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LOCAL EULER OBSTRUCTION, OLD AND NEW, III

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Celebrating 30 years of International Workshop on Real and Complex Singularities

Abstract. The first part of the article is a survey of papers originating from a joint course given by the first and third named authors in São José do Rio Preto. That is an historical journey from Athens to São Carlos, going from the discovery of the Plato polyhedra to characteristic classes of a singular variety, by M.-H. Schwartz and R. MacPherson, from the Euler formula and Poincaré-Hopf Theorem to the study of local Euler obstruction.

In 1965, Marie-Hélène Schwartz defined characteristic classes for singular complex varieties, as cohomology classes of an ambient manifold, with support on the singular varieties. In 1974, Robert MacPherson showed existence of homology characteristic classes for singular varieties, proving a Deligne and Grothendieck conjecture. One of the main ingredients of his definition is the local Euler obstruction, defined by differential forms. An equivalent definition of the local Euler obstruction, using vector fields, has been given by Jean-Paul Brasselet and Marie-Hélène Schwartz in their proof of the coincidence of two previous definitions of characteristic classes via Alexander isomorphism.

In 1998, the first author published a survey, *Local Euler obstruction, old and new* followed in 2010 by a survey by the two first authors *Local Euler obstruction, old and new, II.* The notion of local Euler obstruction was revealed to be very useful to describe the local complexity of stratified singular varieties and developed in many areas, study of foliations, determinantal varieties. Nowadays, a full book would be necessary to write a complete survey on the subject. Many São Carlense researchers published various papers related to local Euler obstruction. Celebrating 30 years of International Workshops on Real and Complex Singularities in São Carlos is the occasion to “take stock” of the successes they achieved in this area alone or with coauthors. That is the second part of the article.

Part 1. With Euler from Athens to São Carlos

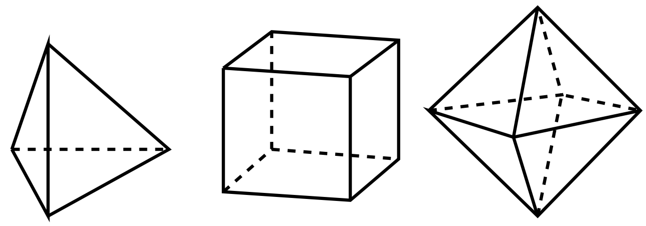
1. The Greek period

Since the beginning of the history of mathematics, mathematicians have sought to classify characteristic surfaces by attributing philosophical and esoteric properties to them.

The story begins with Pythagoras of Samos ($\sim 570 - 495$ B.C.) who knew three of the regular convex polyhedra: Tetrahedron (4 faces), hexahedron (cube, 6 faces) and octahedron (8 faces).

A regular convex polyhedron is a polyhedron whose faces are all identical (that is regular) and in such a way that the segment linking any two points of the polyhedron is completely included in the polyhedron (that is convex).

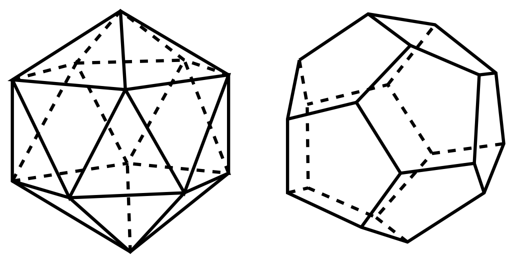
We thank the referees for their fruitful reports that helped us to improve the text making it clear and uniform. We thank all collaborators and members of the obstructionist club of São Carlos for their help and support. The universities and institutions provided supports by inviting the authors, respectively Aix-Marseille University and I2M CNRS, France, ICMC-USP at São Carlos and IBILCE-UNESP at São José do Rio Preto, Brazil. The second author benefited from the grant FAPESP 2019/21181-0.



Tetrahedron V=4, E=6, F=4 Hexahedron V=8, E=12, F=6 Octahedron V=6, E=12, F=8

FIGURE 1. V = number of vertices, E = number of edges, F = number of faces

The following two: icosahedron (20 faces) and dodecahedron (12 faces) have been discovered by Theaetetus of Athens ($\sim$ 415 - 365 B.C.) who gave a mathematical description of all five.



Icosahedron V=12, E=30, F=20 Dodecahedron V=20, E=30, F=12

FIGURE 2.

They have been popularized by Plato ($\sim$ 428 - 348 B.C.) in his philosophical dialogue "Timaeus".

Euclid of Alexandria ($\sim$ 300 B.C.) completely mathematically describes the five Platonic solids in the "Elements".

2. Descartes - Euler

In this section, we provide a short history of the contributions of Maurolico (1494 - 1575), Descartes (1596 - 1650) and Euler (1707 - 1783) (see [Add] and [Ri]).

Francesco Maurolico (1494 - 1575) was an Italian priest, who lived mainly in Messine, Sicily and studied Mathematics and Astronomy. In an unpublished manuscript *Compaginationes solidorum regularium* (1537), Maurolico stated the so-called "Euler's formula" for Platonic solids (see [Add, Pages 291 and 295]):

Formula 2.1 (Maurolico, December 26, 1537)**.** *"Item manifestum est in unoquoque regularium solidorum, numerum basium coniunctum cum numero cacuminum conflare numerum, qui binario excedit numerum laterum."*
In the same way it is evident that, in each regular solid, the number of faces added to that of the vertices exceeds by two the number of edges, i.e. consider a Platonic solid with V vertices, E edges and F faces, then

$$V - E + F = 2. \tag{2.1}$$

In his manuscript *De solidorum elementis,* $\sim$ 1622, at age around 25, Descartes proves the following result:

Theorem 2.2 (Descartes)**.** *The sum of the angles of all faces of a convex polyhedron is equal to* $2(V-2)\pi$ *where* V *is the number of vertices.*

Descartes did not publish the result at that time, and he died in Stockholm on February, 11, 1650.

In 1672, in Paris, Leibniz copied the manuscript with the idea of publishing it. But he died in 1716 without publishing the (copy of the) Descartes' manuscript.

On the one hand, after being in the possession of several people, the original documents were eventually lost.

The legend says that, trying to classify convex polyhedra, Leonhard Euler observed that formula (2.1) was verified for any convex polyhedron. Euler mentioned his discovery in a letter to Christian Goldbach (November 14, 1750). Without knowing either Maurolico's manuscript or Descartes' manuscript, both unpublished, Euler writes "It astonishes me that these general properties of stereometry have not, as far as I know, been noticed by anyone else". He later (1753) published two papers on the formula, but unfortunately his argument was not correct.

Adrien-Marie Legendre (1752-1833) gave the first proof of Euler's formula (1794) using a projection of the polyhedron on a sphere.

In the year 1811, Augustin-Louis Cauchy (1789 - 1857) gave the first combinatorial proof of the formula (see §3).

The story comes back to Descartes: In the year 1883, Foucher de Careil, France's Ambassador to Austria-Hungary came to Hanover. He discovered among Leibniz's documents the (copy of) Descartes' manuscript.

Ernest de Fauque de Jonquières published, in 1890 a "Note aux CRAS" in which he claims that Descartes discovered Euler's "formula". In fact, for convex polyedra, Descartes' Theorem 2.2 is equivalent to the formula (2.1) (see for instance [BT1]).

So, the question of knowing who was the first to discover the formula has no clear answer!

3. Cauchy's proof of the Euler formula.

Cauchy begins with a stereographic projection of the convex polyhedron on a plane P removing a face $\mathcal{F}$.

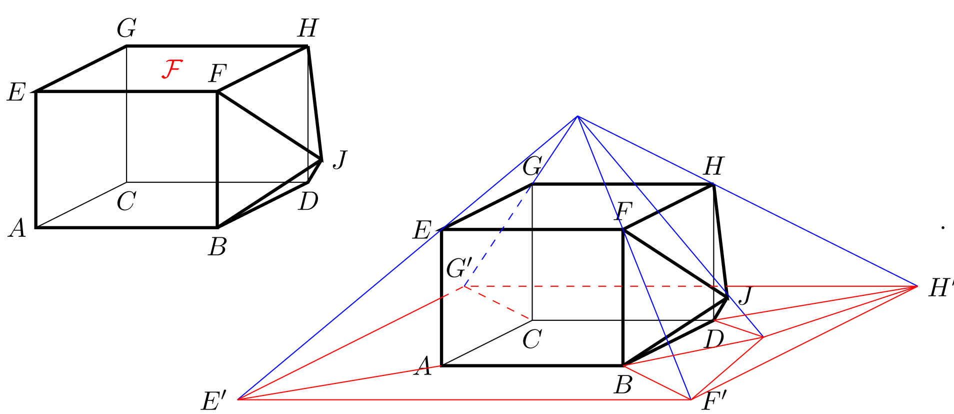


Figure 3.

Cauchy considers a triangulation K of the polygon obtained in the plane and shows that the alternating sum $V - E + F$ is preserved by the subdivision.

Then he extends the "hole" (in blue) by means of two operations that we will describe. This is the stage that is controversial and that we will replace.

Consider a convex triangulated polygon with a hole B that we will extend through two operations, the "Cauchy operations".

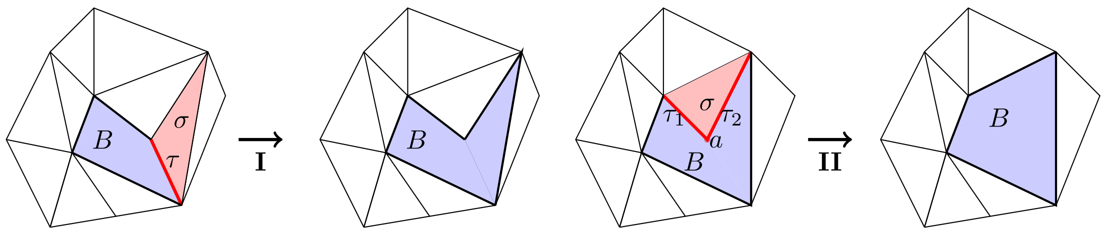


FIGURE 4. Extensions of the hole B using "Cauchy's operations I and II".

Cauchy's operation I: Extending the hole B with a simplex σ having only one edge τ in common with B does not change the alternating sum $V - E + F$.

Cauchy's operation II: Extending the hole B with a simplex σ having two edges τ_1 and τ_2 and a vertex a in common with B does not change the alternating sum $V - E + F$.

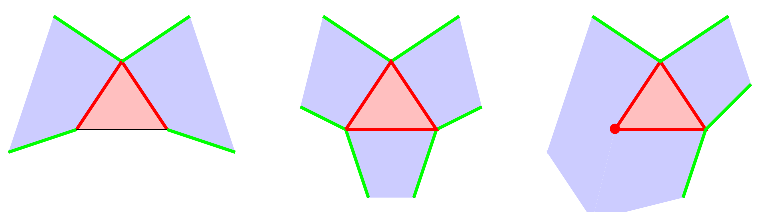
According to Cauchy, at the end of the extension process, only a triangle is left for which: $V - E + F = 3 - 3 + 1 = +1$. Adding the triangle removed at the beginning, we obtain "Euler's formula":

$$V - E + F = +2.$$

4. A controversy

Imre Lakatos (1976) [Lak] makes criticisms in his book "Proofs and Refutations: The Logic of Mathematical Discovery" saying that the Cauchy method can give unsolvable situations.

Elon Lima (Dec. 1985) [Li] criticized Cauchy's proof, saying that there are four possibilities not mentioned by Cauchy. The first three ones (figure 5) do not preserve the alternating sum, the last one preserves the alternating sum.



(a) (b) (c)

Hole B wih boundary: Triangle σ, extension of the hole.

FIGURE 5. Situations which were not mentioned by Cauchy.

According to Elon Lima, in order to avoid these three cases, it is necessary to use deep results of algebraic topology.

5. Elementary proof of Euler's formula using only the Cauchy's method.

In this section we give a sketch of the elementary proof of Euler's formula using Cauchy's method given in [BT1], by Brasselet and Thủy.

Recall the first step of Cauchy's proof : Cauchy begins with a stereographic projection of the convex polyhedron P removing a face $\mathcal{F}$ (Figure 3).

Then Cauchy considers a triangulation K of the polygon in the plane and shows that the alternating sum $V - E + F$ is preserved.

We stop here Cauchy's initial proof and show a theorem that allows us to finish the proof, using only the two Cauchy operations.

Theorem 5.1. [BT1] *Let K be a (finite) triangulated polygon in $\mathbb{R}^2$, homeomorphic to a disc D, with (possible) identification of the simplices in the edge $K_0 = \partial K$, then*

$$V - E + F = V_0 - E_0 + 1,$$

where V_0 and E_0 denote respectively the number of vertices and edges of the boundary K_0 of the polygon, taking into account the identifications.

Example 5.2. *In the following example, one has $V_0 = 4$ vertices and $E_0 = 5$ edges,*

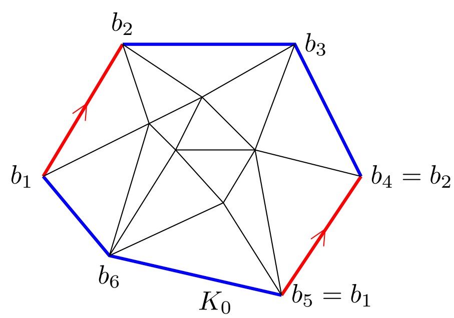


Figure 6. Here, $V - E + F = V_0 - E_0 + 1 = 4 - 5 + 1 = 0$.

Example 5.3. *In the case of the polygon $E'G'H'F'$ of Figure 3, there is no identification for the vertices and edges on the boundary of the polygon, then one has $V_0 = E_0$. Taking into account the removed face $\mathcal{F}$, one has, for the polyhedron P the value $V - E + F = +2$. That is "Euler's formula".*

Sketch of the Proof of Theorem 5.1. (see [BT1]).

