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TOPOLOGICAL CLASSIFICATION OF CIRCLE–VALUED SIMPLE MORSE–BOTT FUNCTIONS ON CLOSED ORIENTABLE SURFACES

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Abstract. In this work we investigate the topological classification of circle-valued simple Morse-Bott functions on connected closed orientable surfaces, up to topological conjugacy. We provide a complete topological invariant, called the MB-Reeb graph. This invariant is based on the generalized Reeb graph and the topological type of singular level sets of these functions. The results presented here extend to those obtained by the authors in a previous work when the surface is the standard sphere.

1. Introduction

Denote by S^n the standard sphere in $\mathbb{R}^{n+1}$.

Let $f, g : M^n \to \mathbb{R}$ be two smooth functions, where M^n denotes an n-dimensional manifold. The term smooth used here will mean "at least three times continuously differentiable" throughout the paper. We say that f and g are *topologically equivalent* if there exist homeomorphisms $h : M^n \to M^n$ and $k : \mathbb{R} \to \mathbb{R}$ such that

$$f = k \circ g \circ h^{-1}.$$

The problem of topological classification of smooth functions is a classical subject in Topology and Singularity theory. However, global results and global invariants are not easy to obtain. Then some restrictions on the manifold M^n or on the function f are considered. For instance, Fukuda [11] shows that there is a finite number of topological equivalence classes if we consider the space of all polynomials $f : M^n = \mathbb{R}^n \to \mathbb{R}$ of limited degree. Prishlyak [21] provides a topological classification of smooth functions on a closed surface M^2 with isolated critical points. About the topological classification of Morse functions on surfaces, we can mention the works of Arnold [4], Kulinich [14], Nicolaescu [18] and Sharko [23, 24], for instance.

In all these works previously cited, the Reeb graph associated to f plays a crucial role. Reeb graphs were introduced by Reeb [22] to study Morse functions from closed surfaces M^2 to $\mathbb{R}$. The classical Reeb graph associated to f is the graph obtained by contracting each connected component of level set (fibers) of f to a point, where the vertices correspond to the singular fibers of f. The Reeb graph allows us to study the evolution and to express the connectivity of the level sets of f. It has many interesting applications in Mathematics as well as in other areas such as Computational Geometry, Computer Graphics, etc. Recently, a generalization of the Reeb graph, called the *generalized Reeb graph*, was introduced in [7, 8]. The generalized Reeb graph has extra additional information compared to the classical Reeb graph. Based on generalized Reeb graphs, the authors in [6] investigated the topological classification of circle-valued Morse-Bott functions, up to topological conjugacy, in the following sense:

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Definition 1. *Two Morse-Bott functions $f, g : M^2 \to S^1$ are said to be* topologically conjugated *if there exist homeomorphisms $h : M^2 \to M^2$ and $k : S^1 \to S^1$ such that $f = k \circ g \circ h^{-1}$ and h sends singular fibers of g to singular fibers of f.*

When $M^2 = S^2$ the topological classification of simple surjective Morse-Bott functions from S^2 to S^1 was done by the authors in [6], where they introduced a complete invariant. If the Morse-Bott functions are not surjective, the topological classification can be reduced to the case from S^2 to $\mathbb{R}$ (via stereographic projection) which had already been obtained in [15, 16]. When $f : M^2 \to S^1$ is a Morse function, the topological classification was done in [20] and in the case of the Morse functions $f : S^1 \to S^1$, the topological classification is related to snakes defined by Arnold (see [2, 3]).

In this paper we extend the topological classification done in [6], now investigating simple Morse-Bott functions from M^2 to S^1, where M^2 is any connected closed orientable surface. We introduce a complete invariant, called the *MB-Reeb graph*. The MB-Reeb graph coincides with the invariant introduced by the authors in [6], when $M^2 = S^2$.

2. Morse-Bott functions

Classical Morse theory deals only with functions whose critical points are all non-degenerate. In particular, the critical points must all be isolated points. However, in many situations, critical points form submanifolds of M^n of dimension greater than or equal to one. Then Bott in [9] studied how to extend the theory of Morse to this situation. In fact, Bott introduced the notion of a non-degenerate critical submanifold: a critical submanifold $S \subset M^n$ is non-degenerate if at any point $p \in S$ the Hessian of f restricted to the normal space to S is non-singular.

