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CRITICAL PRINCIPAL SINGULARITIES OF HYPERSURFACES IN EUCLIDEAN 4-SPACES

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Abstract. In this work will be described the principal foliations of oriented three-dimensional manifolds immersed in Euclidean 4-spaces, near the partially umbilic set when this set is not regular. The two cases considered are generic in one parameter families of immersions and the unfolding also will be analyzed.

1. Introduction

The study of the patterns of principal curvature lines in a neighborhood of an umbilic point of a surface in $\mathbb{R}^3$, at which its principal curvatures coincide, i.e. $k_1 = k_2$, in standard notation, goes back to works of Monge, Dupin, Cayley, Darboux, Gullstrand, among others. See [2], [4], [18], and [24]. In 1982, Gutierrez and Sotomayor introduced into the subject ideas of Structural Stability, Genericity and Bifurcation, proceeding from Dynamical Systems. There they established initial results concerning the *principal configurations* on surfaces. See [8, 9, 10, 11, 12, 22]. An expository account about recent developments on the qualitative analysis of principal curvature lines can be found in [8, 19, 20].

For the case of hypersurfaces immersed in $\mathbb{R}^4$ there are two types of principal singularities: the umbilic points, where the three principal curvatures coincide, in standard notation $k_1 = k_2 = k_3$, and the *partially umbilic points*, where only two principal curvatures coincide. Generically there are no umbilic points and the set of partially umbilic points, when non empty, occurs along regular curves. The geometric and analytic characterization of the patterns of principal curvature foliations near generic partially umbilic points was achieved in [5, 6] and [15].

Referring to a property, in the present work the term *generic* means that it is valid for an open and dense set.

For the case of surfaces in $\mathbb{R}^3$, the bifurcations of umbilic points (principal singularities) in generic one parameter families of immersions was carried out in [7].

For hypersurfaces in $\mathbb{R}^4$ there are five types of generic principal singularities: D_1, D_2, D_3 (Figure 1), D_{12} (Figure 2) and D_{23}, (Figure 3). See [15].

For a global description of principal foliations on tridimensional quadric hypersurfaces the reader is addressed to [16] and [21].

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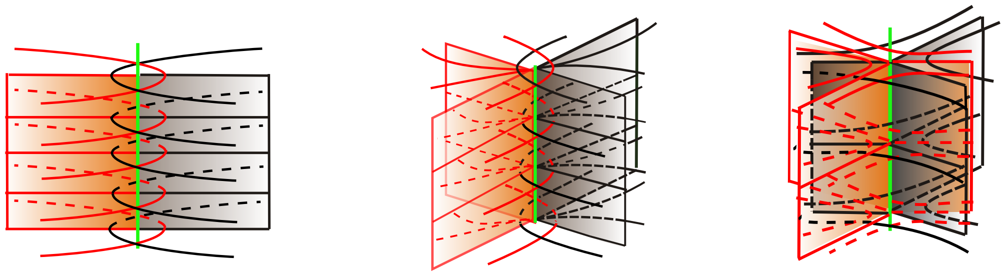

FIGURE 1. Principal foliations $\mathcal{F}_1(\alpha)$ and $\mathcal{F}_2(\alpha)$ in a neighborhood of partially umbilic points D_1, left, D_2, center, and D_3, right.

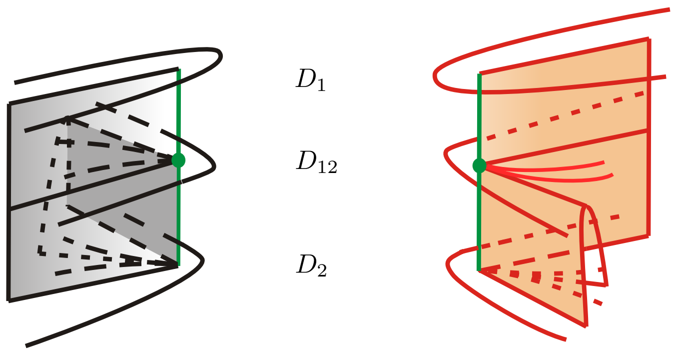

FIGURE 2. Principal foliations $\mathcal{F}_1(\alpha)$ and $\mathcal{F}_2(\alpha)$ in a neighborhood of partially umbilic point D_{12}, transition of arcs D_1 and D_2.

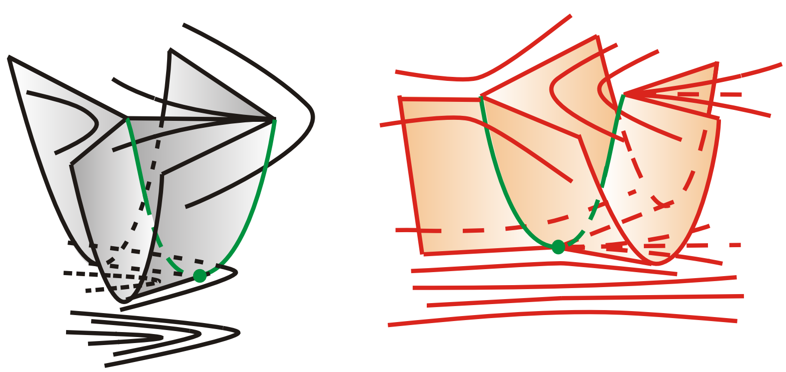

FIGURE 3. Principal foliations $\mathcal{F}_1(\alpha)$ and $\mathcal{F}_2(\alpha)$ in a neighborhood of partially umbilic point D_{23}, transition of arcs D_2 and D_3.