The first path is to take a triangle (here with vertices u_1, u_2, u_3), situated in the interior of the polygon, and to lift the polygon, linearly, keeping the boundary at the level of the plane P and the triangle at the level of a parallel plane P_0. We obtain a pyramid (Figure 8 left) that we cut by horizontal planes passing by the vertices x_i (liftings of the vertices in K).

The next step is to triangulate the pyramid in a compatible way with the cuttings of the planes (Figure 8 right).

Removing the face (2-simplex) (u_1, u_2, u_3). There remain then V vertices, E edges and $(F-1)$ faces.

Going down from the plane P_i to the plane P_{i+1}, and using Cauchy's operations I and II, the sum $V - E + (F - 1)$ does not change. We also observe that the intersection of the pyramid

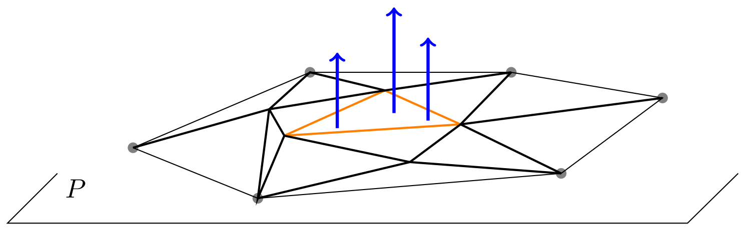


FIGURE 7. Lifting the polygon.

with each plane is homeomorphic to a circle (with the same number of vertices and edges) and in between two planes there is a band with the same number of edges and faces.

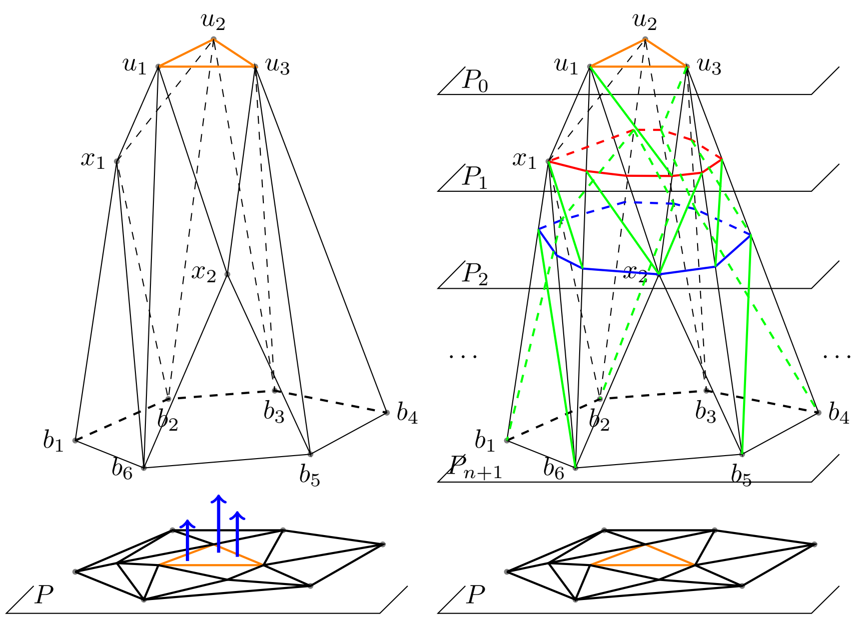


FIGURE 8. The pyramid Π (with horizontal planes).

Coming to the last plane P_{n+1}, and taking into account the identifications on the boundary, one has $V - E + (F - 1) = V_0 - E_0$, that is

$$V - E + F = V_0 - E_0 + 1.$$

□

6. Applications of the Theorem to classical examples.

For a full description of the following examples, see [BT1].

Considering a triangulation of any (smooth) surface orientable or not, one can "cut" the surface according to curves in order to obtain a planar representation of the surface.

By taking the precaution of sub-triangulating the surface in a way compatible with the cut-off curves, one then obtains a triangulation of the planar representation.

Applying Theorem 5.1 to the planar representation provides the value of the Euler-Poincaré characteristic $V - E + F$ of the surface.

Example 6.1. With appropriate cut-off curves along (demi-)meridians, the sphere can be also viewed in that way.

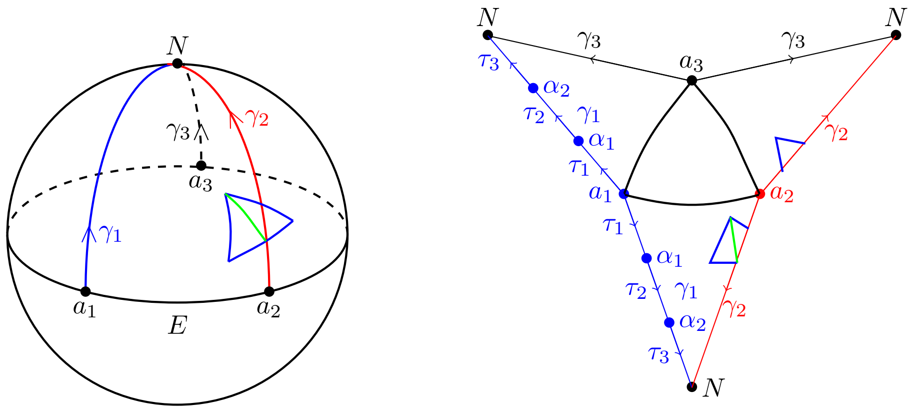


FIGURE 9. Planar representation of the sphere.

The vertex N is common to all curves γ_i, so one has already $+1$ for V_0.

For each curve γ_i one has the same number of vertices and adges, then $V^{\gamma_i} - E^{\gamma_i} = 0$ for $i = 1, 2, 3$.

So, one has $V_0 - E_0 = +1$ and, by Theorem 5.1, for the triangulation K of the sphere:

$$V - E + F = +2.$$

Example 6.2 (The torus)**.** *Given a triangulation K of the torus, we consider a meridian M and a parallel P crossed at a point A. We consider a sub-triangulation K' of K such that for each simplex σ the intersections $\sigma \cap M$ and $\sigma \cap P$ are simplices of K'. The sum $V - E + F$ does not change.*

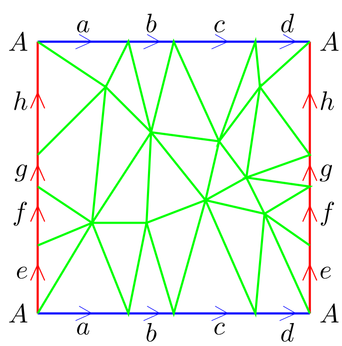


FIGURE 10. Planar representation of the torus.

By Theorem 5.1, one has:

$$V - E + F = V_0 - E_0 + 1 = 0.$$

Example 6.3 (The projective space)**.** *The projective space $\mathbb{P}^2$ is the set of all lines of the Euclidean space $\mathbb{R}^3$ passing through the origin. One way to represent the projective space $\mathbb{P}^2$ is to consider in $\mathbb{R}^3$ the quotient of the sphere $\mathbb{S}^2$ of radius 1 by symmetry with respect to the center of the sphere.*

Let K be a triangulation of the sphere symmetrical relative to the center of the sphere. One considers a sub-triangulation K' compatible with the equator (see [BT1]).

The straight lines of the plane $0xy$ meet the north hemisphere at two diametrically opposite points, which we have to identify. We obtain a representation of the projective space $\mathbb{P}^2$ as the north hemisphere with identification of diametrically opposite points on its edge.

With the projection in the plane of the equator, we obtain a triangulated disk with identifications on the edge. One has $V_0 - E_0 = 0$. Then one has: $V - E + F = +1$.

Example 6.4 (The Klein bottle)**.** *In the same way as the torus, given a triangulation K of the Klein bottle, we consider a "meridian" M and a "parallel" P crossed at a point A. We consider a sub-triangulation K' of K such that for each simplex σ the intersections $\sigma \cap M$ and $\sigma \cap P$ are simplices of K'. Cutting along M and P provides a planar representation of the Klein bottle (see [BT1]).*

On the boundary K_0, one has $V_0 - E_0 = -1$. For the Klein bottle $\chi(X) = V - E + F = 0$.

Example 6.5 (The pinched torus)**.** *Given a triangulation K of the pinched torus, and a "parallel" passing through the pinched point A, on the boundary K_0, one has $V_0 - E_0 = 0$ (see [BT1]).*

For the pinched torus $\chi(X) = V - E + F = +1$.

Part 2. Where the local Euler obstruction arrives

7. Chern classes for singular varieties

The notion of the Euler characteristic $\chi(X) = V - E + F$ for a (compact) surface has been extended by Poincaré as $\chi(X) = \sum_{i=0}^{n} (-1)^i k_i$ for any compact polyhedron X where k_i is the number of i-simplices of any triangulation K of X.

The Poincaré-Hopf Theorem says that the Euler-Poincaré characteristic is the obstruction to the construction of a vector field tangent to a smooth compact manifold (without boundary), see [BT2] for a simple proof of the Poincaré-Hopf Theorem.

Stiefel-Whitney classes (for the real case) and Chern classes (for the complex case) are a measure of the obstruction to the construction of r-frames tangent to smooth (real - complex) manifolds. For a long time, there was no equivalent of the Poincaré-Hopf Theorem or characteristic classes for singular varieties.

In 1965, Marie-Hélène Schwartz showed that considering the so called "radial" stratified vector fields, obtained by a method of radial extension she defined (see [B2]), it is possible to recover a Poincaré-Hopf Theorem for singular stratified varieties. She defined classes in cohomology as the obstruction to the construction of radial stratified r-frames tangent to a complex analytic variety X embedded in a smooth complex manifold M. These are called Schwartz classes, and denoted by $c^S(X) \in H^*(M, M \setminus X)$.

The next year, in his seminar, Grothendieck conjectured the existence of (homology) Chern classes for complex algebraic varieties. The conjecture was taken up by Deligne and called the "Deligne-Grothendieck conjecture". In 1974 MacPherson proved the existence and the uniqueness of Chern classes for possibly singular compact complex algebraic varieties, answering the conjecture. Characteristic classes are the subject of the course given by Brasselet at IMPA [B2] and an extensive survey in the Handbook of Geometry and Topology of Singularities [B].

One defines the functor $\mathbf{F}$ such that for any singular algebraic complex variety X, then $\mathbf{F}(X)$ is the set of constructible functions $\alpha : X \to \mathbb{Z}$.

Theorem 7.1 (MacPherson)**.** *There exists a natural transformation from the functor* $\mathbf{F}$ *to homology which, on a nonsingular variety* X*, assigns to the constant function* $\mathbf{1}_X$ *the Poincaré dual of the total Chern class of* X*.*

In other words, the theorem asserts that we can assign to any constructible function $\alpha : X \to \mathbb{Z}$ on a compact complex algebraic variety X an element $c_*(\alpha)$ of $H_*(X)$ satisfying the following three conditions:

(1) $f_* c_*(\alpha) = c_* f_*(\alpha)$,
(2) $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta)$,
(3) $c_*(1_X) = c(X) \cap [X]$ if X a smooth variety.

Here, $[X]$ is the fundamental class of X and the pushforward f_* is defined on characteristic functions $\mathbf{1}_A$ for $A \subset X$ by:

$$f_*(\mathbf{1}_A)(y) = \chi(f^{-1}(y) \cap A), \qquad y \in Y$$

for a morphism $f : X \to Y$, and linearly extended to elements of $\mathbf{F}(X)$.

The compactness restriction may be dropped with minor modifications of the proof if all maps are taken to be proper and Borel–Moore homology (homology with locally finite supports) is used.

Brasselet and Schwartz proved in [BS], using Alexander's duality isomorphism

$$H^*(M, M \setminus X) \to H_*(X)$$

that the Schwartz classes $c^S(X)$ coincide with MacPherson's classes $c_*(1_X)$, and therefore these classes are called the Chern–Schwartz–MacPherson classes, denoted by $c_{SM}(X) \in H_*(X)$.

Let us now introduce some objects in order to define the Schwartz–MacPherson class.

Suppose X is a representative of a d-dimensional analytic germ $(X, 0) \subset (\mathbb{C}^n, 0)$, such that $X \subset U$, where U is an open subset of $\mathbb{C}^n$. In the case of singular analytic varieties, one may consider Whitney stratifications whose strata are denoted by $\{V_\alpha\}$.

Definition 7.2. *Let us denote by* $TU|_X$ *the restriction to* X *of the tangent bundle of* U*. A stratified vector field* v *on* X *means a continuous section of* $TU|_X$ *such that if* $x \in V_\alpha \cap X$ *then* $v(x) \in T_x(V_\alpha)$*.*

Let $G(d, n)$ denote the Grassmannian of complex d-planes in $\mathbb{C}^n$. On the regular part X_{reg} of X the Gauss map $\phi : X_{\text{reg}} \to U \times G(d, n)$ is well defined by $\phi(x) = (x, T_x(X_{\text{reg}}))$.

Definition 7.3. *The Nash transformation (or Nash blow up) of* X *denoted by* $\widetilde{X}$ *is the closure of the image* $\mathrm{Im}(\phi)$ *in* $U \times G(d, n)$*. It is a (usually singular) complex analytic space endowed with an analytic projection map* $\nu : \widetilde{X} \to X$ *which is biholomorphic away from* $\nu^{-1}(\mathrm{Sing}(X))$*.*

The fibre of the tautological bundle $\mathcal{T}$ over $G(d, n)$, at point $P \in G(d, n)$, is the set of vectors v in the d-plane P. We still denote by $\mathcal{T}$ the corresponding trivial extension bundle over $U \times G(d, n)$. Let $\widetilde{T}$ be the restriction of $\mathcal{T}$ to $\widetilde{X}$, with projection map π. The bundle $\widetilde{T}$ on $\widetilde{X}$ is called *the Nash bundle* of X.

