f and monomial holomorphic differential forms ω and ω' to the reader. Of course, by conjugation, the Bernstein polynomial of the fresco (f, ω') also gives constraints on the possible poles of this Mellin transform, as in Corollary 3.2.2.

4.3. **Examples.** The control of the Bernstein polynomial of a fresco will use Theorem 4.2.3 and Proposition 2.3.2.

4.3.1. $f_\lambda := x^5 + y^5 + z^5 + \lambda xyz^2$. We assume that λ is a non-zero complex number. Then 0 is the only singular point of the hypersurface $\{f = 0\}$:
as on the set $\Sigma := \{df = 0\} \subset \mathbb{C}^3$ we have

$$f(x,y,z) = \frac{1}{5}\lambda xyz^2,$$

we easily deduce that $\Sigma \cap \{f = 0\} = \{0\}$.

Now using the method explained above we obtain, after some elementary computations, for each monomial form ω below a degree 4 polynomial multiple of the Bernstein polynomial of the fresco F_ω.
Note that such a fresco has rank at most equal to 4 (recall that for a fresco with rank k its Bernstein polynomial has degree k; see [8]) and if the rank is equal to 4 then we obtain the Bernstein polynomial itself.
Of course, the reader interested by more monomials can easily complete this list, where | means "divides" :

- $\omega = dx \wedge dy \wedge dz \qquad B_1(\xi) \,|\, (\xi + \frac{7}{10})(\xi + \frac{4}{5})^2(\xi + \frac{6}{5})$.
- $\omega = xdx \wedge dy \wedge dz \qquad B_x(\xi) \,|\, (\xi + \frac{9}{10})(\xi + 1)(\xi + \frac{6}{5})(\xi + \frac{7}{5})$.
- $\omega = zdx \wedge dy \wedge dz \qquad B_z(\xi) \,|\, (\xi + 1)^3(x + \frac{3}{2})$.
- $\omega = z^2dx \wedge dy \wedge dz \qquad B_{z^2}(\xi) \,|\, (\xi + \frac{6}{5})^2(\xi + \frac{13}{10})(\xi + \frac{9}{5})$.
- $\omega = xydx \wedge dy \wedge dz \qquad B_{x.y}(\xi) \,|\, (\xi + \frac{11}{10})(\xi + \frac{7}{5})^2(\xi + \frac{8}{5})$.
- $\omega = x^2dx \wedge dy \wedge dz \qquad B_{x^2}(\xi) \,|\, (\xi + \frac{6}{5})(\xi + \frac{8}{5})^2(\xi + \frac{11}{10})$.
- $\omega = xzdx \wedge dy \wedge dz \qquad B_{x.z}(\xi) \,|\, (\xi + \frac{6}{5})^2(\xi + \frac{7}{5})(\xi + \frac{17}{10})$.
- $\omega = xyzdx \wedge dy \wedge dz \qquad B_{x.y.z}(\xi) \,|\, (\xi + \frac{7}{5})(\xi + \frac{8}{5})^2(\xi + \frac{19}{10})$ etc ...

Note that in this example the differential forms corresponding to degree 2 monomials in x, y, z are global holomorphic $3-$forms on the fibers of the family of compact surfaces given, for λ fixed, by the fibers of the map $\pi_\lambda(s, (x, y, z, t)) = s$, sending

$$\mathcal{X}_\lambda := \{(s, (x, y, z, t)) \in \mathbb{C} \times \mathbb{P}_3(\mathbb{C}) \ / \ st^5 = x^5 + y^5 + z^5 + \lambda xyz^2t\}$$

to $\mathbb{C}$. As, moreover, the map π_λ has no singular point at infinity, the affine computation controls also the global case for these forms.
Remark that the global computation for these forms gives here the same frescos as in the affine case because f_λ has an isolated singularity at the origin.