A point $p \in M^n$ is called a *singular point* of f if $rank(\mathrm{d}f(p))$ is not maximum, where $\mathrm{d}f(p)$ denotes the differential of f at $p \in M^n$. Otherwise, p is called a *regular point* of f. A point $b \in \mathbb{R}$ is called a *singular value* of f if $f^{-1}(b)$ contains a singular point of f. The *singular set* of f, denoted by $Sing(f)$, is the set of all singular points of f. The image of $Sing(f)$ by f is called the *discriminant set* of f, denoted by Δ_f.

For each $a \in \mathbb{R}$ consider the level set $I_a(f) = f^{-1}(a)$. Notice that $I_a(f)$ is a union of connected components, $i_a^k(f)$, $k = 1, \dots, m(a)$, called *fibers*. A *singular fiber* is a connected component of a level set $I_a(f)$ which contains a singular point of f and it is denoted by $s_a(f)$.

If all nearby fibers around a singular fiber are homeomorphic to it then this fiber is called *reducible*. See [1] for details.

The nullity of a singular point p of f is the dimension of the kernel of the Hessian of f evaluated at p, $Hess_p f$. A smooth submanifold $S \subset Sing(f)$ is a non-degenerate singular submanifold of f if the followings conditions are satisfied (see [9]):

(i) $\partial S = \emptyset$;
(ii) S is compact and connected;
(iii) For all $s \in S$, the nullity of s is equal to the dimension of S.

Definition 2. *Let M^n be a closed orientable n-dimensional manifold. A smooth function $f : M^n \to \mathbb{R}$ is called a* Morse-Bott function *(from now on $\mathcal{MB}$ function) if the set $Sing(f)$ is a disjoint union of connected non-degenerate singular submanifolds of dimension $\leq n - 1$.*

It follows from the Morse-Bott Lemma (see [5]) that Morse functions are $\mathcal{MB}$ functions with isolated singular points. Since M^n is compact then these functions have a finite number of isolated singular points.

We can find many examples of $\mathcal{MB}$ functions in the references [6, 15, 16].

From now on, we will work with connected closed orientable 2-dimensional manifolds M^2.

Considering the dimension of the singular submanifolds and using the Morse-Bott Lemma we note that the singular set $Sing(f)$ of an $\mathcal{MB}$ function $f : M^2 \to \mathbb{R}$ can be subdivided into three subsets:

(i) Points on singular submanifolds which are homeomorphic to S^1. In these circles, the function takes extreme values. Such singular submanifolds are called *singular circles*.
(ii) Isolated singular points that are extreme points of f (maximum or minimum).
(iii) Saddle points, that is, isolated singularities of index 1 of f.

3. Circle-valued Morse-Bott functions

In this section, we will investigate simple Morse-Bott functions on M^2, but now taking values on S^1 instead of $\mathbb{R}$. We call such functions $f : M^2 \to S^1$ *circle-valued Morse-Bott functions*. Recently, the authors investigated the simple circle-valued Morse-Bott functions defined on the standard sphere $M^2 = S^2$ (see [6]). Also, a similar approach was done for stable circle-valued functions in [7, 8].

Since S^1 is locally homeomorphic to $\mathbb{R}$ then circle-valued functions may be seen locally as real-valued functions, and then all the local notions of the Morse-Bott theory are transported immediately to the framework of circle-valued functions.

Definition 3. *A circle-valued function $f : M^2 \to S^1$ is called a circle-valued $\mathcal{MB}$ function if for any $x \in M^2$ we can choose a neighborhood V of $f(x) \in S^1$, and a diffeomorphism $\phi : V \to \mathbb{R}$, such that $\phi \circ (f|_U)$, where $U = f^{-1}(V)$, is a real-valued Morse-Bott function.*

Example 1. *Let $M^2 = T_2$ be the bitorus and consider the radial projection as indicated in Figure 1. Then we have an example of a circle-valued $\mathcal{MB}$ function with two saddle points and two singular circles.*

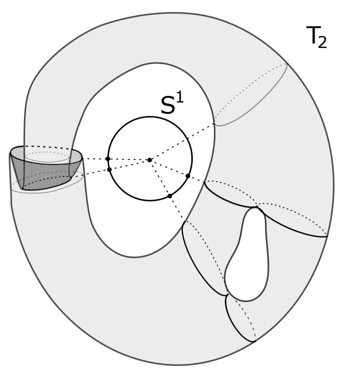

Figure 1. Circle-valued Morse-Bott function of the bitorus.