An element of $\widetilde{T}$ is written (x, P, v) where $x \in U$, P is a d-plane in $\mathbb{C}^n$ based at x and v is a vector in P. We have the following diagram:

$$\begin{array}{ccc} \widetilde{T} & \hookrightarrow & \mathcal{T} \\ \pi\downarrow & & \downarrow \\ \widetilde{X} & \hookrightarrow & U \times G(d,n) \\ \nu\downarrow & & \downarrow \\ X & \hookrightarrow & U. \end{array}$$

MacPherson defined the Mather classes, by the formula

$$c_{\mathrm{Ma}}(X) = \nu_*(c(\widetilde{T}) \cap [\widetilde{X}]),$$

where $[\widetilde{X}]$ denotes the fundamental (orientation) homology class of $\widetilde{X}$.

An algebraic cycle on a variety X is a finite formal linear sum $\sum n_i[X_i]$, where the n_i are integers and the X_i are irreducible subvarieties of X. We may define c_{Ma} on any algebraic cycle of X by

$$c_{\mathrm{Ma}}(\sum n_i[X_i]) = \sum n_i c_{\mathrm{Ma}}(X_i),$$

where by abuse of notation we denote $\mathrm{incl}_{i*}c_{\mathrm{Ma}}(X_i)$ by $c_{\mathrm{Ma}}(X_i)$.

An important object introduced by MacPherson in his work is the local Euler obstruction. This invariant was deeply investigated by many authors; for an overview about it see [B]. Brasselet and Schwartz presented in [BS] an alternative definition for the local Euler obstruction using stratified vector fields.

By Whitney condition (a) one has the following:

Lemma 7.4 (See [BS])**.** *Every stratified vector field v nowhere zero on a subset $A \subset X$ has a canonical lifting as a nowhere zero section $\tilde{v}$ of the Nash bundle $\widetilde{T}$ over $\nu^{-1}(A) \subset \widetilde{X}$).*

Now consider a stratified radial vector field $v(x)$ in a neighbourhood of $\{0\}$ in X, *i.e.* there is ε_0 such that for every $0 < \varepsilon \leq \varepsilon_0$, $v(x)$ is pointing outwards from the ball B_ε over the boundary $S_\varepsilon := \partial B_\varepsilon$.

The following interpretation of the local Euler obstruction has been given by Brasselet and Schwartz in [BS].

Definition 7.5. *Let v be a stratified vector field on $X \cap S_\varepsilon$ pointing outwards from a small ball B_ε centered at $\{0\}$ and $\tilde{v}$ the lifting of v on $\nu^{-1}(X \cap S_\varepsilon)$ to a section of the Nash bundle $\widetilde{T}$.*

The local Euler obstruction (or simply the Euler obstruction), denoted by $\mathrm{Eu}_X(0)$*, is defined to be the obstruction to extending $\tilde{v}$ as a nowhere zero section of $\widetilde{T}$ over $\nu^{-1}(X \cap B_\varepsilon)$.*

More precisely, let

$$\mathcal{O}(\tilde{v}) \in Z^{2d}(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon); \mathbb{Z})$$

be the obstruction cocycle to extending $\tilde{v}$ as a nowhere zero section of $\widetilde{T}$ inside $\nu^{-1}(X \cap B_\varepsilon)$. The Euler obstruction $\mathrm{Eu}_X(0)$ is defined as the evaluation of the class $[\mathcal{O}(\tilde{v})]$ on the fundamental class of the topological pair $(\nu^{-1}(X \cap B_\varepsilon), \nu^{-1}(X \cap S_\varepsilon))$. The Euler obstruction is an integer and is independent of all choices.

Let v_α be a vector field tangent to a stratum V_α with an isolated singularity at the point $a \in V_\alpha$ with index $I(v_\alpha, a)$ also denoted by $I(v_\alpha, a; V_\alpha)$. By the "radial extension" method (see [BS]), M.-H. Schwartz defined in a neighbourhood of a in the manifold M the vector field v_{rad} with an isolated singularity at a with index $I(v_{\mathrm{rad}}, a; M) = I(v_\alpha, a; V_\alpha)$.

Theorem 7.6 (Proportionality Theorem)**.** [BS] *Take a vector field v_α tangent to the stratum V_α with an isolated singularity at the point $a \in V_\alpha$ with index $I(v_\alpha, a)$. Then, for the stratified vector field v_{rad} obtained by radial extension of the vector field v_α, one has:*

$$I(v_{\mathrm{rad}}, a; M) = Eu_X(a) \cdot I(v_\alpha, a : V_\alpha).$$

Using the local Euler obstruction, MacPherson defined a map T from the algebraic cycles on X to the constructible functions on X by

$$T(\sum n_i X_i)(x) = \sum n_i \, \mathrm{Eu}_{X_i}(x).$$

MacPherson proved the fundamental Theorem (Theorem 2 of [MP2]):

Theorem 7.7. *T is a well-defined isomorphism from the group of algebraic cycles to the group of constructible functions and $c_{\mathrm{Ma}}T^{-1}(1_X)$ satisfies the requirements for c_* in the Deligne–Grothendieck conjecture.*

The computation of the local Euler obstruction is not easy. In order to simplify the computation many authors proposed formulae to compute this invariant. With the aid of Gonzalez–Sprinberg's purely algebraic interpretation of the local Euler obstruction ([Gon]), Lê and Teissier in [LT] showed that the local Euler obstruction is an alternating sum of the multiplicities of the local polar varieties. This important formula for computing the local Euler obstruction will be used in this paper.

For a sufficiently general flag $\mathcal{D}$ in $\mathbb{C}^m$

$$D_d \subset \cdots \subset D_0 = \mathbb{C}^m, \tag{$\mathcal{D}$}$$

where $\operatorname{codim}_{\mathbb{C}} D_i = i$, the k-th Schubert variety

$$c_k(\mathcal{D}) = \{P \in G(n,m) : \dim(P \cap D_{n-k+1}) \geq k\}$$

is well defined and its (complex) codimension in $G(n,m)$ is k.

The k-th polar variety of X is defined by $P_k(\mathcal{D}) = \nu(\widetilde{\gamma}^{-1}(c_k(\mathcal{D})))$, where $\widetilde{\gamma}$ is the Gauss map $\widetilde{\gamma} : \tilde{X} \to G(n,m)$.

Theorem 7.8. [LT] *The local Euler obstruction of X at a is equal to*

$$\mathrm{Eu}_X(a) = \sum_{i=0}^{n-1} (-1)^{n-1-i} m_a(P_{n-1-i}(\mathcal{D})), \tag{7.8}$$

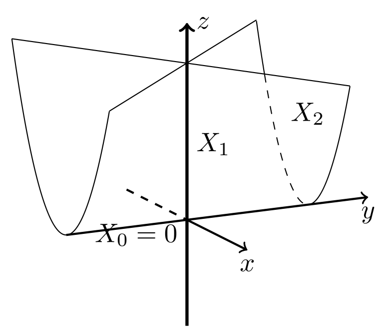
where $m_a(Q)$ is the multiplicity of a variety Q at the point a.

Example 7.9. *The Whitney umbrella.*

An example of application of the formula is given by the surface X in $\mathbb{C}^3$ whose equation is

$$x^2 - y^2 z = 0$$

.



The strata are : the origin $X_0 = \{0\}$, the axis $X_1 = \{x = y = 0\}$ minus the point $\{0\}$ and the complement $X_2 = X_{\mathrm{reg}}$.

One has $\mathrm{Eu}_X(x) = 1$ *if* $x \in X_2 = X_{\mathrm{reg}}$, $\quad \mathrm{Eu}_X(a) = 2$ *if* $a \in X_1$ $\quad$ *and* $\mathrm{Eu}_X(0) = 1$.

An interesting formula for the local Euler obstruction due to Brasselet, Lê and Seade, [BLS] shows that the Euler obstruction, as a constructible function, satisfies the Euler condition relative to generic linear forms. More precisely:

Theorem 7.10. *Let X be an equidimensional complex analytic variety in $\mathbb{C}^m$.*
Let V_α, $\alpha = 1, \ldots, d$, be the (connected) strata of a Whitney stratification containing 0 in their closure.
For every sufficiently general linear form ℓ on $\mathbb{C}^m$, there is ε_0 such that, for all ε, $0 < \varepsilon < \varepsilon_0$ and sufficiently small $t_0 \neq 0$, then :

$$Eu_X(0) = \sum_{\alpha=1}^{d} \chi(V_\alpha \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(t_0))\, Eu_X(V_\alpha),$$

where $\mathbb{B}_\varepsilon$ is the ball of radius ε centered at 0 and $Eu_X(V_\alpha)$ is the value of the local Euler obstruction of X at every point of V_α.

Remark 7.11 (Remark on the notation of Local Euler Obstruction)**.** *The notation* $\mathrm{Eu}_X(x)$ *used in this section is not the original one (*[MP2, BS]*), which is* $\mathrm{Eu}_x(X)$*. The present notation emphasizes the fact that the local Euler obstruction is a (constructible) function of the point $x \in X$. The original notation is more adapted if the point x is fixed and we consider the strata V_α containing x in their closure, obtaining a function* $\mathrm{Eu}_x(\overline{V_\alpha})$ *of the strata. In the following, according to the situation, we will use one or the other of both notations, without risk of error. Another possible notation would be* $\mathrm{Eu}(X, x)$*.*

8. Computations of the Local Euler Obstruction

There are many papers dealing with the computation of the local Euler obstruction in different situations. In the following we review the results obtained by Brazilians, in particular Sãocarlense, researchers and collaborators.

8.1. **Stable types.** Callejas-Bedregal, Saia and Tomazella in [CST] compute the local polar multiplicities of a germ at zero of an analytic variety Y in $\mathbb{C}^p$, which is the image by a finite morphism $f : Z \to Y$, of a d-dimensional isolated complete intersection singularity Z in $\mathbb{C}^n$. They compute the local Euler obstruction of Y at zero in the case that it is reduced. For this they apply the Lê -Teissier formula 7.8.

In [PRS], Pérez, Rizziolli and Saia determine a minimal set of invariants whose constancy guarantees the Whitney equisingularity of families of finitely determined holomorphic map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^3, 0)$, with $n > 3$. As an application, they get explicit algebraic formulae to calculate the local Euler obstruction of the stable types that appear in the singular set $\Sigma(f)$ and also in the discriminant $\Delta(f) = f(\Sigma(f))$ of corank one map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^3, 0)$, with $n > 3$.

Such a minimization was obtained by Pérez, Levcovitz and Saia in [PLS] in the case of a one parameter deformation of corank one finitely determined holomorphic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. The authors describe how the source and target invariants are related and reduce the number using these relations. They show an algebraic formula for the local Euler obstruction in terms of the polar multiplicities and show that the Euler obstruction is an invariant for the Whitney equisingularity.

We also mention the work of Pérez and Saia [PS], where the main goal is to show how to compute the local Euler obstruction of the stable types which appear in a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. The authors apply the Lê -Teissier formula 7.8 to compute the local Euler obstruction of the stable types using the polar multiplicities. In [RS], Rizziolli and Saia consider corank 1, quasi-homogeneous and finitely determined map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, with $n \geq 3$. They obtain formulae for the polar multiplicities defined on some stable types of f in terms of the weights and degrees of f. Examples of such stable types are the discriminant $\Delta(f)$ and images by f of the subvarieties of $\Sigma(f)$ defined by Morin [Mo].

8.2. **Toric and multitoric varieties.** An important class of varieties is the class of toric varieties. In the case where X is a toric surface, Gonzalez-Sprinberg showed in [Gon] that the Euler obstruction of X at 0 depends only on the minimum embedded dimension of this surface.

Theorem 8.1. [Gon] *Let $\sigma \subset \mathbb{R}^2$ be the cone generated by the vectors $v_1 = e_2$ and $v_2 = pe_1 - qe_2$, where $0 < q < p$ and p, q are coprimes. Consider X_σ the toric surface associated to σ and assume that the minimum embedded dimension of X_σ is k, that is, suppose that*

$$\frac{p}{p-q} = [[a_2, a_3, \dots, a_{k-1}]],$$

where the integers $a_2, \dots, a_{k-1}$ satisfy $a_i \geq 2$, for $i = 2, \dots, k-1$, then $\mathrm{Eu}_{X_\sigma}(0) = 3 - k$.

More recently, Matsui and Takeuchi, using Newton's polyhedra, generalized the Gonzalez-Sprinberg result, presenting in [MT] a formula for Euler's obstruction of a n-dimensional toric variety X.

In [BGS] Barbosa, Grulha and Saia show how to obtain the minimal Whitney stratification of the discriminant of finitely determined map germs from $(\mathbb{C}^{n+p}, 0)$ to $(\mathbb{C}^p, 0)$, of corank one if $n < p$, and with only A_k singularities if $n \geq 0$. The authors apply the theory developed by Gaffney which shows how to define a Whitney stratification of discriminants of any finitely determined holomorphic map germ in the nice dimensions of Mather, or in its boundary. For the pairs cited above they show that both stratifications coincide. The authors also compute the local Euler obstruction at 0 for a class of discriminants of finitely determined map germs from $\mathbb{C}^{n+p}$ to $\mathbb{C}^p$ with $n \geq 0$ and with only A_k singularities.

In [DG] Dalbelo and Grulha introduce the notion of multitoric varieties:

Definition 8.2. We will say that a n-dimensional variety $Y \subset \mathbb{C}^k$ is a *multitoric variety* if there is an action $\varphi : \mathbb{T}^n \times \mathbb{C}^k \to \mathbb{C}^k$ of $\mathbb{T}^n$ in $\mathbb{C}^k$ such that φ gives each irreducible component of Y a structure of n-dimensional toric variety.

Dalbelo and Grulha provide a formula for the Euler obstruction of multitoric surfaces. As applications, they compute the Euler obstruction for some families of determinantal surfaces and give some remarks about Milnor number on toric surfaces.

In particular, one has the following results.

Theorem 8.3. *Let $Y \subset \mathbb{C}^k$ be a multitoric surface, $Y = Y_1 \cup \dots \cup Y_m \cup Y_{m+1} \cup \dots \cup Y_{m+s}$, where $Y_{m+1}, \dots, Y_{m+s}$ are the irreducible components of Y with singularity isolated at the origin. Then*

$$\mathrm{Eu}_Y(0) = m + 3s - m_1 - \dots - m_s,$$

where m_i is the smallest dimension of each component Y_{m+i} with singularity.