The purpose of this paper is to characterize the principal foliations around principal curvature singularities when the regularity properties of the partially umbilic curve fails. The mildest patterns for such failure will be studied here.

In Section 2 the main results are stated about the qualitative behavior of the principal curvature lines around the partially umbilic points $D^1_{23\bullet}$ and $D^1_{23\times}$.

In Section 3 will be reviewed essential analytic preliminaries used in the proofs of the main results of this work, formulated in Section 2.

In Section 5 the bifurcations of the partially umbilic points $D^1_{23\bullet}$ and $D^1_{23\times}$ will be analyzed.

In Section 6 are reviewed the coordinate expressions of the geometric functions used in this work.

The proofs of the main results of this work are given in Section 4 and Section 5.

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2. Main Results

In this section the main results describing the local behavior of principal curvature lines near critical singularities of the partially umbilic set will be stated.

Let $\mathbf{p} \in \mathbb{M}^3$ be a partially umbilic point of an immersion $\alpha : \mathbb{M}^3 \to \mathbb{R}^4$ such that

$$k_1(\mathbf{p}) = k_2(\mathbf{p}) = k(\mathbf{p}) < k_3(\mathbf{p}).$$

That is $\mathbf{p} \in \mathcal{S}_{12}(\alpha)$.
Let $(u_1, u_2, u_3) : \mathbb{M}^3 \to \mathbb{R}^3$ be a local Monge chart such that:

$$\alpha(u_1, u_2, u_3) = (u_1, u_2, u_3, h(u_1, u_2, u_3)),$$

where:

$$\begin{aligned} h(u_1,u_2,u_3) &= \tfrac{1}{2}k(u_1^2+u_2^2) + \tfrac{1}{2}k_3u_3^2 \\ &\quad + h_3(u_1,u_2,u_3) + h_4(u_1,u_2,u_3) + h.o.t., \text{ with} \\ h_3(u_1,u_2,u_3) &= \tfrac{1}{6}au_1^3 + \tfrac{1}{2}bu_1u_2^2 + \tfrac{1}{6}cu_2^3 + \tfrac{1}{6}q_{003}u_3^3 + \tfrac{1}{2}q_{012}u_2u_3^2 + q_{111}u_1u_2u_3 \\ &\quad + \tfrac{1}{2}q_{021}u_2^2u_3 + \tfrac{1}{2}q_{102}u_1u_3^2 + \tfrac{1}{2}q_{201}u_1^2u_3 \\ h_4(u_1,u_2,u_3) &= \tfrac{A}{24}u_1^4 + \tfrac{B}{6}u_1^3u_2 + \tfrac{C}{4}u_1^2u_2^2 + \tfrac{D}{6}u_1u_2^3 + \tfrac{E}{24}u_2^4 + \tfrac{Q_{004}}{24}u_3^4 \\ &\quad + \tfrac{Q_{013}}{6}u_2u_3^3 + \tfrac{Q_{103}}{6}u_1u_3^3 + \tfrac{Q_{022}}{4}u_2^2u_3^2 + \tfrac{Q_{202}}{4}u_1^2u_3^2 + \tfrac{Q_{112}}{2}u_1u_2u_3^2 \\ &\quad + \tfrac{Q_{031}}{6}u_2^3u_3 + \tfrac{Q_{301}}{6}u_1^3u_3 + \tfrac{Q_{121}}{2}u_1u_2^2u_3 + \tfrac{Q_{211}}{2}u_1^2u_2u_3 \end{aligned} \tag{1}$$

Notice that in this presentation, the coefficient of the term $u_1^2u_2$ does not appear. It has been eliminated by a rotation.

2.1. **Principal foliations near the critical partially umbilic points.** Consider the Monge chart given by Equation (1).

Definition 1 (Partially umbilic points $D^1_{23\bullet}$ and $D^1_{23\times}$)**.** *The partially umbilic point* $\mathbf{p}$ *of* α *is called* $D^1_{23\times}$ *if* $b = q_{111} = 0$, $ac(q_{021} - q_{201}) = ac\delta \neq 0$ *and* $\zeta < 0$, *see Equation (29) in Section 7. It is called* $D^1_{23\bullet}$ *if* $b = q_{111} = 0$, $ac(q_{021} - q_{201}) = ac\delta \neq 0$ *and* $\zeta > 0$.

We notice that condition $b = q_{111} = 0$ is an intrinsic geometric condition, i.e., it is independent of coordinates.

It can be expressed geometrically by saying that the functions $L_r(u,v,w)$ and $M_r(u,v,w)$ defining the partially umbilic set, see definition in Equation (7), are quasi transversal at 0. This means that the germ map $F:(\mathbb{R}^3,0)\to(\mathbb{R}^2,0)$, $F(u,v,w)=(L_r(u,v,w),M_r(u,v,w))$ has rank 1 at 0 and F and is equivalent to a germ of the form $(u, v^2 \pm w^2)$, or equivalently, F has Boardman symbol $\Sigma^{2,0}$, see [17, pag. 83].

Under the conditions in Definition 1, it follows that $M_r(u,v,w)=0$ defines a regular surface and $L_r(0)=\nabla L_r(0)=0$. The contact between them is of Morse saddle (resp. center) type when $\zeta<0$ (resp. $\zeta>0$). Another equivalent condition to characterize critical singularities is expressed by: $a=b, a(q_{201}-q_{021})+cq_{111}=0$.