Proposition 4. *Let $f : M^2 \to S^1$ be a circle-valued Morse-Bott function, with Euler characteristic $\chi(M^2) \neq 0$. Then f is always non-regular.*

Proof. Suppose f is regular. Then, f should be surjective and from Ehresmann's fibration theorem [10], f should be a locally trivial fibration. In particular, if F is a fiber of this fibration, then it must happen that

$$0 \neq \chi(M^2) = \chi(S^1)\chi(F) = 0,$$

which is absurd.

Then f is always non-regular. □

Definition 5. *We say that $f : M^2 \to S^1$ is a* simple *circle-valued $\mathcal{MB}$ function if there is only one connected component containing the isolated singular points or the non-degenerate singular submanifolds in the singular level. It is contained in a singular fiber $s_a(f) \subset I_a(f)$ for each $a \in \mathbb{R}$.*

In particular, the function defined in Example 1 is a simple circle-valued $\mathcal{MB}$ function.

From now on, we will always consider simple circle-valued $\mathcal{MB}$ functions. Sometimes, for simplicity, when there is no doubt that f takes values on S^1 (i.e., $f : M^2 \to S^1$) we will just write simple $\mathcal{MB}$ function for f.

The proof of the next proposition follows from the same ideas as in Proposition 5 in [6]. Then, its proof will be omitted.

Proposition 6. *Let $f : M^2 \to S^1$ be a simple $\mathcal{MB}$ function. Then the non-degenerate singular submanifolds of f are homeomorphic either S^1 or points. Moreover, there is a finite number of them on M^2.*

It follows from Proposition 6, that the singular set $Sing(f)$ associated to a simple $\mathcal{MB}$ function $f : M^2 \to S^1$ can also be divided into three subsets, according to the dimension of the singular submanifolds and their indices. The possibilities are: singular circles (homeomorphic to S^1), maximum or minimum points and saddle points.

4. The MB-Reeb graph of a simple circle-valued $\mathcal{MB}$ function

It is well-known that the Reeb graph is a powerful tool for studying the topological classification of functions. For instance, Arnold [4], Kulinich [14], Nicolaescu [18] and Sharko [23, 24] investigate the classification of Morse functions on surfaces, using Reeb graphs with some additional information. Prishlyak [21] classified smooth functions with isolated critical points on closed surfaces, also working with Reeb graphs. Sharko [25], Masumoto-Saeki [17] and recently Michalak [19], investigate the problem of the realization of a given graph as the Reeb graph associated to a smooth function $f : M^n \to \mathbb{R}$ with finitely many critical points, where M^n is a closed manifold.

In this paper, our goal is to extend the topological invariant introduced in [6], now for simple circle-valued $\mathcal{MB}$ functions defined on any connected closed orientable surface M^2. This new invariant, the MB-Reeb graph, will be used to provide a complete topological classification, up to topological conjugacy, of simple circle-valued $\mathcal{MB}$ functions from M^2 to S^1.

The MB-Reeb graph is inspired by the notion of generalized Reeb graphs which, in turn, are a generalization of classical Reeb graphs (cf. [6, 7, 8]). Its definition also takes into account the notions of circles and separatrix eights associated to saddle points. These circles and separatrix eights were first introduced in [15, 16] to investigate the classification of singular level sets of $\mathcal{MB}$ functions from M^2 to $\mathbb{R}$ and also the singular leaves of $\mathcal{MB}$ foliations on orientable surfaces. In this work we will use this classification (see Theorems 11 and 12) in order to define a global invariant for simple circle-valued $\mathcal{MB}$ functions on M^2.

Let $f : M^2 \to S^1$ be a simple $\mathcal{MB}$ function. Consider the following equivalence relation on M^2: $x \sim y$ if and only if $f(x) = f(y)$, where x and y are in the same connected component of $f^{-1}(f(x))$.