As an application of the previous formula, we can highlight the next three theorems. Analyzing that the families of determinant surfaces below are families of multitoric surfaces, the authors proved the following formulae.

Theorem 8.4. *Let $Y \subset \mathbb{C}^k$ be the determinant surface given by the minors 2×2 of the matrix*

$$\begin{pmatrix} z_1 & z_2 & \cdots & z_{k-3} & z_{k-2}^b & z_{k-2}^{b-1} z_{k-1}^c \\ z_2^a & z_3 & \cdots & z_{k-2} & z_{k-1} & z_k \end{pmatrix},$$

where a, b, c are positive integers with $b \geq 2$. Then Y is a bitoric surface (multitoric surface Y with two irreducible components) and $\mathrm{Eu}_Y(0) = 4 - k$.

Theorem 8.5. *Let $Y \subset \mathbb{C}^k$ be the determinant surface given by the minors 2×2 of the matrix*

$$C = \begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_{k-3} & z_{k-2}^c & z_{k-2}^{c-1} z_{k-1}^d \\ z_2^a z_3^{b-1} & z_3^b & z_4 & \cdots & z_{k-2} & z_{k-1} & z_k \end{pmatrix},$$

where a, b, c, and d are positive integers with $b, c \geq 2$. Then Y is a multitoric surface and $\mathrm{Eu}_Y(0) = 5 - k$.

Theorem 8.6. *Let $Y \subset \mathbb{C}^{k+1}$ be the determinant surface given by the minors 2×2 of the matrix*

$$\begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_{k-1} & z_k^b z_{k+1} \\ z_2^a & z_3 & z_4 & \cdots & z_k & z_{k+1}^2 \end{pmatrix},$$

where a, b, are positive integers. Then Y is a multitoric surface and $\mathrm{Eu}_Y(0) = 5 - 2k$.

8.3. **Determinantal varieties.** The definition of generic determinantal variety is the following:

Definition 8.7. *Let $n, k, s \in \mathbb{Z}$, $n \geq 1$, $k \geq 0$ and $\mathrm{Mat}_{(n,n+k)}(\mathbb{C})$ be the set of all $n \times (n+k)$ matrices with complex entries, $\Sigma^s \subset \mathrm{Mat}_{(n,n+k)}(\mathbb{C})$ the subset formed by matrices that have rank less than s, with $1 \leq s \leq n$. The set Σ^s is called the generic determinantal variety.*

Remark 8.8. *The following properties of the generic determinantal varieties are fundamental.*

1. *Σ^s is an irreducible singular algebraic variety.*
2. *The codimension of Σ^s in the ambient space is $(n-s+1)(n+k-s+1)$.*
3. *The singular set of Σ^s is exactly Σ^{s-1}.*
4. *The stratification of Σ^s given by $\{\Sigma^t \setminus \Sigma^{t-1}\}$, with $1 \leq t \leq s$, is locally analytically trivial and hence it is a Whitney stratification of Σ^s.*

Definition 8.9. *Let $F : U \subset \mathbb{C}^q \longrightarrow \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+k})$ be an analytic map, where U is an open neighbourhood of 0 and $F(0) = 0$. Let s such that $1 \leq s \leq n$ and denote by $X = F^{-1}(\Sigma^s)$ the subvariety in $\mathbb{C}^q$. If $\mathrm{codim}(X) = \mathrm{codim}\Sigma^s$, then X is called a determinantal variety in U of type $(n+k, n, s)$.*

In [EG] Ebeling and Gusein–Zade introduced the notion of a determinantal variety with an *essentially isolated determinantal singularity* (EIDS) ([EG, Section 1]).

Definition 8.10. *A determinantal variety $X \subset U$, where U is an open neighbourhood of 0 in $\mathbb{C}^q$, defined by $X = F^{-1}(\Sigma^s)$, $1 \leq s \leq n$, where $F : U \subset \mathbb{C}^q \longrightarrow \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+k})$ is an analytic map, is an essentially isolated determinantal singularity (EIDS) of type $(n+k, n, s)$ if F is transverse to the rank stratification of $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+k})$ except possibly at the origin.*

If X is an EIDS in U of type $(n+k, n, s)$, the singular set of X is given by $F^{-1}(\Sigma^{s-1})$. The regular part of X is given by $F^{-1}(\Sigma^s \setminus \Sigma^{s-1})$ and denoted by X_{reg}. As mentioned by Ebeling and Gusein–Zade, an EIDS X has an isolated singularity at the origin if and only if $q \leq (n-s+2)(n+k-s+2)$.

A deformation $\tilde{F} : U \subset \mathbb{C}^q \to \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+k})$ of F which is transverse to the rank stratification is called a *stabilization* of F. According to Thom's Transversality Theorem (see [KZ, Tri]), F always admits a stabilization $\widetilde{F}$.

The variety $\widetilde{X} = \widetilde{F}^{-1}(\Sigma^s)$ is called an *essential smoothing* of X ([EG, Section 1]). Ebeling and Gusein–Zade also remarked that, in the specific case $q < (n-s+2)(n+k-s+2)$, then the essential smoothing is a genuine smoothing.

In [Cha] Chachapoyas–Siesquén studies the Euler obstruction of essentially isolated determinantal singularities (EIDS). The author obtains formulae to calculate the Euler obstruction for the determinantal varieties whose singular set is an ICIS.

The paper [GGR] by Gaffney, Grulha and Ruas, has two complementary parts, in the first part the authors compute the local Euler obstruction of generic determinantal varieties and apply this result to compute the Chern-Schwartz-MacPherson class of such varieties. In the second part they compute the Euler characteristic of the stabilization of an essentially isolated determinantal

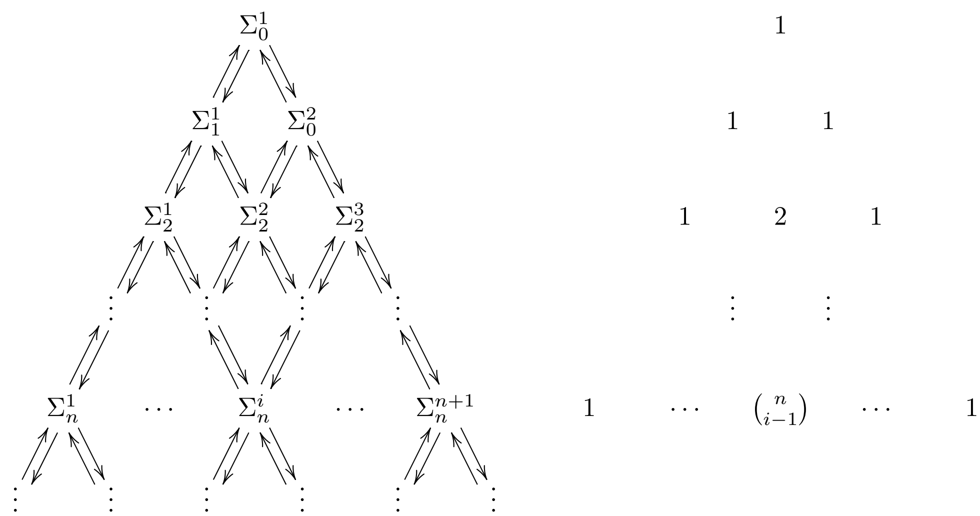


Figure 11.

singularity (EIDS). The formula is given in terms of the local Euler obstruction and Gaffney's m_d multiplicity (see [GR, Definition 3.3]).

Let us denote the topological Euler-Poincaré characteristic by χ and the reduced Euler-Poincaré characteristic $\overline{\chi}$ by $\overline{\chi} = \chi - 1$.

Proposition 8.11. [EG, Prop.3] *Let $\ell : \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+k}) \to \mathbb{C}$ be a generic linear form. Then, for $s \leq n$, one has*

$$\overline{\chi}(\Sigma^s \cap \ell^{-1}(1)) = (-1)^s \binom{n-1}{s-1}.$$

In order to find the Chern–Schwartz–MacPherson class of a generic determinantal variety, first we calculate its local Euler obstruction.

Theorem 8.12. [GGR] *Let $\Sigma^s \subset \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+k})$ be a generic determinantal variety defined as above, we have*

$$Eu_{\Sigma^s}(0) = \binom{n}{s-1},$$

for $1 \leq s \leq n$.

For this part, fix $k \in \mathbb{Z}^+$, $k \geq 1$, and for $i \in \mathbb{Z}^+$, $1 \leq i \leq n+1$, let us denote $\Sigma^i \subset \mathrm{Hom}(n, n+k)$ by Σ_n^i.

On the one hand, on the left figure 11, we have a triangle of spaces and maps.

Description is given in [GGR, Remark 1.18]: In the apex of the triangle (row zero) we have $\{0\} \in \mathrm{Hom}(\mathbb{C}^0, \mathbb{C}^k)$. Elements in row 1 are $\{0\} \in \mathrm{Hom}(\mathbb{C}^1, \mathbb{C}^{k+1})$ and $\mathrm{Hom}(\mathbb{C}^1, \mathbb{C}^{k+1})$. We have maps from the element in row 0 to each element in row 1 given by the inclusions of $\mathbb{C}^k$ to $\mathbb{C}^{k+1}$, and projection of $\mathbb{C}^1$ to $\mathbb{C}^0$. Row 2 is $\Sigma^1 = \{0\} \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^{k+2})$, $\Sigma^2 \subset \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^{k+2})$, and $\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^{k+2})$.

Again there are maps given by projection and inclusion from elements of row 1 into adjacent pairs of elements of row 2.

Then row n consists of the spaces $\Sigma_n^i \subset \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+k})$, $1 \leq i \leq n+1$, with maps from the previous row to adjacent pairs of elements of this row. The triangle on the right is Pascal's triangle. Then Theorem 8.12 says that the local Euler obstruction takes the (left) triangle of spaces to the (right) Pascal triangle.

Using this last proposition and MacPherson definition, we can calculate the Chern–Schwartz–MacPherson class of Σ^s as follows.

Theorem 8.13. [GGR, Theorem 1.22] *In the same setting as above, the (total) Chern–Schwartz–MacPherson class of Σ^s is*

$$c_{\mathrm{SM}}(\Sigma^s) = \sum_{j=0}^{s-1} (-1)^{s-1+j} \binom{n-j-1}{s-j-1} c_{\mathrm{Ma}}(\Sigma^{j+1}),$$

where c_{Ma} denotes the total Chern–Mather class.

The key tool for the next results is the theory of integral closure of modules andmultiplicity of pairs of modules. Based on this theory, in this section, we compute the Euler characteristic of the stabilization of an essentially isolated determinantal singularity (Def. 8.10). The results of this part are mainly based on [GRa, GR].

In [EG2], Ebeling and Gusein–Zade studied the radial index and the Euler obstruction of 1-form on a singular variety. The authors presented a formula expressing the radial index of a 1-form in terms of the Euler obstructions of the 1-form on different strata.

One of the main ingredients to prove results about stabilization, as remarked in [GR], in the EIDS context, instead of a smoothing, we have a stabilization – a determinantal deformation of X to the generic fibre. Then the multiplicity of the polar curve of $JM_z(\mathcal{X})$ over the parameter space at the origin in a stabilization is the number of critical points that a generic linear form has on the complement of the singular set on a generic fibre. Call this number $m_d(X)$, where $d = \dim X$. The m_d multiplicity was defined by Gaffney in [Gaff3] for the study of isolated complete intersection singularities (ICIS), and for isolated singularities whose versal deformation have a smooth base in [Gaff1].

Now, from [GGR] and using that if X is an EIDS of type $(n+k, n, s)$, defined by the analytic map $F : U \subset \mathbb{C}^q \to \mathrm{Hom}(n, n+k)$, when $q > n(n+k)$ we have a submersion on the strata different from $\{0\}$, so we have a fibred structure and when $q < n(n+k)$ we have an immersion, using the transversality of F we have the following results.

Before stating the next results let us fix the following notation: ${}_iX = F^{-1}(\Sigma^i), 1 \leq i \leq n$.

Proposition 8.14. [GGR, Proposition 2.14]*Let $X \subset \mathbb{C}^q$ be an EIDS of type $(n+k, n, s)$, defined by the analytic map $F : U \subset \mathbb{C}^q \to \mathrm{Hom}(n, n+k)$, with $F(0) = 0$, $1 \leq i \leq n$ and $n(n+k) > q$. In this setting we have*

$$\mathrm{Eu}_0(X) = e(s-1, n-1) + \sum_{i=2}^{s} \overline{\chi}_*(i, n) \binom{n-i}{s-i},$$

where $\overline{\chi}_*(i, n) = \overline{\chi}({}_iX \cap l^{-1}(t_0) \cap B_\varepsilon(0))$ and l is a generic linear form.

Proposition 8.15. [GGR, Proposition 2.15] *Let $X \subset \mathbb{C}^q$ be an EIDS of type $(n+k, n, s)$, defined by the analytic map $F : U \subset \mathbb{C}^q \to \mathrm{Hom}(n, n+k)$, with $F(0) = 0$ and $q > n(n+k)$. In this setting we have*

$$\mathrm{Eu}_0(X) = \binom{n}{s-1} + \overline{\chi}({}_1X \cap H) \binom{n-1}{s-1} + \sum_{i=2}^{s} \overline{\chi}_*(i, n) \binom{n-1}{s-1}.$$

Remark 8.16. *Note that in this case, when $q > n(n+k)$ there are 2 additional terms. Also note that ${}_1X$ is always an ICIS so $\overline{\chi}({}_1X \cap H)$ is $\mu({}_1X \cap H)$ up to a sign. Note also that if $n = s = 2$*

we get

$$\mathrm{Eu}_0(X) = 2 + \overline{\chi}({}_1X \cap H) + 1 + \overline{\chi_*}(2,2) = 2 + \overline{\chi}({}_1X \cap H) + \tilde{\chi_*}(2,2),$$

which is Siesquén's formula [Cha].