4.3.2. $f = xy^3 + yz^3 + zx^3 + \lambda xyz$. The singularity set of the hypersurface $\{f = 0\}$ is the origin: It is easy to see that any monomial of f is a linear combination of f and $x\frac{\partial f}{\partial x}, y\frac{\partial f}{\partial y}, z\frac{\partial f}{\partial z}$, so that each monomial in f has to vanish on the singular set of $\{f = 0\}$. Then this implies easily our claim.
Again using Theorem 4.2.3 and Proposition 2.3.2 allows us, after some elementary computations, to find for each monomial form ω below a degree 3 polynomial dividing the Bernstein polynomial of the fresco F_ω.

- $\omega = dx \wedge dy \wedge dz \qquad B_1(\xi) \,|\, (\xi + 1)^3$.
- $\omega = xdx \wedge dy \wedge dz \qquad B_x(\xi) \,|\, (\xi + \frac{8}{7})(\xi + \frac{9}{7})(\xi + \frac{11}{7})$.
- $\omega = x^2dx \wedge dy \wedge dz \qquad B_{x^2}(\xi) \,|\, (\xi + \frac{9}{7})(\xi + \frac{11}{7})(\xi + \frac{15}{7})$.
- $\omega = xydx \wedge dy \wedge dz \qquad B_{x.y}(\xi) \,|\, (\xi + \frac{10}{7})(\xi + \frac{12}{7})(\xi + \frac{13}{7})$.
- $\omega = xyzdx \wedge dy \wedge dz \qquad B_{x.y.z}(\xi) \,|\, (\xi + 2)^3$.
- $\omega = x^7dx \wedge dy \wedge dz \qquad B_{x^7}(\xi) \,|\, (\xi + 5)(\xi + 3)(\xi + 2)$.

4.3.3. $f := xy^2z^3 + yz^2t^3 + zt^2x^3 + tx^2y^3 + \lambda xyzt$. In this case the singularity set is not isolated: the singular set of $\{f = 0\}$ contains the union of the planes $\{x = z = 0\}$ and $\{y = t = 0\}$.

Lemma 4.3.1. *The estimate for the Bernstein polynomial associated to the monomial* 1 *(so of the fresco* F_ω *with* $\omega := dx \wedge dy \wedge dz \wedge dt$*) is* $B_1(\xi) \,|\, (\xi + 1)^4$.

PROOF. Write m_1, m_2, m_3, m_4, m_5 the monomials in f. As we have

$$6\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\\0\end{pmatrix} + \begin{pmatrix}0\\1\\2\\3\end{pmatrix} + \begin{pmatrix}3\\0\\1\\2\end{pmatrix} + \begin{pmatrix}2\\3\\0\\1\end{pmatrix}$$

the element in $\mathcal{A}$ given by Theorem 4.2.3 which kills $dx \wedge dy \wedge dz \wedge dt$ (noted by (1) in the sequel) is deduced from the equality

$$\lambda^6 m_1 m_2 m_3 m_4 = m_5^6$$

and its initial part in (a,b) corresponds to the monomial $m_1 m_2 m_3 m_4$ in $\mathcal{A}(1)$. We shall make the computation explicit (following the proof of the theorem) in this very interesting example.

The inverse of the matrix :

$$M = \begin{pmatrix}1 & 0 & 3 & 2 & 1\\ 2 & 1 & 0 & 3 & 1\\ 3 & 2 & 1 & 0 & 1\\ 0 & 3 & 2 & 1 & 1\\ 1 & 1 & 1 & 1 & 1\end{pmatrix}$$

is the matrix:

$$M^{-1} = \frac{1}{8}\begin{pmatrix}1 & 1 & 3 & -1 & -4\\ -1 & 1 & 1 & 3 & -4\\ 3 & -1 & 1 & 1 & -4\\ 1 & 3 & -1 & 1 & -4\\ -4 & -4 & -4 & -4 & 24\end{pmatrix}$$