Proposition 7. *Let $f : M^2 \to S^1$ be a simple $\mathcal{MB}$ function. Then the quotient space $\mathcal{R}_f = M^2/\sim$ admits a structure of a connected graph in the following way:*

1. *the vertices are the connected components of level curves $f^{-1}(v)$, where $v \in S^1$ is a critical value of f;*
2. *each edge is formed by points corresponding to connected components of level curves $f^{-1}(v)$, where $v \in S^1$ is a regular value of f.*

The proof of Proposition 7 is analogous to the proof of Proposition 6 in [6]. Then it will be omitted here.

Each vertex of the graph obtained as in Proposition 7 can be of four types, depending on if the connected component has a saddle point, maximum/minimum critical point, regular points whose images are critical values or singular circles. Vertices corresponding to singular circles will be identified by white vertices " $\circ$ ".

So, the possible incidence rules of edges and vertices when $f : M^2 \to S^1$ is a simple $\mathcal{MB}$ function and according to the topological type of level curve of f are given in Figure 2.

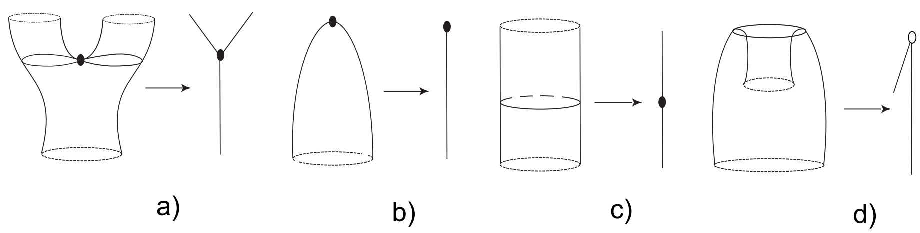

FIGURE 2. Incidence rules of edges and vertices.

Consider $\mathcal{B}$ a singular fiber containing a saddle point of a simple circle-valued $\mathcal{MB}$ function as in Figure 3. For simplicity, we call this curve a *saddle curve* (which is homeomorphic to an eight). Denote by $V_{\mathcal{B}}$ a regular closed neighborhood of $\mathcal{B}$ (for details about regular neighborhoods, see [12, 13]).

Lemma 8. *Let $f : M^2 \to S^1$ be an $\mathcal{MB}$ function having a saddle point x. Then a closed regular neighborhood $V_{\mathcal{B}}$ of a singular fiber $\mathcal{B}$ containing x is homeomorphic to a sphere minus three disks.*

Proof. Since f is an $\mathcal{MB}$ function and $x \in M^2$ is a saddle point of f, it follows from Definition 3 that we can choose a neighborhood V of $f(x) \in S^1$, and a diffeomorphism $\phi : V \to \mathbb{R}$, such that $\tilde{f} = \phi \circ (f|_U)$, where $U = f^{-1}(V)$, is a real-valued Morse-Bott function and x is a saddle point of $\tilde{f}$. Applying [16, Lemma 3.3], we have that a closed regular neighborhood $V_{\mathcal{B}}$ of $\mathcal{B}$ must be homeomorphic to a sphere minus three disks. $\square$

If $f : M^2 \to S^1$ is an $\mathcal{MB}$ function then a closed regular neighborhood of a saddle curve $\mathcal{B}$ has three boundary curves in $V_{\mathcal{B}}$ (see Figure 3). One of them is contractible to $\mathcal{B}$ and this respective boundary curve will be denoted by $J_{\mathcal{B}}$ (see Figure 3).

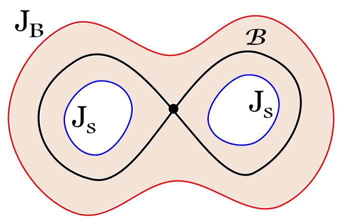

FIGURE 3. A regular closed neighborhood $V_{\mathcal{B}}$ of a saddle curve $\mathcal{B}$.

4.1. **Classification of circles and separatrix eights.**

In this subsection we bring referential elements of the topological classification (up to homeomorphisms) of circles and eights on closed orientable surfaces. This classification was done in [15] and, as mentioned in the Introduction, it will be a key point to define the MB-Reeb graph.

We start with two general definitions:

Definition 9. *An embedded circle on M^2 is the image of an embedding $\psi : S^1 \to M^2$.*

Definition 10. *We will say that two circles (or two eights) in M^2 are* topologically equivalent *if there is a homeomorphism on M^2 that sends one of them to the other.*

Now, in coherence with the notations and results given in [15] we present here some more notations.