8.4. **Euler obstruction of a module.** In [GGR] Gaffney, Grulha and Ruas generalize the definition of polar varieties as follows:

Given a submodule M of the free $\mathcal{O}_{X^d}$ module $\mathcal{O}^p_{X^d}$ of rank p, we can associate the Rees algebra $\mathcal{R}(M)$ of M, that is the subalgebra of the symmetric $\mathcal{O}_{X^d}$ algebra on p generators. Then $\mathrm{Projan}(\mathcal{R}(\mathrm{M}))$, the projective analytic spectrum of $\mathcal{R}(M)$ is the closure of the projectivised row spaces of M at points where the rank of a matrix of generators of M is maximal (see [Gaff1], [GGR, page 25]).

Definition 8.17. *Let us consider a submodule M of the free $\mathcal{O}_{X^d}$ module $\mathcal{O}^p_{X^d}$ of rank p whose generic rank is g. The polar variety of codimension k of M in X, denoted $P_k(M)$, is constructed by intersecting* $\mathrm{Projan}(\mathcal{R}(\mathrm{M}))$ *with $X \times H_{g+k-1}$, where H_{g+k-1} is a general plane of codimension $g+k-1$, then projecting to X.*

Let us define the local Euler obstruction of a module $M \subset \mathcal{O}^p_{X^d}$.

Definition 8.18. *Assume X equidimensional, generically reduced. Given a sheaf of modules $M \subset \mathcal{O}^p_{X^d}$, M with the same generic rank on each component of X. We define*

$$\mathrm{Eu}_0(M) = \sum_{i=0}^{d-1} (-1)^i m_0(P_i(M)),$$

where $P_i(M)$ is the polar variety of M of codimension i. Since X is generically reduced, $P_0(M) = X$.

Remark 8.19. *When M is the Jacobian module $JM(X)$ the generalization of the polar varieties ($Def.$ 8.17) coincides with the classical notion of polar varieties used by Lê and Teissier. In other words, in this case we have* $\mathrm{Eu}_0(JM(X)) = \mathrm{Eu}_0(X)$.

Theorem 8.20. *Given $F : \mathbb{C}^q \to \mathrm{Hom}(n, n+k)$, with $0 < q \le n(n+k)$, such that F defines a EIDS X. Let $M_i = JM((F\mid_{H^{c(r)+i}})^{-1}(\Sigma^r)) = JM(F^{-1}(\Sigma^r) \cap H^{c(r)+i}), i > 0$. Here $H^{c(r)+i}$ is a generic plane of dimension $c(r)+i$, where $c(r)$ is the codimension of Σ^r in $\mathrm{Hom}(n, n+k)$. Let $N_i = (F\mid_{H^{c(r)+i}})^*(JM(\Sigma^r))$. We let $M_0, N_0 = 0$. Then,*

$$\mathrm{Eu}_0(JM(X)) = \mathrm{Eu}_0(X) = \sum_{i=0}^{d-1} (-1)^i e(M_i, N_i; \mathcal{O}_{X \cap H^{c(r)+i}}) + \mathrm{Eu}_0(F^*(JM(\Sigma^r))).$$

For the following Corollary, proved in [GGR], it is convenient to change our notation a little since the main tool is based on [GRa], so we match the notation there. We let $\overline{\Sigma}_r$ denote Σ^{r+1}, that is we let Σ_r denote the matrices of kernel rank r.

Corollary 8.21. *Suppose that $X \subset \mathbb{C}^q$ and its generic plane sections are good approximations to $\overline{\Sigma}_r \subset Hom(n, n+k)$. Suppose that $n(n+k) > q > \dim(\Sigma^r)$. Then $Eu_0(X) = Eu_0(\overline{\Sigma}_r)$.*

8.5. **Ruled surfaces.** Ruled surfaces are also very interesting objects in mathematics that have many applications. Consider two complex curves $\alpha : D \to \mathbb{C}^3$ and $\beta : D \to \mathbb{C}^3$, where $D \subset \mathbb{C}$ is a disk centered at the origin, we can consider the two curves together as a map $(\alpha, \beta) : D \to \mathbb{C}^3 \times \mathbb{C}^3$. We say that $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ and $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t))$ is a *primitive parameter pair* for (α, β) if it cannot be re-parameterized by powers of a new variable.

A ruled surface in $\mathbb{C}^3$ is locally the image of the application: $f : D \times \mathbb{C} \to \mathbb{C}^3$ given by

$$f(t, u) = \alpha(t) + u\beta(t),$$

where α and β are complex spatial curves with $\beta \neq 0$. We call $\alpha : D \to \mathbb{C}^3$ the *base curve* and $\beta : D \to \mathbb{C}^3$ the *steering curve*.

Lemma 8.22. [MN] *Given the germ of a ruled surface in $\mathbb{C}^3$, we can choose coordinates in $\mathbb{C}^3$ such that the surface is parametrized in the form $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$*

$$f(t, u) = (0, \alpha_1(t), \alpha_2(t)) + u(1, \beta_1(t), \beta_2(t)). \tag{8.22}$$

Given a pair of primitive parametrizations (α, β) of complex plane curves, we will denote by $f_{(\alpha,\beta)}$ a parametrization of the ruled surface associated with these curves as in (8.22).

Definition 8.23. *Given a pair $(\gamma^{(1)}, \gamma^{(2)}) : D \to \mathbb{C}^2 \times \mathbb{C}^2$, the multiplicity* (of the pair) *at the origin $0 \in D \subset \mathbb{C}$ is $(m(\gamma^{(1)}), m(\gamma^{(2)}))$ with $m(\gamma^{(j)}) = min\{ord_t \gamma_i^{(j)}(t); i = 1, 2\}$.*

In [GEM] given two integers $m \geq n \geq 0$ the authors exhibit (ruled) surfaces with multiplicity m and Euler obstruction n.

Theorem 8.24. *If $(X, 0)$ is the germ of a ruled surface given by $f_{(\alpha,\beta)}$ as in (8.22) where (α, β) is a pair of primitive parametrizations of complex plane curves, with pair of multiplicities (n_0, n_1) with $n_0 \geq n_1 \geq 0$, then $Eu_X(0) = n_1$.*

As a consequence of this result we see that given a positive integer n it is possible to find a germ of a ruled surface $(X, 0)$ such that $Eu_X(0) = n$.

Corollary 8.25. *Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be the germ of a ruled surface, if the Euler obstruction $Eu_X(0)$ is greater than 1, then $(X, 0)$ has no isolated singularity at the origin.*

9. The Euler obstruction of a function and the Brasselet number

In [BG], Brasselet and Grulha write a continuation of the first author's survey Local Euler Obstruction, Old and New (1998). It takes into account recent results obtained by various authors, in particular with regard to the extensions of Euler's local obstruction to frames, functions and maps and for differential forms and collections of them.

Definition 9.1. *(see Section 7)* Let $Y \subset \mathbb{R}^n$ be a semi-algebraic set. A constructible function $\alpha : Y \to \mathbb{Z}$ is a $\mathbb{Z}$-valued function that can be written as a finite sum:

$$\alpha = \sum_{i \in I} m_i \mathbf{1}_{Y_i},$$

where Y_i is a semi-algebraic subset of Y and $\mathbf{1}_{Y_i}$ is the characteristic function on Y_i.

The sum and the product of two constructible functions on Y are again constructible. The set of constructible functions on Y is thus a commutative ring, denoted by $\mathbf{F}(Y)$.

Definition 9.2. If α is a constructible function in $\mathbf{F}(Y)$ and $W \subset Y$ is a semi-algebraic set then the Euler characteristic $\chi(W, \alpha)$ is defined by

$$\chi(W, \alpha) = \sum_{i \in I} m_i \chi_c(W \cap Y_i),$$

where $\alpha = \sum_{i \in I} m_i \mathbf{1}_{Y_i}$ and χ_c is the Euler characteristic in Borel-Moore homology.

In what follows, the complex link is a key object in the study of the topology of complex analytic sets. It is analogous to the Milnor fibre and was studied first in [Le1]. It plays a crucial role in complex stratified Morse theory (see [GM2]) and appears in general bouquet theorems for the Milnor fibre of a function with isolated singularity (see [Le2]).

Let V be a stratum of the stratification $\mathcal{V}$ of X and let x be a point in V. Let

$$g : (\mathbb{C}^n, x) \to (\mathbb{C}, 0)$$

be an analytic complex function-germ such that the differential form $Dg(x)$ is not a degenerate covector of $\mathcal{V}$ at x. Let $N^{\mathbb{C}}_{x,V}$ be a normal slice to V at x, *i.e.* $N^{\mathbb{C}}_{x,V}$ is a closed complex submanifold of $\mathbb{C}^n$ which is transversal to V at x and $N^{\mathbb{C}}_{x,V} \cap V = \{x\}$.

Definition 9.3 ([GM2], p.161)**.** The complex link $\mathcal{L}^X_V$ of V is defined by

$$\mathcal{L}^X_V = X \cap N^{\mathbb{C}}_{x,V} \cap B_\epsilon(x) \cap \{g = \delta\},$$

where $0 < |\delta| \ll \epsilon \ll 1$. Here $B_\epsilon(x)$ is the closed ball of radius ϵ centered at x.

The normal Morse datum $\mathrm{NMD}(V)$ of V is the pair of spaces

$$\mathrm{NMD}(V) = \left(X \cap N^{\mathbb{C}}_{x,V} \cap B_\epsilon(x), X \cap N^{\mathbb{C}}_{x,V} \cap B_\epsilon(x) \cap \{g = \delta\}\right),$$

where $x \in X$.

The fact that these two notions are well-defined, *i.e.* independent of all the choices made to define them, is explained in [GM2].

Definition 9.4. [SchuTib, Definition 2.2] *Let $\alpha \in F(X)$ be a constructible function with respect to the stratification $\mathcal{V}$. Its normal Morse index $\eta(V, \alpha)$ along V is defined by*

$$\eta(V, \alpha) = \chi(\mathrm{NMD}(V), \alpha).$$

Moreover, the key role of the Euler obstruction comes from the following identities (see [SchuTib] p.34 or [Schu] p.292 and p.323-324):

$$\eta(V', \mathrm{Eu}_{\overline{V}}) = 1 \text{ if } V' = V$$

and

$$\eta(V', \mathrm{Eu}_{\overline{V}}) = 0 \text{ if } V' \neq V.$$

The Euler obstruction is a constructible function and there are two distinguished bases for the free abelian group of constructible functions: the characteristic function $\mathbf{1}_{\overline{V}}$ and the Euler obstruction $\mathrm{Eu}_{\overline{V}}$ of the closure $\overline{V}$ of all strata V. The Euler characteristic $\chi(W, \alpha)$ is also called the Euler integral of α and denoted by $\int_W \alpha d\chi_c$. Here we follow the terminology and notations used in [BMPS, DG, ST].

In [BLS], Brasselet, Lê and Seade study the Euler obstruction using hyperplane sections, following ideas of Dubson and Kato. Let us assume that 0 belongs to X.

Theorem 9.5 ([BLS], Theorem 3.1)**.** *For each generic linear form l, there is ϵ_0 such that for any ϵ with $0 < \epsilon < \epsilon_0$, the Euler obstruction of $(X, 0)$ is equal to:*

$$\mathrm{Eu}_X(0) = \chi\big(X \cap B_\epsilon(0) \cap l^{-1}(\delta), \mathrm{Eu}_X\big),$$

where $0 < |\delta| \ll \epsilon \ll 1$.

Let $f : X \to \mathbb{C}$ be a holomorphic function. We assume that f has an isolated singularity (or an isolated critical point) at 0, *i.e.* that f has no critical point in a punctured neighbourhood of 0 in X.

In [BMPS] Brasselet, Massey, Parameswaran and Seade introduced an invariant which measures, in a way, how far the equality given in Theorem 9.5 is from being true if we replace the generic linear form l with some other function on X with at most an isolated stratified critical point at 0. This number is called the Euler obstruction of a function and denoted by $\mathrm{Eu}_{f,X}(0)$.

Let $f : X \to \mathbb{C}$ be an analytic function, restriction of an analytic function $F : U \to \mathbb{C}$, with an isolated singular point at $0 \in X$. The gradient vector field of F allows to construct a stratified vector field denoted by $\overline{\nabla}_X f$ using the Marie-Hélène Schwartz construction and Whitney conditions.

Definition 9.6. *The local Euler obstruction of f on X, at the point 0, denoted by $\mathrm{Eu}_{f,X}(0)$, is the local Euler obstruction $\mathrm{Eu}(\overline{\nabla}_X f, 0, X)$ of the stratified vector field $\overline{\nabla}_X f$ at $0 \in X$.*

The following result of Brasselet, Massey, Parameswaran and Seade [BMPS] compares, in the same point, the local Euler obstruction with the Euler obstruction of a function.

Theorem 9.7 ([BMPS], Theorem 3.1)**.** *Let $f : X \to \mathbb{C}$ be a function with an isolated singularity at 0. For $0 < |\delta| \ll \varepsilon \ll 1$ we have:*

$$\mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0) = \chi\big(X \cap B_\epsilon(0) \cap f^{-1}(\delta), \mathrm{Eu}_X\big),$$

where $0 < |\delta| \ll \epsilon \ll 1$.

In [STV], the authors show that the Euler obstruction of f is closely related to the number of Morse points of a Morsification of f, as it is stated in the next proposition.

Proposition 9.8 ([STV] Proposition 2.3)**.** *Let $f : X \to \mathbb{C}$ be an analytic function with isolated singularity at the origin. Then:*

$$\mathrm{Eu}_{f,X}(0) = (-1)^d n_{\mathrm{reg}},$$

where n_{reg} is the number of Morse points on X_{reg} in a stratified Morsification of f lying in a small neighbourhood of 0.