Theorem 1. *Let $\mathbf{p}\in\mathcal{S}_{12}$ be a partially umbilic point of α and of type $D^1_{23\bullet}$. Then the behavior of the principal foliations $\mathcal{F}_1(\alpha)$ and $\mathcal{F}_2(\alpha)$ in the neighborhood of $\mathbf{p}$ are as illustrated in Figure 4 and described below.*

- *i) The partially umbilic set consists only of the point $\mathbf{p}$.*
- *ii) The principal foliation $\mathcal{F}_1(\alpha)$ has two leaves asymptotic to the partially umbilic point and all other leaves are regular curves.*
- *iii) The principal foliation $\mathcal{F}_2(\alpha)$ has a two-dimensional funnel whose leaves are asymptotic to the partially umbilic point and all all other leaves are regular curves.*
- *iv) There exists a local surface $W_1(\mathbf{p})$ (non unique) passing through the partially umbilic point $\mathbf{p}$ and $W_1(\mathbf{p})$ is invariant by one principal foliation $\mathcal{F}_2(\alpha)$. Moreover, there exists a leaf $F_1\subset W_1(\mathbf{p})$ of $\mathcal{F}_2(\alpha)$ which is asymptotic to $\mathbf{p}$.*

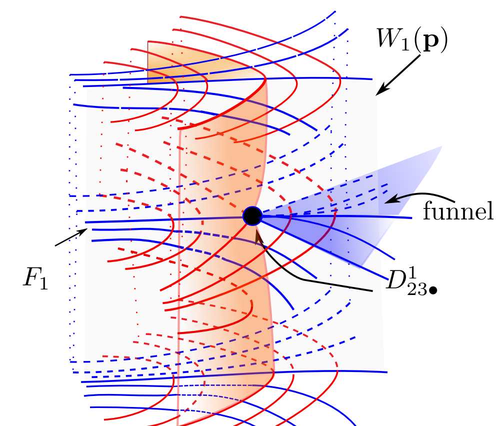

FIGURE 4. Principal foliations $\mathcal{F}_1(\alpha)$ and $\mathcal{F}_2(\alpha)$ in a neighborhood of a partially umbilic point $D^1_{23\bullet}$.

Theorem 2. *Let $\mathbf{p}$ be a partially umbilic point of type $D^1_{23\times}$. Then the behavior of the principal foliations of $\mathcal{F}_1(\alpha)$ and $\mathcal{F}_2(\alpha)$ in the neighborhood of $\mathbf{p}$ is as illustrated in Figure 5. More precisely,*

- *i) The partially umbilic set consists of two regular curves which are quasi transversal at $\mathbf{p}$.*
- *ii) The principal foliation $\mathcal{F}_1(\alpha)$ has two arcs of D_2 points and other two arcs of D_3 points (Figure 5) and a there are two three-dimensional wedge sectors whose leaves are asymptotic to the partially umbilic set. The invariant surfaces shown in Figure 5 are fibered by principal lines.*

iii) *The principal foliation $\mathcal{F}_2(\alpha)$ has two arcs of partially umbilic points of type D_2 and other two arcs of type D_3 (Figure 5). There is a three-dimensional wedge sector whose leaves are asymptotic to the partially umbilic set. The invariant surfaces shown in Figure 5 are fibered by principal lines.*

iv) *There exists a local surface $W_1(\mathcal{S}_{12})$ containing the partially umbilic set $\mathcal{S}_{12}$ with the following property: one of the connected components of $W_1(\mathcal{S}_{12}) \setminus \mathcal{S}_{12}$ is invariant by $\mathcal{F}_1(\alpha)$ and the other is invariant by $\mathcal{F}_2(\alpha)$.*

Moreover $F_1 \subset W_1$ is a leaf of $\mathcal{F}_2(\alpha)$ asymptotic to the $D^1_{23\times}$ partially umbilic point $\mathbf{p}$.

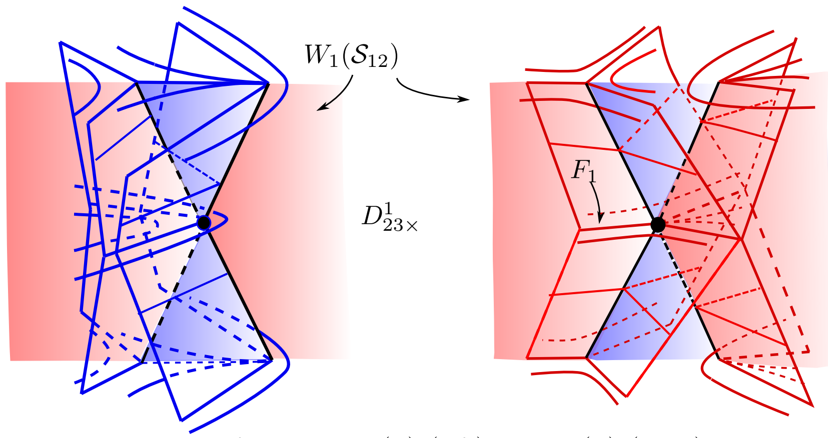

FIGURE 5. Principal foliations $\mathcal{F}_1(\alpha)$ (left) and $\mathcal{F}_2(\alpha)$ (right) in a neighborhood of a partially umbilic point $D^1_{23\times}$.

3. Preliminaries on Partially Umbilic Points

In this section, we will review the techniques developed to describe the local behavior of curvature lines near Darbouxian and semi-Darbouxian partially umbilic points. For more details see [15].

Let $\mathbb{M}^3$ be a C^∞, oriented, compact, $3-$dimensional manifold.