Let $E\left(a\right)$ be the largest integer not greater than a and $C\left(a\right)$ be the smallest integer not less than a.

If $\psi(S^1) = l$ is the image of an embedding, then $M^2 \setminus l$ could have one or two connected components. Taking the closure of these components, we obtain a compact surface with holes, denoted by N (or N_j, $j = 1, 2$ in the case of two connected components). We will say that l is of type l_0 if l is homotopic to zero. Also, we say that l is of type l_i if it is homologous to zero but not homotopic to zero and $M^2 \setminus l$ is the disjoint union of N_1 and N_2, with $i = min(g_1, g_2)$, where g_1 and g_2 are the genus of N_1 and N_2, respectively. Finally, we say that l is of type l_K if the curve is not homologous to zero.

The next two theorems give the topological classification of circles and eights on closed orientable surfaces.

Theorem 11. *[[15]] Let M^2 be a closed orientable surface of genus g. Then, the number of non-equivalent embeddings of S^1 on M^2 is*

(i) *1 if $g = 0$,*
(ii) *$E\left(\frac{g}{2}\right) + 2$ with representatives $l_0, l_1, \ldots, l_{E\left(\frac{g}{2}\right)}, l_K$ if $g > 0$.*

Theorem 12. *[[15]] Let M^2 be a closed orientable surface of genus g. Then, the number of topological types of eights on M^2 is*

(1) *$3g + 1$, if $g = 0, 1$,*
(2) *$E\left(\frac{g}{2}\right) C\left(\frac{g}{2}\right) + E\left(\frac{g}{2}\right) + 2g + 3$, if $g \geq 2$.*

Although an eight is homeomorphic to two circles glued by a point, the topological type of these circles is not sufficient to distinguish the topological type of the eight (see cases 11 and 12 in the Table 1 in [15]). Hence the topological type of an eight must be included in any complete invariant associated with $\mathcal{MB}$ function from M^2 to $\mathbb{R}$. It was done in [15]. In this work, we will take a similar approach carrying this topological information in the Reeb graph, as follows.

Let ξ be the function that associates each saddle vertex of $\mathcal{R}_f$ with the edge containing the $J_{\mathcal{B}}$ curve. Let us indicate the effects of ξ in $\mathcal{R}_f$ by the symbol $\star$ on the corresponding edge. See Figure 4.

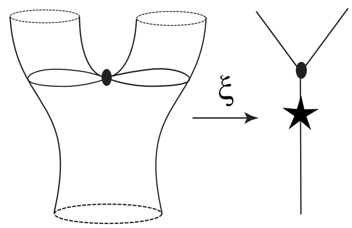

FIGURE 4. The function ξ.

Lemma 13. *Let ξ be a function on $\mathcal{R}_f$ that associates each saddle vertex of degree 3 with the edge containing the $J_{\mathfrak{B}}$ curve. The graph $\mathcal{R}_f$ and ξ determine the topological type of saddle curve $\mathfrak{B}$.*

Proof. Locally, each saddle vertex of degree 3 of $\mathcal{R}_f$ has exactly the same behavior as a saddle vertex of degree 3 in a classical Reeb graph associated to an $\mathcal{MB}$ function from M^2 to $\mathbb{R}$. Applying the Proposition 3.10 in [16], we have that the graph $\mathcal{R}_f$ and ξ determine the topological type of saddle curve $\mathfrak{B}$. □

Let $v_1, \ldots, v_r \in S^1$ be the critical values of f. We choose a base point $v_0 \in S^1$ and an orientation. We can reorder the critical values such that $v_0 \leq v_1 < \ldots < v_r$ and we label each vertex with the index $i \in \{1, \ldots, r\}$, if it corresponds to the critical value v_i.