Definition 9.9 ([Ma], p.971)**.** A good stratification of X relative to f is a stratification $\mathcal{V}$ of X which is adapted to X^f, (*i.e.* X^f is a union of strata) , where $X^f = X \cap f^{-1}(0)$, such that $\{V_i \in \mathcal{V}; V_i \not\subset X^f\}$ is a Whitney stratification of $X \setminus X^f$ and such that for any pair of strata (V_a, V_b) such that $V_a \not\subset X^f$ and $V_b \subset X^f$, the Thom (a_f) condition is satisfied.

In this section, we recall several formulae proved by Dutertre and Grulha in [DG3] that relate the number of critical points of a Morsification of a polynomial function f on an algebraic set X, to the global Brasselet numbers and the Brasselet numbers at infinity of f. We note that when $X = \mathbb{C}^n$, similar formulae have already appeared in the literature in the work of many authors as Artal, Luengo, Melle, Tibar, Parusinski, Siersma, Suzuki and others.

Durtertre and Grulha [DG1] defined the Brasselet number as follows.

Definition 9.10. *Suppose that X is equidimensional. Let $\mathcal{V} = \{V_i\}_{i=0}^q$ be a good stratification of X relative to f. The Brasselet number, $\mathrm{B}_{f,X}(0)$, is defined by*

$$\mathrm{B}_{f,X}(0) = \sum_{i=1}^{q} \chi\big(V_i \cap B_\varepsilon(0) \cap f^{-1}(\delta)\big) \cdot \mathrm{Eu}_X(V_i),$$

where $0 < |\delta| \ll \varepsilon \ll 1$.

Remark 9.11. Note that if f has a stratified isolated singularity at the origin then, by Theorem 9.7, we have that $\mathrm{B}_{f,X}(0) = \mathrm{Eu}_X(0) - \mathrm{Eu}_{f,X}(0)$.

In [DG2] the authors present an alternative proof of the Brasselet, Massey, Parameswaran and Seade formula for the Euler obstruction of a function using Ebeling and Gusein-Zade's results on the radial index and the Euler obstruction of 1-forms.

Grulha compares in [Gru, GruE] the local Euler obstruction with some generalizations of the Milnor number, in particular the Milnor number of Lê, and the Milnor number of Bruce and Roberts.

Let $\mathcal{O}_n$ the ring of germs of analytic functions $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and $I(X)$ the ideal of $\mathcal{O}_n$ of the germs of functions vanishing on X. A germ of vector field v_n on $(\mathbb{C}^n, 0)$, can be seen as a derivation $v_n : \mathcal{O}_n \to \mathcal{O}_n$. It is tangent to X if

$$dg(v_n) = v_n g \in I(X), \forall g \in I(X).$$

Definition 9.12. *Let $f \in \mathcal{O}_n$ and let θ_X be the set of vector fields tangent to X, the Milnor number of Bruce and Roberts is defined by*

$$\mu_{BR}(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{df(\theta_X)}.$$

For a logarithmic stratification of X, we denote by $LC(V)$ the union of the conormal spaces of the strata V_α. There are subspaces of $T_0^*(\mathbb{C}^m)$ of vanishing forms on TV_α . One denotes by m_α the multiplicity of T^*V_α in $LC(V)$.

Theorem 9.13 ([Gru, GruE])**.** *Let $(X, 0)$ be a germ of a reduced equidimensional analytic variety in $\mathbb{C}^m$ and $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ an analytic function with isolated singularity at 0.*
If $LC(V)$ is Cohen-Macaulay then

$$\mu_{BR}(f) = \sum_\alpha (-1)^{\dim_{\mathbb{C}} V_\alpha} \, m_\alpha \, Eu_{f,\overline{V}_\alpha}(0).$$

By [GM2], given a stratification $\mathcal{S}$ of X, one can refine $\mathcal{S}$ to obtain a Whitney stratification $\mathcal{V}$ of X which is adapted to X^f. Moreover, by [BMM, Theorem 4.3.2], the refinement $\mathcal{V}$ satisfies the Thom a_f condition. This means that good stratifications always exist.

For instance, if $\mathcal{V}$ is a Whitney stratification of X and $f : X \to \mathbb{C}$ has a stratified isolated critical point at $\{0\}$, then

$$\{V_\alpha \setminus X^f, \quad V_\alpha \cap X^f \setminus \{0\}, \quad \{0\}\}, \quad \text{with } V_\alpha \in \mathcal{V},$$

is a good stratification for f. We call it the *good stratification induced by f*.

Definition 9.14. *The critical locus of f relative to $\mathcal{V}$, $\Sigma_{\mathcal{V}} f$, is defined by the union*

$$\Sigma_{\mathcal{V}} f = \bigcup_{V_\lambda \in \mathcal{V}} \Sigma(f|_{V_\lambda}).$$

In [DG1] the authors proved that the Brasselet number satisfies a Lê-Greuel type formula, which relates this invariant with the number of Morse critical points. To explain this property we introduce the following definition.

Definition 9.15. *Let $\mathcal{V}$ be a good stratification of X relative to f. We say that $g : (X, 0) \to (\mathbb{C}, 0)$ is prepolar with respect to $\mathcal{V}$ at the origin if the origin is a stratified isolated critical point of g.*

Given f and g function-germs defined on $(X, 0)$, the Thom (a_f) condition in Definition 9.10 together with the hypothesis of g be prepolar guarantee that $g : X \cap f^{-1}(\delta) \cap B_\varepsilon \to \mathbb{C}$ has no critical points on $\{g = 0\}$ [Ma, Proposition 1.12] and so the number of stratified Morse critical points on the top stratum $V_q \cap f^{-1}(\delta) \cap B_\varepsilon(0)$ appearing in a Morsification of

$$g : X \cap f^{-1}(\delta) \cap B_\varepsilon(0) \to \mathbb{C}$$

does not depend on the Morsification.

The following result shows that the Brasselet number satisfies a Lê-Greuel type formula [DG, Theorem 4.4].

Theorem 9.16. *Suppose that X is equidimensional and that g is prepolar with respect to $\mathcal{V}$ at the origin. For $0<|\delta|\ll\varepsilon\ll 1$, we have*

$$\mathrm{B}_{f,X}(0)-\mathrm{B}_{f,X^g}(0)=(-1)^{d-1}m,$$

where m is the number of stratified Morse critical points of a Morsification of

$$g: X\cap f^{-1}(\delta)\cap B_\varepsilon\to\mathbb{C}$$

appearing on $X_{\mathrm{reg}}\cap f^{-1}(\delta)\cap\{g\neq 0\}\cap B_\varepsilon$. In particular, this number does not depend on the Morsification.

In [San], Santana considered the case where the function g has a stratified singular set of dimension 1 and proved that in this case the difference of the Brasselet numbers $\mathrm{B}_{f,X}(0)$ and $\mathrm{B}_{f,X^g}(0)$ is still related with the number of Morse critical points on the regular part of the Milnor fibre of f appearing in a Morsification of g. To prove this result the author considered that the function-germ g is tractable.

The notion of tractability uses the following auxiliary definition.

Definition 9.17. *If $\mathcal{V}=\{V_\alpha\}$ is a stratification of X, the symmetric relative polar variety of f and g with respect to $\mathcal{V}$, $\tilde{\Gamma}_{f,g}(\mathcal{V})$, is the union $\cup_\alpha\tilde{\Gamma}_{f,g}(V_\alpha)$, where $\Gamma_{f,g}(V_\alpha)$ denotes the closure in X of the critical locus of $(f,g)|_{V_\alpha\setminus(X^f\cup X^g)}$.*

Definition 9.18. *A function $g:(X,0)\to(\mathbb{C},0)$ is tractable at the origin with respect to a good stratification $\mathcal{V}$ of X relative to $f:(X,0)\to(\mathbb{C},0)$ if the dimension of $\tilde{\Gamma}^1_{f,g}(\mathcal{V})$ is less than or equal to 1 in a neighbourhood of the origin and, for all strata $V_\alpha\subseteq X^f$, $g|_{V_\alpha}$ has no critical points in a neighbourhood of the origin except perhaps at the origin itself.*

Theorem 9.19. [San, Theorem 3.2] *Suppose that g is tractable at the origin with respect to $\mathcal{V}$ relative to f. Then, for $0<|\delta|\ll\varepsilon\ll 1$,*

$$B_{f,X}(0)-B_{f,X^g}(0)-\sum_{j=1}^{r}m_{f,b_j}\cdot(\mathrm{Eu}_X(b_j)-\mathrm{Eu}_{X^g}(b_j))=(-1)^{d-1}m,$$

where m is the number of stratified Morse critical points of a partial Morsification of

$$g: X\cap f^{-1}(\delta)\cap B_\varepsilon\to\mathbb{C}$$

appearing on $X_{\mathrm{reg}}\cap f^{-1}(\delta)\cap\{g\neq 0\}\cap B_\varepsilon$, $\Sigma_{\mathcal{V}}g=\{0\}\cup b_1\cup\ldots\cup b_r$ and m_{f,b_j} is the multiplicity of $f|_{b_j}$.

The authors in [ANOT] prove that given an analytic function germ $f:(X,0)\to\mathbb{C}$ on an isolated determinantal singularity or on a reduced curve, one has formulae relating the local Euler obstruction of f to the vanishing Euler characteristic of the fibre $X\cap f^{-1}(0)$ and to the Milnor number of f. Restricting ourselves to the case where X is a complete intersection, one obtains an easy way to calculate the local Euler obstruction of f as the difference between the dimension of two algebras.

The concept of the evanescent Euler characteristic was extended and applied in the context of normal toric surfaces. In [DGP], the authors give a formula to calculate it, and associate this number with the second polar multiplicity of X_σ. They also present a formula for Euler's obstruction of a function and for the difference between the Euler obstruction of the toric surface X_σ and the Euler obstruction of a function f. As an application of this result they compute the Euler obstruction of polynomials of a certain type on a family of determinantal surfaces. In the same direction, in [DaPe] Dalbelo and Pereira present a formula to compute the Euler obstruction of a function $f:(X,0)\to(\mathbb{C},0)$ and its Brasselet number, where X is a multitoric surface. As an application of this formula, the authors compute the Euler obstruction of a function on some families of determinantal surfaces.

In [DHa] the authors present a formula to compute the Brasselet number of $f : (Y, 0) \to (\mathbb{C}, 0)$, where $Y \subset X$ is a non-degenerate complete intersection in a toric variety X. As applications one establishes several results concerning invariance of the Brasselet number for families of non-degenerate complete intersections. Moreover, when $(X, 0) = (\mathbb{C}^n, 0)$ one derives sufficient conditions to obtain the invariance of the Euler obstruction for families of complete intersections with an isolated singularity contained in X.

Since the introduction of the notion of free divisors by Saito, people have discovered how commonplace they are. Discriminants of the versal unfoldings of isolated hypersurface and complete intersection singularities are free divisors and the bifurcation sets associated to the versal unfoldings of isolate hypersurfaces singularities are also free divisors, for instance. One way to investigate these objects is to compute and understand the behavior of some invariants on them. In [Gru2] the author uses the local Euler obstruction in order to investigate free divisors.

10. Global Euler Obstruction

We assume $X \subset \mathbb{C}^n$ to be a reduced algebraic set of dimension d, equipped with a finite Whitney stratification $\mathcal{V} = \{V_1, \dots, V_t\}$. In [STV], Seade, Tibăr and Verjovsky introduced a global analogue of the Euler obstruction called the global Euler obstruction and denoted by $\mathrm{Eu}(X)$. Let us denote by $\widetilde{X} \xrightarrow{\nu} X$ the Nash modification of X (Definition 7.3), and let us consider a stratified real vector field v on a subset $V \subset X$: this means that the vector field is continuous and tangent to the strata. The restriction of v to V has a well-defined canonical lifting $\widetilde{v}$ to $\nu^{-1}(V)$ as a section of the real bundle underlying the Nash bundle $\widetilde{T} \to \widetilde{X}$.

Definition 10.1 ([STV], Definition 2.1)**.** We say that the stratified vector field v on X is radial-at-infinity if it is defined on the restriction to X of the complement of a sufficiently large ball B_M centered at the origin of $\mathbb{C}^N$, and it is transversal to S_R, pointing outwards, for any $R > M$. In particular, v does not vanish on $X \setminus B_M$.

The "sufficiently large" radius M is furnished by the following well-known result.

Lemma 10.2 ([STV], Lemma 2.2)**.** *There exists $M \in \mathbb{R}$ such that, for any $R \geq M$, the sphere S_R centered at the origin of $\mathbb{C}^N$ and of radius R is stratified transversal to X, i.e. transversal to all strata of the stratification $\mathcal{V}$.*

Using this last lemma and inspired by [BS] and [STV], Seade, Tibăr and Verjovsky defined the global Euler obstruction as follows:

Definition 10.3 ([STV], Definition 2.3)**.** Let $\tilde{v}$ be the lifting to a section of the Nash bundle $\tilde{T}$ of a radial-at-infinity stratified vector field v over $X \setminus B_R$. We call global Euler obstruction of X, and denote it by $\mathrm{Eu}(X)$, the obstruction for extending $\tilde{v}$ as a nowhere zero section of $\widetilde{T}$ within $\nu^{-1}(X \cap B_R)$.

To be precise, the obstruction to extend $\tilde{v}$ as a nowhere zero section of $\widetilde{T}$ within $\nu^{-1}(X \cap B_R)$ is in fact a relative cohomology class

$$o(\tilde{v}) \in H^{2d}(\nu^{-1}(X \cap B_R), \nu^{-1}(X \cap S_R)) \simeq H^{2d}_c(\widetilde{X}).$$

The global Euler obstruction of X is the evaluation of $o(\tilde{v})$ on the fundamental class of the pair $(\nu^{-1}(X \cap B_R), \nu^{-1}(X \cap S_R))$. Thus $\mathrm{Eu}(X)$ is an integer and does not depend on the radius of the sphere defining the link at infinity of X. Since two radial-at-infinity vector fields are homotopic as stratified vector fields, it does not depend on the choice of v either.

Remark 10.4. The global Euler obstruction has the following properties (see [STV] p. 396):

((1)) $\mathrm{Eu}(X) = \chi(X, \mathrm{Eu}_X)$,
((2)) if X is non-singular, then $\mathrm{Eu}(X) = \chi(X)$.