An immersion α of $\mathbb{M}^3$ into $\mathbb{R}^4$ is a map such that $D\alpha_p : T\mathbb{M}^3_p \to \mathbb{R}^4$ is one to one, for every $p \in \mathbb{M}^3$. Denote by $\mathrm{Imm}^k(\mathbb{M}^3, \mathbb{R}^4)$ the set of C^k - immersions of $\mathbb{M}^3$ into $\mathbb{R}^4$ endowed with the C^k-topology, see [14].

Associated to every $\alpha \in \mathrm{Imm}^k(\mathbb{M}^3, \mathbb{R}^4)$ is defined the normal map $\mathcal{N}_\alpha : \mathbb{M}^3 \to \mathbb{S}^3$:

$$\mathcal{N}_\alpha = (\alpha_1 \wedge \alpha_2 \wedge \alpha_3) \,/\, |\ \alpha_1 \wedge \alpha_2 \wedge \alpha_3\ |,$$

where $(u_1, u_2, u_3) : (M, p) \to (\mathbb{R}^3, 0)$ is a positive chart of $\mathbb{M}^3$ around $\mathbf{p}$, $\wedge$ denotes the wedge product of vectors determined by a once for all fixed orientation in $\mathbb{R}^4$. This space is endowed with the Euclidean norm $|\cdot| = \langle \cdot, \cdot \rangle^{\frac{1}{2}}$. Also $\alpha_1 = \dfrac{\partial \alpha}{\partial u_1}$, $\alpha_2 = \dfrac{\partial \alpha}{\partial u_2}$, $\alpha_3 = \dfrac{\partial \alpha}{\partial u_3}$, $\dfrac{\partial^2 \alpha}{\partial u_i \partial u_j} = \alpha_{ij}$.

Clearly, $\mathcal{N}_\alpha$ is well defined and of class C^{k-1} in $\mathbb{M}^3$.

Since $D\mathcal{N}_\alpha(p)$ has its image contained in that of $D\alpha(p)$, the endomorphism $\omega_\alpha : T\mathbb{M}^3 \to T\mathbb{M}^3$ is well defined by

$$D\alpha.\omega_\alpha = D\mathcal{N}_\alpha.$$

It is well known that ω_α is a self adjoint endomorphism, when $T\mathbb{M}^3$ is endowed with the metric $\langle \cdot, \cdot \rangle_\alpha$ induced by α pulling back the metric in $\mathbb{R}^4$. See [23].

The opposite values of the eigenvalues of ω_α are called *principal curvatures* of α and will be denoted by $k_1 = k_1(\alpha) \le k_2 = k_2(\alpha) \le k_3 = k_3(\alpha)$.

The *principal singularities* of the immersion α are defined as follows:

- *Umbilic Points*: $\mathcal{U}_\alpha = \{p \in \mathbb{M}^3 : k_1(p) = k_2(p) = k_3(p)\}$,
- *Partially Umbilic Points*: $\mathcal{S}_\alpha = \mathcal{S}_{12}(\alpha) \cup \mathcal{S}_{23}(\alpha)$, where
- $\mathcal{S}_{12}(\alpha) = \{p \in \mathbb{M}^3 : k_1(p) = k_2(p) < k_3(p)\}$,
- $\mathcal{S}_{23}(\alpha) = \{p \in \mathbb{M}^3 : k_1(p) < k_2(p) = k_3(p)\}$.

The eigenspaces associated to the principal curvatures, when simple, define three line fields $\mathcal{L}_i(\alpha)$, $(i = 1, 2, 3)$ mutually orthogonal in $T\mathbb{M}^3$ (endowed with the metric $\langle \cdot, \cdot \rangle_\alpha$), called *principal line fields* of α. They are characterized by Rodrigues's equations (see [23]):

$$\{v \in T\mathbb{M}^3 : \omega_\alpha v + k_i v = 0,\ (i = 1,\ 2,\ 3)\}.$$

These line fields are well defined and smooth outside their respective sets of *principal singularities*, as follows: $\mathcal{L}_1(\alpha)$ is of class C^{k-2} outside $\mathcal{U}_\alpha \cup \mathcal{S}_{12}(\alpha)$, $\mathcal{L}_3(\alpha)$ is of class C^{k-2} outside $\mathcal{U}_\alpha \cup \mathcal{S}_{23}(\alpha)$, $\mathcal{L}_2(\alpha)$ is of class C^{k-2} outside $\mathcal{U}_\alpha \cup \mathcal{S}_{12}(\alpha) \cup \mathcal{S}_{23}(\alpha)$.

This follows from the smooth dependence of simple eigenvalues and corresponding one-dimensional eigenspaces.

The foliations by integral curves of $\mathcal{L}_i$, $(i = 1, 2, 3)$ are called the *principal foliations* $\mathcal{F}_i(\alpha)$ of α.

In a local chart $(u_1, u_2, u_3) : M^3 \to \mathbb{R}^3$ the first and the second fundamental forms associated to the immersion α are given, respectively, by $I_\alpha = \sum g_{ij} du_i du_j$ and $II_\alpha = \sum \lambda_{ij} du_i du_j$, where $g_{ij} = \langle \alpha_i, \alpha_j \rangle$, $\lambda_{ij} = \langle \alpha_{ij}, \mathcal{N}_\alpha \rangle$ and $\mathcal{N}_\alpha$ is the unit positive normal vector to the immersion α.

According to [15], see also [4], the differential equations of lines of curvature of $\mathcal{F}_i(\alpha)$ in terms of the fundamental forms in matrix form are given by

$$(\Lambda - k_i G)du = 0,\ du = (du_1, du_2, du_3)^T,\ (i = 1,\ 2,\ 3) \tag{2}$$

where $\Lambda = (\lambda_{ij})$ is the second fundamental form $G = (g_{ij})$ is the first fundamental form and k_i $(i = 1, 2, 3)$ are the principal curvatures.