Then we have:

$$\begin{pmatrix}b(1)\\b(1)\\b(1)\\b(1)\\a(1)\end{pmatrix} = \begin{pmatrix}1 & 0 & 3 & 2 & 1\\ 2 & 1 & 0 & 3 & 1\\ 3 & 2 & 1 & 0 & 1\\ 0 & 3 & 2 & 1 & 1\\ 1 & 1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}m_1\\m_2\\m_3\\m_4\\m_5\end{pmatrix}$$

and so

$$\begin{pmatrix}m_1\\m_2\\m_3\\m_4\\m_5\end{pmatrix} = \frac{1}{8}\begin{pmatrix}1 & 1 & 3 & -1 & -4\\ -1 & 1 & 1 & 3 & -4\\ 3 & -1 & 1 & 1 & -4\\ 1 & 3 & -1 & 1 & -4\\ -4 & -4 & -4 & -4 & 24\end{pmatrix}\begin{pmatrix}b(1)\\b(1)\\b(1)\\b(1)\\a(1)\end{pmatrix}$$

we obtain:

$$m_1 = m_2 = m_3 = m_4 = \frac{1}{2}(b-a)(1) \quad m_5 = -2b(1) + 3a(1) = (3a-2b)(1)$$

and then

$$\begin{pmatrix}2b(m_1)\\3b(m_1)\\4b(m_1)\\b(m_1)\\a(m_1)\end{pmatrix} = \begin{pmatrix}1 & 0 & 3 & 2 & 1\\ 2 & 1 & 0 & 3 & 1\\ 3 & 2 & 1 & 0 & 1\\ 0 & 3 & 2 & 1 & 1\\ 1 & 1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}m_1^2\\m_1m_2\\m_1m_3\\m_1m_4\\m_1m_5\end{pmatrix}.$$

$$\begin{pmatrix} m_1^2 \\ m_1m_2 \\ m_1m_3 \\ m_1m_4 \\ m_1m_5 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 1 & 1 & 3 & -1 & -4 \\ -1 & 1 & 1 & 3 & -4 \\ 3 & -1 & 1 & 1 & -4 \\ 1 & 3 & -1 & 1 & -4 \\ -4 & -4 & -4 & -4 & 24 \end{pmatrix}\begin{pmatrix} 2b(m_1) \\ 3b(m_1) \\ 4b(m_1) \\ b(m_1) \\ a(m_1) \end{pmatrix}.$$

And so:

$$\begin{aligned} m_1^2 &= \frac{1}{2}(4b-a)(m_1) = \frac{1}{4}(4b-a)(b-a)(1) \\ m_1m_2 &= \frac{1}{2}(2b-a)(m_1) = \frac{1}{4}(2b-a)(b-a)(1) \\ m_1m_3 &= \frac{1}{2}(2b-a)(m_1) = m_1m_2 \\ m_1m_4 &= \frac{1}{2}(2b-a)(m_1) = m_1m_2 \\ m_1m_5 &= (3a-5b)(m_1) = \frac{1}{2}(3a-5b)(b-a)(1) \end{aligned}$$

We have $m_1m_2 = xy^3z^5t^3$ and then

$$m_1m_2m_3 = \frac{1}{8}\begin{pmatrix} 3 & -1 & 1 & 1 & 1 & -4 \end{pmatrix}\begin{pmatrix} 2b(m_1m_2) \\ 4b(m_1m_2) \\ 6b(m_1m_2) \\ 4b(m_1m_2) \\ a(m_1m_2) \end{pmatrix} = \frac{1}{2}(3b-a)(m_1m_2).$$