A natural question is to know if the concepts of the Euler obstruction and the Brasselet number of a function could be extended to the global setting, as Seade, Tibăr and Verjovsky did for the local Euler obstruction. In the case of positive answer, what kind of information could we obtain with these new global invariants?

As the idea for defining the global Euler obstruction is to consider balls and spheres whose radius R goes to infinity, to answer the previous question, it is natural to consider results concerning singularities at infinity. The main references we use in this setting are [DRT, Ti] and we refer to these papers for details.

We consider $X \subset \mathbb{C}^N$ a reduced algebraic set of dimension d. We use coordinates $(x_1, \ldots, x_N)$ for the space $\mathbb{C}^N$ and coordinates $[x_0 : x_1 : \cdots : x_N]$ for the projective space $\mathbb{P}^N$. We consider the algebraic closure $\overline{X}$ of X in the complex projective space $\mathbb{P}^N$ and we denote by

$$H^\infty = \big\{[x_0 : x_1 : \cdots : x_N] \mid x_0 = 0\big\}$$

the hyperplane at infinity of the embedding $\mathbb{C}^N \subset \mathbb{P}^N$.

One may endow $\overline{X}$ with a semi-algebraic Whitney stratification $\mathcal{W} = \{\mathcal{W}_\alpha\}$ such that X_{reg} is a stratum and the part at infinity $\overline{X} \cap H^\infty$ is a union of strata.

Since $\overline{X}$ is projective and since the stratification of $\overline{X}$ is locally finite, it follows that $\mathcal{W}$ has finitely many strata. We denote by X_{sing} the set of singular points of X, *i.e.* $X_{\mathrm{sing}} = X \setminus X_{\mathrm{reg}}$.

In order to recall the definition of t-regularity, let us recall the definition of the conormal spaces.

Definition 10.5 ([DRT], Definition 2.1)**.** We denote by $C(X)$ the conormal modification of X, defined as:

$$C(X) = \text{closure}\big\{(x, H) \in X_{\mathrm{reg}} \times \check{\mathbb{P}}^{N-1} \mid T_x X_{\mathrm{reg}} \subset H\big\} \subset \overline{X} \times \check{\mathbb{P}}^{N-1}.$$

Let $\pi : C(X) \to \overline{X}$ denote the projection $\pi(x, H) = x$.

Definition 10.6 ([DRT], Definition 2.2)**.** Let $g : X \to \mathbb{C}$ be an analytic function defined in some neighbourhood of X in $\mathbb{C}^N$. Let X_0 denote the subset of X_{reg} where g is a submersion. The relative conormal space of g is defined as follows:

$$C_g(X) = \text{closure}\Big\{(x, H) \in X_0 \times \check{\mathbb{P}}^{N-1} \mid T_x g^{-1}(g(x)) \subset H\Big\} \subset \overline{X} \times \check{\mathbb{P}}^{N-1},$$

together with the projection $\pi : C_g(X) \to \overline{X}$, $\pi(x, H) = x$.

Let $F : \mathbb{C}^N \to \mathbb{C}$ be a polynomial function and $f : X \to \mathbb{C}$ defined by $f = F_{|X}$. Let $\mathbb{X} = \overline{\mathrm{graph} f}$ be the closure of the graph of f in $\mathbb{P}^N \times \mathbb{C}$ and let $\mathbb{X}^\infty = \mathbb{X} \cap (H^\infty \times \mathbb{C})$. One has an isomorphism $\mathrm{graph}(f) \simeq X$.

We consider the affine charts $U_j \times \mathbb{C}$ of $\mathbb{P}^N \times \mathbb{C}$, where

$$U_j = \{[x_0 : \cdots : x_N] \mid x_j \neq 0\}, j = 0, 1, \ldots, N.$$

Identifying the chart U_0 with the affine space $\mathbb{C}^N$, we have $\mathbb{X} \cap (U_0 \times \mathbb{C}) = \mathbb{X} \setminus \mathbb{X}^\infty = \mathrm{graph} f$, and $\mathbb{X}^\infty$ is covered by the charts $U_1 \times \mathbb{C}, \ldots, U_N \times \mathbb{C}$.

If g denotes the projection to the variable x_0 in some affine chart $U_j \times \mathbb{C}$, then the relative conormal space $C_g(\mathbb{X} \setminus \mathbb{X}^\infty \cap U_j \times \mathbb{C}) \subset \mathbb{X} \times \check{\mathbb{P}}^N$ is well defined.

With the projection $\pi(y, H) = y$, let us then consider the space $\pi^{-1}(\mathbb{X}^\infty)$, which is well defined for every chart $U_j \times \mathbb{C}$ as a subset of $C_g(\mathbb{X} \setminus \mathbb{X}^\infty \cap U_j \times \mathbb{C})$.

Definition 10.7 ([DRT], Definition 2.4)**.** We call space of characteristic covectors at infinity the set $C^\infty = \pi^{-1}(\mathbb{X}^\infty)$. For any $p_0 \in \mathbb{X}^\infty$, we denote $C^\infty_{p_0} := \pi^{-1}(p_0)$.

By Lemma 2.8 in [Ti], these notions are well-defined, *i.e.* they do not depend on the chart U_j.

Let us denote by $\tau : \mathbb{P}^N \times \mathbb{C} \to \mathbb{C}$ the second projection. One defines the relative conormal space $C_\tau(\mathbb{P}^N \times \mathbb{C})$ as in Definition 10.6 where the function g is replaced by the mapping τ.

Definition 10.8 ([DRT], Definition 2.5)**.** We say that f is t-regular at $p_0 \in \mathbb{X}^\infty$ if

$$C_\tau(\mathbb{P}^N \times \mathbb{C}) \cap C^\infty_{p_0} = \emptyset.$$

We say that $f^{-1}(t_0)$ is t-regular if f is t-regular at all points $p_0 \in \mathbb{X}^\infty \cap \tau^{-1}(t_0)$.

Let us now recall the definition of ρ-regularity. Let $K \subset \mathbb{C}^N$ be some compact (possibly empty) set and let $\rho : \mathbb{C}^N \setminus K \to \mathbb{R}_{\geq 0}$ be a proper analytic submersion.

Definition 10.9 (ρ-regularity at infinity, [DRT], Definition 5.2)**.** We say that f is ρ-regular at $p_0 \in \mathbb{X}^\infty$ if there is an open neighbourhood $U \subset \mathbb{P}^N \times \mathbb{C}$ of p_0 and an open neighbourhood $D \subset \mathbb{C}$ of $\tau(p_0)$ such that, for all $t \in D$, the fibre $f^{-1}(t) \cap X_{\mathrm{reg}} \cap U$ intersects all the levels of the restriction $\rho_{|U \cap X_{\mathrm{reg}}}$ and this intersection is transversal.

We say that the fibre $f^{-1}(t_0)$ is ρ-regular at infinity if f is ρ-regular at all points $p_0 \in \mathbb{X}^\infty \cap \tau^{-1}(t_0)$. We say that t_0 is an asymptotic ρ-non-regular value if $f^{-1}(t_0)$ is not ρ-regular at infinity.

Let $X \subset \mathbb{C}^n$ be a reduced algebraic set of dimension d, equipped with a finite Whitney stratification $\mathcal{V} = \{V_i\}_{i=1}^t$. We assume that $V_1, \ldots, V_{t-1}$ are connected, $\overline{V_1}, \ldots, \overline{V_t}$ are reduced and that $V_t = X_{\mathrm{reg}}$, where X_{reg} has dimension d. Let $f : X \to \mathbb{C}$ be a complex polynomial, restriction to X of a polynomial function $F : \mathbb{C}^n \to \mathbb{C}$, *i.e.* $f = F_{|X}$. We assume that f has a finite number of critical points, which means that for $i = 1, \ldots, t$, $F_{|V_i}$ has a finite number of critical points. We denote by $\{q_1, \ldots, q_s\}$ the set of critical points of f and by $\{a_1, \ldots, a_r\}$ the set of stratified asymptotic non-ρ_E-regular values of f.

For simplicity, we will write B_R for the ball $B_R(0)$ and S_R for ∂B_R.

Lemma 10.10. *Let $\alpha : X \to \mathbb{Z}$ be a constructible function with respect to $\mathcal{V}$. The function $c \mapsto \chi(f^{-1}(c), \alpha)$ is constant on $\mathbb{C} \setminus \Big(\{f(q_1), \ldots, f(q_s)\} \cup \{a_1, \ldots, a_r\}\Big)$.*

Definition 10.11. When X is equidimensional, we define the global Brasselet number of f at c by

$$\mathrm{B}^X_{f,c} = \chi(f^{-1}(c), \mathrm{Eu}_X)$$

and the global Euler obstruction of f at c by

$$\mathrm{Eu}^X_{f,c} = \mathrm{Eu}(X) - \mathrm{B}^X_{f,c}.$$

We start to compare the global Brasselet numbers of f and the Euler obstructions of the fibres of f.

Proposition 10.12. *Let $a \in \mathbb{C}$, we have*

$$\mathrm{B}^X_{f,a} = \mathrm{Eu}(f^{-1}(a)) + \sum_{j \ | \ f(q_j)=a} \mathrm{Eu}_X(q_j) - \mathrm{Eu}_{f^{-1}(a)}(q_j).$$

Note that for a regular value c of f, $\mathrm{B}^X_{f,c} = \mathrm{Eu}(f^{-1}(c))$. Furthermore if $X = \mathbb{C}^n$ then $\mathrm{Eu}_X(q_j) = 1$ and $\mathrm{Eu}_{f^{-1}(a)} = 1 + (-1)^{n-2}\mu'(f, q_j)$, where $\mu'(f, q_j)$ is the first Milnor-Teissier number of f at q_j, so

$$\mathrm{B}^X_{f,a} = \chi(f^{-1}(a)) = \mathrm{Eu}(f^{-1}(a)) + (-1)^{n-1} \sum_{j \ | \ f(q_j)=a} \mu'(f, q_j).$$

Remark 10.13. As remarked in [DG3], in the equality of Proposition 10.12, we can distinguish between two kinds of critical points: those lying in $V = X_{\text{reg}}$ and those lying in a lower dimensional stratum. Note that the collection of lower dimensional strata gives a Whitney stratification of X_{sing}, the singular locus of X. Hence the critical points of f that lie on X_{sing} depend on the stratification of X_{sing}. However, the formula of Proposition 10.12 implies that the sum

$$\sum_{\substack{j \ | \ f(q_j)=a \\ q_j \in X_{\text{sing}}}} \mathrm{Eu}_X(q_j) - \mathrm{Eu}_{f^{-1}(a)}(q_j)$$

does not depend on the stratification of X_{sing}.

A direct corollary of the previous proposition is a global relative version of the local index formula of Brylinski, Dubson and Kashiwara [BDK].

Corollary 10.14. *Let $\alpha : X \to \mathbb{Z}$ be a constructible function with respect to $\mathcal{V}$. For any $a \in \mathbb{C}$, we have*

$$\chi(f^{-1}(a), \alpha) = \sum_{i=1}^{t} \mathrm{B}_{f,a}^{\overline{V_i}} \eta(V_i, \alpha).$$

As before, $X \subset \mathbb{C}^n$ is a reduced algebraic set of dimension d, equipped with a finite Whitney stratification $\mathcal{V} = \{V_i\}_{i=1}^{t}$ such that $V_1, \ldots, V_{t-1}$ are connected, $\overline{V_1}, \ldots, \overline{V_t}$ are reduced and $V_t = X_{\text{reg}}$; $f : X \to \mathbb{C}$ is a complex polynomial, restriction to X of a polynomial function $F : \mathbb{C}^n \to \mathbb{C}$. We assume that f has a finite number of critical points, which means that for $i = 1, \ldots, t$, $F_{|V_i}$ has a finite number of critical points. We denote by $\{q_1, \ldots, q_s\}$ the set of critical points of f and by $\{a_1, \ldots, a_r\}$ the set of stratified asymptotic non-ρ_E-regular values of f.

Definition 10.15. We say that $\tilde{f} : X \to \mathbb{C}$ is a Morsification of f if $\tilde{f}$ is a small deformation of f which is a local (stratified) Morsification at all isolated critical points of f.

Let $\tilde{f}$ be a Morsification of f. As in the local case, we can take $\tilde{f}$ of the form $f + tl$ where t is a sufficiently small complex number and l is the restriction to X of a generic linear form (see Theorem 2.2 in [Le2]). Note that $\tilde{f}$ has two kinds of critical points: those appearing in a small neighbourhood of a critical point of f and those appearing at infinity, *i.e.* outside a ball of sufficiently big radius. We will only consider the first ones.

Let n_i, $i = 1, \ldots, t$, be the number of critical points of $\tilde{f}$ appearing in a small neighbourhood of a critical point of f on the stratum V_i. Note that

$$n_i \geq \mu^T(f_{|V_i}) = \sum_{j \ | \ q_j \in V_i} \mu(f_{|V_i}, q_j),$$

where $\mu(f_{|V_i}, q_j)$ is the Milnor number of $f_{|V_i}$ at q_j, since we do not assume that f is general with respect to $\mathcal{V}$.

The next result, proved in [DG3] relates the number of stratified critical points of $\tilde{f}$ appearing on the stratum V_i to the topology of X and a generic fibre of f.

Theorem 10.16. *Let $c \in \mathbb{C}$ be a regular value of f, which is not a stratified asymptotic non-ρ_E-regular value. We have*

$$\chi(X) - \chi(f^{-1}(c)) = \sum_{i=1}^{t} (-1)^{d_i} n_i \Big(1 - \chi(\mathcal{L}_{V_i}^X)\Big) - \lambda_f^{X,\infty}.$$

Moreover if f is general with respect to $\mathcal{V}$, then we have

$$\chi(X) - \chi(f^{-1}(c)) = \sum_{i=1}^{t} (-1)^{d_i} \mu^T(f_{|V_i}) \Big(1 - \chi(\mathcal{L}^X_{V_i})\Big) - \lambda_f^{X,\infty}.$$

An interesting application occurs when X is equidimensional, then by [STV], Proposition 2.3, the term $(-1)^{d_i} n_{ij}$ that appears in Equality $(*)$ is equal to $\mathrm{Eu}_{f,\overline{V_i}}(q_j)$, if $\alpha = \mathrm{Eu}_X$.