Consider the plane $\mathcal{P}_3(q)$ passing through $q \in M$ having the principal direction $e_3(q)$ (associated to $k_3(q)$) as normal vector:

$$\begin{aligned} \mathcal{P}_3(q) = &\{(du_1, du_2, du_3); \langle (du_1, du_2, du_3), G \cdot (e_3(q))^T \rangle = 0\} \\ =&\{(du_1, du_2, du_3) : \omega_3(du_1, du_2, du_3) = 0\}, with \\ \omega_3 = &[g_{11}U_1 + g_{12}V_1 + g_{13}W_1]du_1 + [g_{12}U_1 + g_{22}V_1 + g_{23}W_1]du_2 \\ &+ [g_{13}U_1 + g_{23}V_1 + g_{33}W_1]du_3\}, \end{aligned} \tag{3}$$

where

$$\begin{aligned} U_1 &= (g_{12}g_{23} - g_{22}g_{13})\, k_3^2 + (-g_{12}\lambda_{23} - g_{23}\lambda_{12} + \lambda_{22}g_{13} + g_{22}\lambda_{13})\, k_3 \\ &\quad + \lambda_{23}\lambda_{12} - \lambda_{22}\lambda_{13} \\ V_1 &= (-g_{11}g_{23} + g_{13}g_{12})\, k_3^2 + (\lambda_{11}g_{23} + g_{11}\lambda_{23} - \lambda_{13}g_{12} - g_{13}\lambda_{12})\, k_3 \\ &\quad + \lambda_{13}\lambda_{12} - \lambda_{11}\lambda_{23} \\ W_1 &= \left(g_{11}g_{22} - g_{12}^2\right) k_3^2 + (-\lambda_{11}g_{22} - g_{11}\lambda_{22} + 2\lambda_{12}g_{12})\, k_3 + +\lambda_{11}\lambda_{22} - \lambda_{12}^2. \end{aligned} \tag{4}$$

The equation $\omega_3 = 0$, under suitable conditions fulfilled here, will be also written as

$$du_3 = \mathcal{U} du_1 + \mathcal{V} du_2. \tag{5}$$

Let E_r, F_r, G_r and e_r, f_r, g_r be the coefficients of the first and second fundamental forms of α restricted to $\mathcal{P}_3(q)$. That is:

$$I_r(du_1, du_2) = I_\alpha\Big|_{\mathcal{P}_3(q)} = E_r du_1^2 + 2F_r du_1 du_2 + G_r du_2^2,$$
$$II_r(du_1, du_2) = II_\alpha\Big|_{\mathcal{P}_3(q)} = e_r du_1^2 + 2f_r du_1 du_2 + g_r du_2^2.$$

Here it is assumed that, in the chart (u_1, u_2, u_3), it holds: $g_{13}U_1 + g_{23}V_1 + g_{33}W_1 \neq 0$.

The principal directions $e_1(q)$ and $e_2(q)$ associated to $k_1(q)$ and $k_2(q)$ belonging to the plane $\mathcal{P}_3(q)$ (the kernel of ω_3) are defined by the implicit differential equation

$$\begin{aligned} &(F_r g_r - f_r G_r)P^2 + (E_r g_r - e_r G_r)P + E_r f_r - e_r F_r = 0,\ P = du_2/du_1 \\ &\omega_3(du_1, du_2, du_3) = U_1(u_1, u_2, u_3)du_1 + V_1(u_1, u_2, u_3)du_2 \\ &\qquad + W_1(u_1, u_2, u_3)du_3 = 0 \end{aligned} \tag{6}$$

written in a chart (u_1, u_2, u_3).

This follows from the fact that the quadratic form in Equation (6) is the Jacobian of the forms $I_r(du_1, du_2)$ and $II_r(du_1, du_2)$. Here we use the equivalence between principal directions as eigendirections of the derivative of the normal map and as the extremal directions of the normal curvature. See [4], [23], [24] .

Define,

$$L_r = F_r g_r - f_r G_r, \quad M_r = E_r g_r - e_r G_r, \quad N_r = E_r f_r - e_r F_r. \tag{7}$$

Remark 1. *The partially umbilic points ($k_1 = k_2 \neq k_3$) are defined by $L_r(u_1, u_2, u_3) = 0$ and $M_r(u_1, u_2, u_3) = 0$.*

We say that the partially umbilic set of α is **regular** (resp. **critical**) if 0 is a regular (resp. not a regular) value of (L_r, M_r).

In the (u_1, u_2, u_3, p)-space consider the hypersurface (variety)

$$\mathcal{L} = \{(u_1, u_2, u_3, p) :\ \mathcal{L}(u_1, u_2, u_3; p) = 0\} \tag{8}$$

where

$$\mathcal{L}(u_1, u_2, u_3; p) = L_r p^2 + M_r p + N_r,$$

called *Lie Cartan hypersurface* of the principal curvature lines orthogonal to $\mathcal{P}_3$.