So

$$m_1m_2m_3 = \frac{1}{8}(3b-a)(2b-a)(b-a)(1).$$

As $m_1m_2m_3 = x^4y^3z^6t^5$ we obtain:

$$m_1m_2m_3m_4 = \frac{1}{8}\begin{pmatrix} 1 & 3 & -1 & 1 & -4 \end{pmatrix}\begin{pmatrix} 5b(m_1m_2m_3) \\ 4b(m_1m_2m_3) \\ 7b(m_1m_2m_3) \\ 6b(m_1m_2m_3) \\ a(m_1m_2m_3) \end{pmatrix} = \frac{1}{2}(4b-a)(m_1m_2m_3)$$

we obtain $m_1m_2m_3m_4 = \frac{1}{16}(4b-a)(3b-a)(2b-a)(b-a)(1)$ ■

4.3.4. $f := xy^2 + x^2y + zt^3 + tz^3 + \lambda xyzt$. Again we assume that λ is a non-zero complex number. The hypersurface $\{f = 0\}$ has an isolated singularity at the origin :
If $\Sigma := \{df = 0\} \subset \mathbb{C}^4$ we have on Σ the relations

$$xy^2 = x^2y = \frac{-1}{3}\lambda xyzt \quad \text{and} \quad zt^3 = z^3t = \frac{-1}{4}\lambda xyzt.$$

So on $\Sigma \cap \{f = 0\}$ we have $xy = 0 = zt$ and this implies that $\Sigma \cap \{f = 0\} = \{0\}$.

Now we shall use again Theorem 4.2.3 and Proposition 2.3.2 in order to give a polynomial of degree 12 which is a multiple of the Bernstein polynomial of the fresco F_ω for $\omega := dx\wedge dy\wedge dz\wedge dt$.

The relation between the monomials of f is

$$\lambda^{12}(xy^2)^4(yx^2)^4(zt^3)^3(z^3t)^3 = (\lambda xyzt)^{12}.$$

So to compute the initial form in (a,b) of the polynomial in $\mathcal{A}$ constructed in Theorem 4.2.3 annihilating $[\omega]$ in $E_f[\lambda]$, it is enough to compute the homogeneous in (a,b) polynomial P of degree 12 satisfying in $E_f[\lambda]$ the relation $P[\omega] = [(\lambda xyzt)^{12}\omega]$.
Note $m_1, \dots, m_4$ the first monomials in f and $m := \lambda xyzt$. Then we have in $E_f[\lambda]$ the equalities for any integer $k \geq 0$ (where ω is omitted)

- $\big((k+1)b(m^k) - m^{k+1}\big) = (m_1 + 2m_2)m^k$
- $\big((k+1)b(m^k) - m^{k+1}\big) = (2m_1 + m_2)m^k$
- $\big((k+1)b(m^k) - m^{k+1}\big) = (m_3 + 3m_4)m^k$
- $\big((k+1)b(m^k) - m^{k+1}\big) = (3m_3 + m_4)m^k$

and so we obtain

$$\big(a - \frac{7}{6}(k+1).b\big)(m^k) = \frac{-1}{6}m^{k+1}.$$

Then the initial form of the polynomial annihilating $[\omega]$ is equal to the product ordered from left to right by decreasing k

$$\prod_{k=0}^{11} \big(a - \frac{7}{6}(k+1)b\big)(m^k).$$

This gives the following estimate for the Bernstein polynomial

$$B(\xi) \mid \prod_{k=0}^{11} (\xi + \frac{k+7}{6})$$

The reader interested by another holomorphic monomial form can follow the same line to obtain an analogous result.

In a forthcoming article on this subject we shall examine the **effective contribution** of the roots in $\xi + \mathbb{Z}$ of the Bernstein polynomial of the fresco associated to a volume form ω to the poles of the meromorphic extension of the integrals

$$\frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^j \rho\omega \wedge \bar{\omega}'$$

assuming the hypothesis $H(\xi, 1)$ and where ω' is in Ω_0^{n+1} and ρ is $\mathscr{C}^\infty$ on X, $\rho \equiv 1$ near 0 and has enough small support in order that $\rho\bar{\omega}'$ is $\mathscr{C}^\infty$ on X.

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