In [DG3], the authors show that it is possible to define a global Brasselet number at infinity and they prove the important result that this number satisfies a Brylinski-Dubson-Kashiwara type formula [BDK].

From now on, we assume that X is equidimensional. If $f = l$ is the restriction to X of a generic linear function $L : \mathbb{C}^N \to \mathbb{C}$, then l has no stratified asymptotic non-ρ_E-regular values and moreover l is a stratified Morse function (see [STV], Lemma 3.1).

Keeping the notations introduced in [STV], we denote by $\alpha_X^{(d)}$ the number of (Morse) critical points of l on X_{reg} and by $\alpha_{X,a}^{(d)}$ those not occuring on $l^{-1}(a)$. In this case, if c is a regular value of l then $\mathrm{Eu}^X_{l,c} = (-1)^d \alpha_X^{(d)}$ and if a is a critical value of l, then $\mathrm{Eu}^X_{l,a} = (-1)^d \alpha_{X,a}^{(d)}$. By the relation between $\mathrm{B}^X_{l,a}$ and $\mathrm{Eu}(l^{-1}(a))$, we obtain

$$\mathrm{Eu}(X) - \mathrm{Eu}(l^{-1}(a)) = (-1)^d \alpha_{X,a}^{(d)} + \sum_{j \ \mid \ l(q_j)=a} \mathrm{Eu}_X(q_j) - \mathrm{Eu}_{l^{-1}(a)}(q_j),$$

where the q_j's are the critical points of l. For a regular value c of l, this gives

$$\mathrm{Eu}(X) - \mathrm{Eu}(l^{-1}(c)) = (-1)^d \alpha_X^{(d)},$$

and we remark that we have recovered Equality (2), page 401 in [STV]. Based on this equality, Seade, Tibăr and Verjovsky could express the global Euler obstruction as an alternating sum of global polar invariants. In the sequel, we will establish a relative version of this result for the global Brasselet number. Such results are also proved for the Brasselet numbers at infinity in [DG3].

We consider a polynomial function $f : X \to \mathbb{C}$, restriction to X of a polynomial function $F : \mathbb{C}^N \to \mathbb{C}$. We assume that f has a finite number of critical points, which means that for $i = 1, \dots, t$, $F_{|V_i}$ has a finite number of critical points. We denote by $\{q_1, \dots, q_s\}$ the set of critical points of f. For $a \in \mathbb{C}$, we put $X_a = f^{-1}(a)$. The algebraic set X_a is equidimensional and if $q_1, \dots, q_u$, $u \leq s$, are the critical points of f on $f^{-1}(a)$, then

$$\mathcal{V}_a = \big(\sqcup_{i=1}^t V_i \cap f^{-1}(a) \setminus \{q_1, \dots, q_u\}\big) \cup \big(\sqcup_{j=1}^u \{q_j\}\big)$$

is a Whitney stratification of X_a.

Let $L : \mathbb{C}^N \to \mathbb{C}$ be a linear function and let $l : X \to \mathbb{C}$ be its restriction to X. We denote by $\Gamma^X_{f,l}$ the relative polar variety of f and l defined as follows:

$$\Gamma^X_{f,l} = \overline{\{x \in X_{\mathrm{reg}} \mid \mathrm{rank}[df(x), dl(x)] < 2\}}.$$

It is well-known that for L generic, $\Gamma^X_{f,l}$ is a reduced algebraic curve. Moreover if L is generic, we can assume the following fact:

$$l_{|X_a} : X_a \to \mathbb{C} \text{ is } \rho\text{-regular at infinity and Morse stratified.}$$

Let $I_X(\Gamma^X_{f,l}, X_a)$ be the global intersection multiplicity of $\Gamma^X_{f,l}$ and X_a, namely

$$I_X(\Gamma^X_{f,l}, X_a) = \sum_{p \in \Gamma^X_{f,l} \cap f^{-1}(a)} I_p(\Gamma^X_{f,l}, X_a),$$

where $I_p(\Gamma^X_{f,l}, X_a)$ is the local intersection multiplicity of $\Gamma^X_{f,l}$ and X_a at p. If $\dim(X) = 1$ then $\Gamma^X_{f,l} = X$ and in this case $I_p(\Gamma^X_{f,l}, X_a)$ is the degree of $l : (X, p) \to (\mathbb{C}, a)$, that is the cardinality of $l^{-1}(c) \cap X \cap B_\epsilon(p)$ for $0 < |c - a| \ll \epsilon \ll 1$.

Proposition 10.17. *We have*

$$\mathrm{B}^X_{f,a} - \mathrm{B}^{X\cap H}_{f,a} = (-1)^{d-1} I_X(\Gamma^X_{f,l}, X_a) + \sum_{j=1}^{u} \mathrm{Eu}_{f,X}(q_j),$$

where H is a generic hyperplane given by $H = L^{-1}(g)$ for a regular value g of $l_{|X_a}$ and $l_{|X}$.

By a standard connectivity argument, $I_X(\Gamma^X_{f,l}, X_a)$ does not depend on the choice of the generic linear function L. We denote it by $\gamma^{(d-1)}_{X,a}$. Similarly for $i = 2, \ldots, d$, we define

$$\gamma^{(d-i)}_{X,a} = I_{X\cap H^{i-1}}(\Gamma^{X\cap H^{i-1}}_{f,l}, X_a \cap H^{i-1}),$$

where H^{i-1} is a generic linear space of codimension $i - 1$.

The following statement is a relative version of the Seade-Tibăr-Verjovsky polar formula for the global Euler obstruction.

Corollary 10.18. *We have*

$$\mathrm{B}^X_{f,a} = \sum_{i=1}^{d} (-1)^{d-i} \gamma^{(d-i)}_{X,a} + \sum_{j=1}^{u} \mathrm{Eu}_{f,X}(q_j).$$

Another corollary is a characterization of the Brasselet numbers at infinity in terms of critical points of generic linear forms (see [DG3]).

In [San] it is shown that the Brasselet number of a function f with nonisolated singularities describes numerically the topological information of its generalized Milnor fibre. Using the Brasselet number, H. Santana provides several formulae for germs $f : (X, 0) \to (\mathbb{C}, 0)$ and $g : (X, 0) \to (\mathbb{C}, 0)$ in the case where g has a one-dimensional critical locus. The author also gives applications when f has isolated singularities and when it is a generic linear form.

11. The Euler obstruction of a map and the Chern number of collections of forms

The Euler obstruction of a map, defined by Grulha [Gru1], and the Chern number of collections of forms, defined by Ebeling and Gusein-Zade [EG3], were defined in the first decade of the 21st century and, in [BGR], Brasselet, Grulha and Ruas related these two invariants. The Euler obstruction of a map is a generalization of the Euler obstruction of a function, defined in [BMPS].

The Chern number of collections of forms, defined by Ebeling and Gusein-Zade in [EG3], has a foundation similar to the Euler obstruction of maps; however it has travelled a different trajectory "avoiding" cellular decompositions, which are, in some sense, covered by the study of the loci of collections of forms. That is another way to present a generalization of the local Euler obstruction and, in [GG], Gaffney and Grulha present an algebraic treatment to study the Chern number.

Since the Euler obstruction of a function-germ g at the origin gives important topological information about g, more precisely, it counts the number of Morse points on the regular part of a generic perturbation of g, a natural question has been raised: which kind of topological information could be encoded by the Euler obstruction of a map and how is this invariant related with the Euler obstruction of the coordinate functions of f.

The notion of local Chern obstruction extends the notion of local Euler obstruction in the case of collections of germs of 1-forms. More precisely, Ebeling and Gusein-Zade perform the following construction.

Let $(X,0)\subset(\mathbb{C}^n,0)$ be the germ of a d-equidimensional reduced complex analytic variety at the origin. Let $\{\omega_j^{(i)}\}$ be a collection of germs of 1-forms on $(\mathbb{C}^n,0)$ such that $i=1,\dots,s$; $j=1,\dots,d-k_i+1$, where the k_i are non-negative integers with $\sum_{i=1}^s k_i=d$. Let $\varepsilon>0$ be small enough so that there is a representative X of the germ $(X,0)$ and representatives $\{\omega_j^{(i)}\}$ of the germs of 1-forms inside the ball $B_\varepsilon(0)\subset\mathbb{C}^n$.

Definition 11.1. *For a fixed i, the locus of the subcollection $\{\omega_j^{(i)}\}$ is the set of points $x\in X$ such that there exists a sequence x_n of points from the non-singular part X_{reg} of the variety V such that the sequence $T_{x_n}X_{\mathrm{reg}}$ of the tangent spaces at the points x_n has a limit L (in $G(d,n)$) and the restrictions of the 1-forms $\omega_1^{(i)},\dots,\omega_{d-k_i+1}^{(i)}$ to the subspace $L\subset T_x\mathbb{C}^n$ are linearly dependent.*

Definition 11.2. *A point $x\in X$ is called a special point of the collection $\{\omega_j^{(i)}\}$ if it is in the intersection of the loci of the subcollections $\{\omega_j^{(i)}\}$ for each $i=1,\dots,s$. The collection $\{\omega_j^{(i)}\}$ of 1-forms has an isolated special point at $\{0\}$ if it has no special point on X in a punctured neighbourhood of the origin.*

Note that in Definition 11.2, for each fixed i, we require each subcollection $\{\omega_j^{(i)}\}$ to be linearly dependent when restricted to the same limit plane. Also note that if an element of the collection has less than maximal rank at a point, then it is linearly dependent on all planes passing through the point.

Let $\{\omega_j^{(i)}\}$ be a collection of germs of 1-forms on $(X,0)$ with an isolated special point at the origin. Let $\nu:\tilde{X}\to X$ be the Nash transformation of the variety X and $\tilde{T}$ be the Nash bundle. The collection of 1-forms $\{\omega_j^{(i)}\}$ gives rise to a section $\Gamma(\omega)$ of the bundle

$$\tilde{\mathbb{T}}=\bigoplus_{i=1}^{s}\bigoplus_{j=1}^{d-k_i+1}\tilde{T}^*_{i,j},$$

where $\tilde{T}^*_{i,j}$ are copies of the dual Nash bundle $\tilde{T}^*$ over the Nash transformation $\tilde{X}$.

Let $\mathbb{D}\subset\tilde{\mathbb{T}}$ be the set of pairs $(x,\{\alpha_j^{(i)}\})$ where $x\in\tilde{X}$ and the collection of 1-forms $\{\alpha_j^{(i)}\}$ is such that $\alpha_1^{(i)},\dots,\alpha_{n-k_i+1}^{(i)}$ are linearly dependent for each $i=1,\dots,s$.

Definition 11.3. *Let 0 be a special point of the collection $\{\omega_j^{(i)}\}$. The local Chern obstruction $\mathrm{Ch}_{X,0}\{\omega_j^{(i)}\}$ of the collection of germs of 1-forms $\{\omega_j^{(i)}\}$ on $(X,0)$ at the origin is the obstruction to extend the section $\Gamma(\omega)$ of the fibre bundle $\tilde{\mathbb{T}}\setminus\mathbb{D}\to\tilde{X}$ from $\nu^{-1}(X\cap S_\varepsilon)$ to $\nu^{-1}(X\cap B_\varepsilon)$.*

It is easy to see that the correct obstruction dimension for which the first non-zero homotopy group appears [Ste] is d in our setting.

The following result is a consequence of [EG3, Proposition 3.3].

Proposition 11.4. Let $(X,0)\subset(\mathbb{C}^n,0)$ be the germ of a d-equidimensional reduced complex analytic variety at the origin. Let $\{\omega_j^{(i)}\}$ be a collection of germs of 1-forms on $(\mathbb{C}^n,0)$ such that $i=1,\dots,s$; $j=1,\dots,d-k_i+1$, where the k_i are non-negative integers with $\sum_{i=1}^s k_i=d$. Let 0 be an isolated special point for the collection. If $\omega^{(i)}$, $i=2,\dots,s$, are generic collections of linear forms, then the number $\mathrm{Ch}_{X,0}\{\omega_j^{(i)}\}$ does not depend on the choice of the subcollections $\omega^{(i)}$, $i = 2, \dots, s$.

In [Gru1] Grulha defined a notion of the Euler obstruction of a map, but in his construction the obstruction depends on a certain cellular decomposition. Later, in [BGR], Brasselet, Grulha and Ruas compared the notion of the Euler obstruction of a map and the Chern number, defined by Ebeling and Gusein-Zade [EG3]. In that paper they also proved that the Euler obstruction does not depend on a generic choice in its construction. Based on this, we define the Euler obstruction of a map in terms of collection of forms.

Definition 11.5. *Let X be an equidimensional complex variety of dimension d, $f : (X, 0) \to \mathbb{C}^p$, a holomorphic map, with $0 \le p \le d$ and $\omega_1 = \{df_1, df_2, ..., df_p\}$, with df_i the differential of the coordinate functions of f, and ω_2 a generic collection, in such a way that 0 is a special point of the collection of collections $\omega = \{\omega_1, \omega_2\}$. We define the Euler obstruction of the map f at the origin, denoted by $Eu^*_{f,X}(0) = \mathrm{Ch}_{X,0}\{\omega_j^{(i)}\}$.*

Part 3. The local Euler obstruction in São Carlos

The Local Euler obstruction appears to be an important research topic for all "singularists" and not only singularity researchers. Due to the particular relationship with other invariants it is a promising area of research leading to exciting and interesting results.

In particular, the local Euler obstruction is a subject of intensive studies in Brazil and more precisely in the team of São Carlos and its collaborators.

Celebrating 30 years of the International Workshop on Real and Complex Singularities is the occasion to mention the obstructionist club of São Carlos and their collaborators.

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