Proposition 1. *Consider the vector field*

$$X = X_{\mathcal{L}} = X_1 \frac{\partial}{\partial u_1} + X_2 \frac{\partial}{\partial u_2} + X_3 \frac{\partial}{\partial u_3} + X_4 \frac{\partial}{\partial p}, \tag{9}$$

where:

$$X_1 = \frac{\partial \mathcal{L}}{\partial p},\ X_2 = p\frac{\partial \mathcal{L}}{\partial p},\ X_3 = (\mathcal{U} + p\mathcal{V})\frac{\partial \mathcal{L}}{\partial p},\ X_4 = -\left(\frac{\partial \mathcal{L}}{\partial u_1} + p\frac{\partial \mathcal{L}}{\partial u_2} + \frac{\partial \mathcal{L}}{\partial u_3}(\mathcal{U} + p\mathcal{V})\right),$$

$\mathcal{U} = \frac{U_1}{W_1}$ *and* $\mathcal{V} = \frac{V_1}{W_1}$.

Then, X is of class C^{k-3}, it is tangent to $\mathcal{L}$ and the projections of the integral curves of X by $\pi(u_1, u_2, u_3, p) = (u_1, u_2, u_3)$ are the principal lines of the two principal foliations $\mathcal{F}_1(\alpha)$ and $\mathcal{F}_2(\alpha)$ which are singular along the partially umbilic set $\mathcal{S}_{12}(\alpha)$.

Proof. See [15]. For the case of surfaces this construction was developed in [1]. □

Remark 2. *To cover the whole sub-bundle, of the tangent projective bundle over* $\mathbb{M}^3$*, defined by the lines orthogonal to* $\mathcal{P}_3$*, consider also the equation* $\mathcal{G} = L_r du_2^2 + M_r du_1 du_2 + N_r du_1^2 = 0$*, with the coordinate* $q = du_1/du_2$*. In this case, the Lie Cartan vector field is*

$$Y = (q\mathcal{G}_q)\frac{\partial}{\partial u_1} + (\mathcal{G}_q)\frac{\partial}{\partial u_2} + ((q\mathcal{U}+\mathcal{V})\,\mathcal{G}_q)\frac{\partial}{\partial u_3} - (q\mathcal{G}_{u_1} + \mathcal{G}_{u_2} + \mathcal{G}_{u_3}\,(q\mathcal{U}+\mathcal{V})))\frac{\partial}{\partial q}, \tag{10}$$

where $\mathcal{G}_q = \dfrac{\partial \mathcal{G}}{\partial q}$, $\mathcal{G}_{u_i} = \dfrac{\partial \mathcal{G}}{\partial u_i}(i = 1, 2, 3)$.

4. Principal foliations near critical partially umbilic points

In this section the proofs of Theorems 1 and 2 will be provided.

Proposition 2. *Suppose* $ac(q_{021} - q_{201}) = ac\delta \neq 0$*. There is a change of coordinates* $(u_1, u_2, u_3) = (u, v, w) + \varphi_2(u, v, w)$ *such that the differential equation of principal lines near a partially umbilic point* $D^1_{23\times}$ *or* $D^1_{23\bullet}$ *is given by*

$$\begin{aligned}\mathcal{L} =& (a_1u^2 + b_1v^2 + c_1w^2 + O_1(3))dv^2 + (-au + cv + \delta w + O_2(3))\, dudv\\ & -(a_1u^2 + b_1v^2 + c_1w^2 + O_3(3))du^2 = 0\\ dw =& (\sigma u + O_4(2))du + O_5(2)dv.\end{aligned} \tag{11}$$

The coefficients (a_1, b_1, c_1, σ) *are functions of the fourth jet of the functions* h *given in Equation (1).*

Proof. Consider a change of coordinates

$$\begin{aligned}u_1 =& u + a_{11}u^2 + a_{12}uv + a_{22}v^2 + a_{13}uw + a_{23}vw + a_{33}w^2 + O(3)\\ u_2 =& v + b_{11}u^2 + b_{12}uv + b_{22}v^2 + b_{13}uw + b_{23}vw + b_{33}w^2 + O(3)\\ u_3 =& w + c_{11}u^2 + c_{12}uv + c_{22}v^2 + c_{13}uv + c_{23}vw + c_{33}w^2 + O(3)\end{aligned}$$

Apply this to differential equation of principal lines (6) and to the functions $L_r = F_r g_r - G_r f_r$, $M_r = E_r g_r - G_r e_r$, $N_r = E_r f_r - F_r e_r$ and to the plane field $\omega_3 = 0$ given by equations in Section 6. Using symbolic computation and solving the linear homological equation system the result follows. Explicitly it is obtained that:

$$\begin{aligned}4(k-k_3)c\delta\, a_1 =& -2\,cB\,(k-k_3)(q_{021}-q_{201}) + aC\,(q_{021}-q_{201})\\ & + a^2(Q_{121} + acQ_{211}) + 2\,ak_3[(k-k_3)(q_{021}-q_{201}) + aq_{021}\,q_{102}\\ & + cq_{102}\,q_{201} + q_{021}\,q_{201}\,(q_{021}-q_{201})]\end{aligned}$$

$$\begin{aligned}4(k-k_3)a\delta\, b_1 =& -2\,c(k-k_3)\,(C\,q_{021} - C\,q_{201} - aQ_{121} - cQ_{211})\\ & -2\,(k-k_3)(q_{021}-q_{201})(aD - ck^3)\\ & +2\,c(aq_{021}\,q_{102} + cq_{102}\,q_{201}) - 2\,cq_{021}\,q_{201}\,(q_{021}-q_{201})\end{aligned}$$

$$\begin{aligned}4(k-k_3)ac\,c_1 =&2\,(k-k_3)C\,(q_{021}-q_{201})^2-acQ_{112}\\&+(\,q_{021}-q_{201})(aQ_{121}-aQ_{211})\\&-2\,k^3(q_{021}-q_{201})^2(k-k_3)-4\,acq_{012}\,q_{102}\\&+2\,(q_{021}-q_{201})(aq_{021}\,q_{102}-cq_{012}\,q_{201})+2\,q_{021}\,q_{201}\,(q_{021}-q_{201})^2\end{aligned}$$