Definition 14. *The graph given by $\mathcal{R}_f$ together with the labels, colors of the vertices and with the function ξ, as previously defined, will be called here the* MB-Reeb graph *associated to a simple $\mathcal{MB}$ function $f : M^2 \to S^1$. The MB-Reeb graph associated to f will be denoted by Γ_f.*

Remark 15. *Notice that if $M^2 = S^2$ then there is only one topological type of saddle curves. Then, in this case, it is not necessary to label the edges that contain the points associated to $J_{\mathfrak{B}}$ curves. Hence, the MB-Reeb graph of f introduced here coincides with that graph defined in [6].*

Proposition 16 ([6])**.** *Let $f : S^2 \to S^1$ be a simple $\mathcal{MB}$ function. Then the MB-Reeb graph of f is a tree.*

Example 2. *Let f_1, $f_2 : S^2 \to S^1$ be two simple $\mathcal{MB}$ functions given by the radial projection. Then their respective MB-Reeb graphs are given as they appear in Figure 5.*

The functions f_1 and f_2 in Example 2 have the same classical Reeb graph, but their corresponding MB-Reeb graphs are distinct. In fact, the function f_1 is not surjective while f_2 is surjective. Therefore, f_1 and f_2 are not topologically equivalent. This example shows that the classical Reeb graph is not enough to distinguish between these two simple $\mathcal{MB}$ functions.

Example 3. *Consider the following two simple circle-valued $\mathcal{MB}$ functions of the torus and their respective MB-Reeb graphs, as shown in Figure 6. Notice that without the $\star$ symbol the graphs would be equivalent, but their MB-Reeb graphs are different because the topological type of the saddle curves are not the same.*

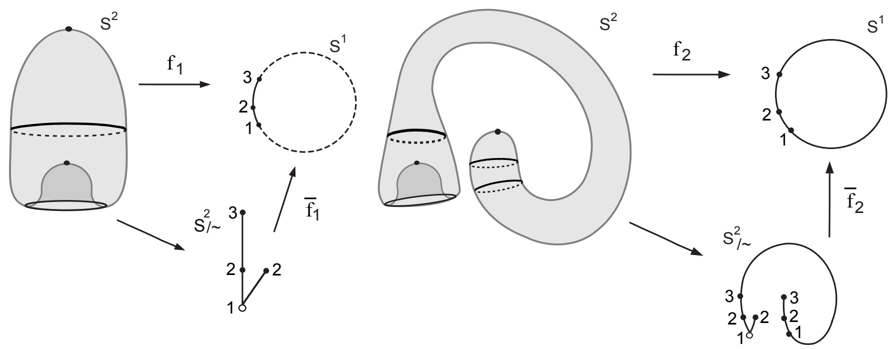

FIGURE 5. MB-Reeb graphs associated to f_1 and f_2.

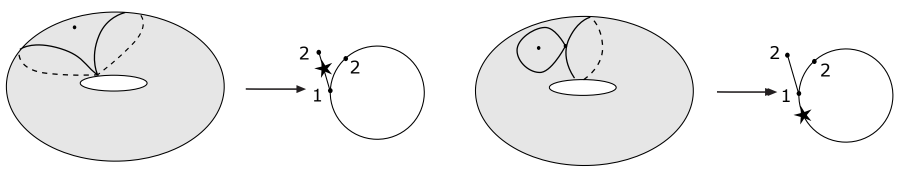

FIGURE 6. Two MB-Reeb graphs associated to distinct simple $\mathcal{MB}$ functions of the torus.

Remark 17. 1. *The MB-Reeb graph was inspired by the invariant used in works [6, 7, 8, 15, 16]. However, the MB-Reeb graph contains some extra information. In fact, it has vertices corresponding to the regular connected components of $f^{-1}(v)$, where v is a critical value; white vertices corresponding to singular circles and the star symbol "$\star$" on the corresponding edge to distinguish the topological type of the saddle curves.*

2. *If $f : M^2 \to S^1$ is not surjective, then f may be regarded as a simple $\mathcal{MB}$ function from M^2 to $\mathbb{R}$ (via stereographic projection) and we can apply the results of [16].*

It is obvious that the labeling of vertices in the MB-Reeb graph is not uniquely determined, since it depends on the chosen orientations and the base points on each S^1. Different choices will produce either a cyclic permutation or an inversion of the labeling in the MB-Reeb graph. This allows us to introduce the following notion of equivalence between MB-Reeb graphs (Definition 19).