$$\begin{aligned}4\delta^2(k-k_3)c^2\sigma =&-(k-k_3)(9\,(a^2-c^2)(q_{021}-q_{201})C+10\,ac(q_{021}-q_{201})B\\&-10\,ac(q_{021}-q_{201})D-4\,a^2(q_{021}-q_{201})E+9\,c(a^2+c^2)Q_{211}\\&+9\,a(a^2+c^2)Q_{121}+4\,c^2(q_{021}-q_{201})A)\\&-3\,k^3(a^2-c^2)(q_{021}-q_{201})(k-k_3)-9\,ac(aq_{012}\,q_{201}+cq_{021}\,q_{102})\\&+3\,a^2q_{021}\,(4\,q_{021}-3\,q_{201})(q_{021}-q_{201})-9\,a^3q_{021}\,q_{102}-9\,c^3q_{012}\,q_{201}\\&+3\,c^2q_{201}\,(q_{021}-q_{201})(3\,q_{021}-4\,q_{201})\end{aligned}$$

It is worth remarking that the coefficients (a_1,b_1,c_1,σ) are linear in the variables

$$(A,B,C,D,E,Q_{112},Q_{121},Q_{211}).$$

□

Lemma 1. *Let*

$$\zeta=b_1c_1a^2+a_1b_1\delta^2+a_1c_1c^2\neq 0 \tag{12}$$

i) *If $\zeta>0$ the singular set of Equation (11) is the origin.*
ii) *If $\zeta<0$ the singular set of Equation (11) is a pair of curves passing through the origin and intersecting quasi-transversally.*

Proof. The singular set of Equation (11) is given by

$$\begin{aligned}L_r(u,v,w)=&a_1u^2+b_1v^2+c_1w^2+O_1(3)=0\\M_r(u,v,w)=&-au+cv+\delta w+O_2(3)=0\end{aligned}$$

By the Implicit Function Theorem we can write $u=u(v,w)$ satisfying $M_r(u(v,w),v,w)=0$ and

$$\begin{aligned}L_2(v,w)=L_r(u(v,w),v,w)=&\frac{1}{a^2}((a^2b_1+a_1c^2)v^2\\&+2a_1c\delta vw+(a^2c_1+a_1\delta^2)w^2+O(3).\end{aligned}$$

The determinant of the Hessian of L_2 at the origin is $\frac{4}{a^2}\zeta$. So the result follows from Morse Lemma. □

Remark 3. *In the Monge chart (adapted by a rotation) we have that ζ is given by Equation (29) of Section 7.*

Proposition 3. *If* $\mathbf{p}$ *is a partially umbilic point of type $D^1_{23\bullet}$ then the partially umbilic set of α, in a neighborhood of* $\mathbf{p}$ *is only the point* $\mathbf{p}$. *This case is given by the condition $\zeta>0$.*

Proof. Direct from Lemma 1. □

Proposition 4. *Let* $\mathbf{p}$ *be a partially umbilic point of type $D^1_{23\times}$. Then the partially umbilic set of α, in the neighborhood of* $\mathbf{p}$, *is formed by two regular curves intersecting quasi- transversally at* $\mathbf{p}$. *This case is given by the condition $\zeta<0$.*

Proof. Direct from Lemma 1. □

Associated to the Equation (16) the Lie Cartan vector field $X_{\mathcal{L}} = (X_1, X_2, X_3, X_4)$, see Equation (9), in affine coordinates (u, v, w, p) is given by

$$
\begin{aligned}
u' =& X_1 = 2\,\left(a_1\, u^2 + b_1\, v^2 + c_1\, w^2\right) p - au + cv + \delta\, w + h.o.t. \\
v' =& X_2 = p\left(2\,\left(a_1\, u^2 + b_1\, v^2 + c_1\, w^2\right) p - au + cv + \delta\, w + h.o.t.\right) \\
w' =& X_3 = \left(2\,\left(a_1\, u^2 + b_1\, v^2 + c_1\, w^2\right) p - au + cv + \delta\, w\right) \sigma\, u + h.o.t \\
p' =& X_4 = -2\, b_1\, vp^3 - (2\;\, c_1\, w\sigma\, u + c + 2\, a_1\, u)\, p^2 + (a + 2\, b_1\, v - \delta\, \sigma\, u)\, p \\
& +2\, a_1\, u + 2\, c_1\, w\sigma\, u + h.o.t.
\end{aligned}
\tag{13}
$$

A similar definition of the Lie Cartan vector field $Y_{\mathcal{L}}$ holds in the affine coordinates (u, v, w, q).

Lemma 2. *Let $\mathcal{S}$ be a partially umbilic set containing a partially umbilic point of type $D^1_{23\times}$ or $D^1_{23\bullet}$. Then,*

(i) *If $p \in \mathcal{S}$ is of type $D^1_{23\times}$, then the set of equilibrium points of the Lie Cartan vector field is formed by six regular curves of singularities γ_i, β_i (i=1,2,3), where γ_i and β_i intersect quasi transversally. Moreover, the pairs of curves $\gamma_1 \cup \beta_1$, $\gamma_2 \cup \beta_2$ and $\gamma_3 \cup \beta_3$ are disjoints, in the neighborhood of axis $\mathbf{p}$, and $\gamma_i \cap \beta_i = \mathcal{A}_i$ where $\mathcal{A}_i$ is a point with coordinates $u = v = w = 0$ and $p = p_i$ with $p_1 = 0$, $p_2 = \frac{a}{c}$ and $p_3 = \infty$ $(Q = 0)$.*