Let $f, g : M^2 \to S^1$ be two simple $\mathcal{MB}$ functions. Let Γ_f and Γ_g be their respective MB-Reeb graphs. Consider the induced quotient maps

$$\bar{f} : \Gamma_f \to S^1_f \quad \text{and} \quad \bar{g} : \Gamma_g \to S^1_g,$$

where S^1_f, S^1_g denote S^1 with the graph structure whose vertices are the critical values of f, g respectively.

Definition 18. *An* isomorphism *between two graphs Γ_1 and Γ_2 is a bijection f from $V(\Gamma_1)$ to $V(\Gamma_2)$, where $V(\Gamma_i) = \{vertices\ of\ \Gamma_i\}$, such that two vertices v and w are adjacent in Γ_1 if and only if $f(v)$ and $f(w)$ are adjacent in Γ_2.*

Definition 19. *Let $f, g : M^2 \to S^1$ be two simple $\mathcal{MB}$ functions with Γ_f and Γ_g their respective MB-Reeb graphs. We say that Γ_f* is equivalent to *Γ_g and we denote it by $\Gamma_f \sim \Gamma_g$, if there exist* *graph isomorphisms $j : \Gamma_f \to \Gamma_g$ preserving the assignments of the functions ξ and $l : S^1_f \to S^1_g$, such that the following diagram is commutative:*

$$\begin{array}{ccc} V_f & \xrightarrow{\bar{f}|V_f} & \Delta_f \\ {\scriptstyle j|_{V_f}}\downarrow & & \downarrow{\scriptstyle l|_{\Delta_f}} \\ V_g & \xrightarrow{\bar{g}|V_g} & \Delta_g \end{array}$$

where $V_f = \{$vertices of $\Gamma_f\}$, $V_g = \{$vertices of $\Gamma_g\}$ and Δ_f and Δ_g are their respective discriminant sets.

5. The topological invariance

In this section, we prove that the MB-Reeb graph is a complete topological invariant, up to topological conjugacy.

First, we remark that for simple circle-valued $\mathcal{MB}$ functions we have the following trivial result:

Proposition 20. *If $f, g : M^2 \to S^1$ are two topologically conjugated simple $\mathcal{MB}$ functions, then the singular fibers $s_a(f)$ and $s_{k(a)}(g)$ are homeomorphic.*

Theorem 21. *Let $f, g : M^2 \to S^1$ be two simple $\mathcal{MB}$ functions. If f and g are topologically conjugated, then their respective MB-Reeb graphs are equivalent.*

Proof. Since f and g are topologically conjugated, there exist homeomorphisms $h : M^2 \to M^2$ and $k : S^1 \to S^1$ such that $f = k \circ g \circ h^{-1}$ and h sends singular fibers of g to singular fibers of f. Hence, h induces a graph isomorphism from Γ_f to Γ_g and k induces a graph isomorphism from S^1_f to S^1_g that provide the equivalence requested. □

The next theorem is the converse of Theorem 21.

Theorem 22. *Let f, $g : M^2 \to S^1$ be two simple $\mathcal{MB}$ functions with Γ_f and Γ_g their respective MB-Reeb graphs. If Γ_f and Γ_g are equivalent, then f and g are topologically conjugated.*

Proof. Since $\Gamma_f \sim \Gamma_g$, there exist graph isomorphisms $j : \Gamma_f \to \Gamma_g$ preserving the assignments of the function ξ and $l : S^1_f \to S^1_g$ as in Definition 19. These isomorphisms j and l induce graph isomorphisms $\bar{h} : \Gamma_f \to \Gamma_g$ preserving the assignments of the function ξ, and $\bar{k} : S^1_f \to S^1_g$ which realize the graph isomorphisms j, l respectively and such that $\bar{g} \circ \bar{h} = \bar{k} \circ \bar{f}$. The graph isomorphism $\bar{k}$ induces a homeomorphism $k : S^1 \to S^1$.

Since $k \circ f$ is topologically conjugated to f then by Theorem 21 we have $\Gamma_{k \circ f} \sim \Gamma_f$. Moreover, these graphs are the same because $\bar{k} \circ \bar{f} = \overline{k \circ f}$. In other words the following diagram is commutative:

$$\begin{array}{ccc} \Gamma_g & \xrightarrow{\bar{g}} & S^1_g \\ {\scriptstyle \bar{h}}\uparrow & \nearrow_{\overline{k \circ f}} & \\ \Gamma_f & & \end{array}$$

For simplicity, we simply write f instead of $k \circ f$. By construction $\bar{h}(V_f) = V_g$, but now f and g have the same critical values $v_1, \ldots, v_n \in S^1$. We choose a base point and an orientation in S^1 and assume that $v_1 < v_2 < \ldots < v_n$.

Denote by $\operatorname{arc}(a, b)$ the oriented arc from a to b, where a and b are regular distinct values of f in S^1, and by $\overline{\operatorname{arc}(a, b)}$ its closure. Then, for both a and b we choose $\epsilon_a > 0$ and $\epsilon_b > 0$

such that $\overline{\mathrm{arc}(a-\epsilon_a, a+\epsilon_a)}$ and $\overline{\mathrm{arc}(b-\epsilon_b, b+\epsilon_b)}$ have just regular values of f. Then we define $A := \overline{\mathrm{arc}(a+\epsilon_a, b-\epsilon_b)}$ and $B := \overline{\mathrm{arc}(b+\epsilon_b, a-\epsilon_a)}$ and consider the restriction maps

$$f_a := f|_{f^{-1}(A)} : f^{-1}(A) \to A, \qquad f_b := f|_{f^{-1}(B)} : f^{-1}(B) \to B,$$
$$g_a := g|_{g^{-1}(A)} : g^{-1}(A) \to A, \qquad g_b := g|_{g^{-1}(B)} : g^{-1}(B) \to B,$$

Since f_a, f_b, g_a and g_b are $\mathcal{MB}$ functions with values in $\mathbb{R}$, it follows from [16] that we can consider graphs associated to these restrictions which here we will denote by Γ_{f_a}, Γ_{f_b}, Γ_{g_a} and Γ_{g_b}, respectively. Notice that the graph isomorphism $\bar{h}$ restricted to Γ_{f_a} (resp. Γ_{f_b}) preserves the assignments of ξ for the saddle vertices. Then the function ξ induces an orientation η in the vertices of Γ_{f_a} and Γ_{g_a} (resp. Γ_{f_b} and Γ_{g_b}). By Theorem 3.17 of [16] there exist homeomorphisms $h_a : f^{-1}(A) \to g^{-1}(A)$ and $h_b : f^{-1}(B) \to g^{-1}(B)$ such that $f_a = g_a \circ h_a$ and $f_b = g_b \circ h_b$.

Since the boundary of the sets $f^{-1}(A)$, $f^{-1}(B)$, $g^{-1}(A)$, $g^{-1}(B)$ is formed by a finite number of disjoint closed curves we can assume that the homeomorphisms h_a and h_b when restricted to the boundary preserve orientation. Then, there exist extensions of the homeomorphisms h_a and h_b to $f^{-1}(\mathrm{arc}(a-\epsilon_a, a+\epsilon_a) \cup \mathrm{arc}(b-\epsilon_b, b+\epsilon_b))$ such that they coincide in

$$f^{-1}(\mathrm{arc}(a-\epsilon_a, a+\epsilon_a) \cup \mathrm{arc}(b-\epsilon_b, b+\epsilon_b)) \quad \text{(see [26])}.$$

Define a map $H : M^2 \to M^2$ given by

$$H(x) = \begin{cases} h_a(x), & \text{if } x \in f^{-1}(\mathrm{arc}(a-\epsilon_a, b+\epsilon_b)), \\ h_b(x), & \text{if } x \in f^{-1}(\mathrm{arc}(b-\epsilon_b, a+\epsilon_a)), \end{cases}$$

and if $x \in f^{-1}(\mathrm{arc}(a-\epsilon_a, a+\epsilon_a) \cup \mathrm{arc}(b-\epsilon_b, b+\epsilon_b))$ then $h_a(x) = h_b(x)$, by previous construction. Therefore, H is well-defined. Moreover, H is a homeomorphism that topologically conjugates f and g. □

Example 4. *The two simple $\mathcal{MB}$ functions considered in Example 3 are not topologically conjugated since their respective MB-graphs are not equivalent.*

Remark 23. *In Example 2, if we do not consider the regular vertices in the MB-Reeb graphs associated to f_1 and f_2, then these graphs would become indistinguishable (and therefore equivalent). However, clearly f_1 and f_2 are not topologically conjugated.*

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