(ii) *If $p \in \mathcal{S}$ is of type $D^1_{23\bullet}$ then the singular set of the Lie Cartan vector field is formed by three points $\mathcal{A}_i$ $(i = 1, 2, 3)$ whose coordinates are $u = v = w = 0$ and $p = p_i$ with $p_1 = 0$, $p_2 = \frac{a}{c}$ and $p_3 = \infty$ $(Q = 0)$.*

Proof. From Equation (13) it follows that $X_L(0, p) = -cp^2 + ap$. In the affine coordinates (u,v,w,q) it follows that $X_L(0, q) = aq^2 - cq$. As by construction the singularities of $X_{\mathcal{L}}$ project on the partially umbilic set, the result follows from Lemma 1. More concretely, near the origin the singularities of $X_{\mathcal{L}}$ are defined by

$$
\begin{aligned}
u =& \quad \frac{cv}{a} + \frac{\delta w}{a} + O(2) \\
p =& \quad -\frac{2a_1\, cv}{a^2} - \frac{2a_1\, \delta\, w}{a^2} + O(2) \\
R(v, w) =& \quad \frac{\left(a^2 b_1 + a_1\, c^2\right) v^2}{a^2} + \frac{2a_1\, c\delta\, vw}{a^2} + \frac{\left(a^2 c_1 + a_1\, \delta^2\right) w^2}{a^2} + O(3)
\end{aligned}
\tag{14}
$$

This is achieved applying the Implicit Function Theorem to the system $\mathcal{L} = X_1 = X_4 = 0$. It follows that

$$\det(\text{Hessian} R(0)) = \frac{4\zeta}{a^2}.$$

Therefore, when $\zeta < 0$ the singularities of $X_{\mathcal{L}}$ in a neighborhood of the origin consists of a pair of regular curves defined by the real solutions of Equation (14). When $\zeta > 0$ the origin is a Morse singularity of index 0 or 2 of R in Equation (14), and so the origin is the unique singularity of $X_{\mathcal{L}}$ near the origin. A similar analysis can be performed near $(0, a/c)$ and $q = 0$ $(p = \infty)$. This ends the proof. □

Proposition 5. *Let $X_{\mathcal{L}}$ be the Lie Cartan vector field restricted to the singular hypersurface $\mathcal{L} = 0$. Then,*

(i) *The condition $D^1_{23\times}$ implies that the phase portrait of $X_{\mathcal{L}}$ in the neighborhood of $\gamma_2 \cup \beta_2$ is topologically as shown in Figure 6 (left). The curves γ_i, β_i $(i = 1, 3)$ are normally hyperbolic of saddle type of the Lie Cartan vector field $X_{\mathcal{L}}$ restricted to the Lie Cartan* *hypersurface* $\mathcal{L}=0$. *Moreover, the Lie Cartan hypersurface has a Morse singular point in* $\gamma_i\cap\beta_i=\mathcal{A}_i$ $(i=1,3)$.

(ii) *The condition* $D^1_{23\bullet}$ *implies that the phase portrait of* $X_{\mathcal{L}}$ *in the neighborhood of* $\mathcal{A}_2$ *is topologically as shown in Figure 6 (right). In the neighbourhood of* $\mathcal{A}_i$ $(i=1,3)$, *there exists a two-dimensional manifold* W_i *(contained in the Lie Cartan hypersurface) where the phase portrait of Lie Cartan vector field is as shown in Figure 7.*

Proof. In the neighborhood of $(0,\frac{a}{c})$ the Lie Cartan hypersurface is regular. In fact,

$$\nabla H(0,0,0,\frac{a}{c})=\left(-\frac{a^2}{c},a,\frac{\delta\, a}{c},0\right)\neq 0.$$

The solution $v=v(u,w,p)$ of the implicit equation $\mathcal{L}(u,v,w,p)=0$ in the neighborhood of $(0,a/c)$ is given by

$$\begin{aligned}v(u,w,p)=&\frac{au}{c}-\frac{\delta\, w}{c}+2\,\frac{b_1\,\delta\,\left(a^2-c^2\right)uw}{c^4}-\frac{\left(a^2-c^2\right)\left(a^2b_1+c^2a_1\right)u^2}{ac^4}\\&-\frac{\left(a^2-c^2\right)\left(\delta^2b_1+c^2c_1\right)w^2}{ac^4}+O(3)\end{aligned}$$

Near $\left(0,\frac{a}{c}\right)$ the vector field $X_{\mathcal{L}}$ restricted to the Lie Cartan hypersurface is given by

$$X_{\mathcal{L}}=\begin{cases}\dot u=X_1=\frac{\left(a^2+c^2\right)\left(a^2b_1+a1\,c^2\right)u^2}{c^3a}-\frac{2b_1\,\delta\,\left(a^2+c^2\right)uw}{c^3}+\frac{\left(a^2+c^2\right)\left(\delta^2b_1+c^2c_1\right)w^2}{c^3a}+O(3)\\[2mm] \dot w=X_3=(\sigma u+O(2))X_1\\[2mm] \dot p=X_4=-\frac{\left(\delta\,a\sigma\,c^3+2\left(a^2-c^2\right)\left(a^2b_1+c^2a_1\right)\right)u}{c^4}-a\left(p-\frac{a}{c}\right)+\frac{2a\,\delta\,b_1\,w\left(a^2-c^2\right)}{c^4}+O(2)\end{cases}$$

Therefore, the eigenvalues of $DX_{\mathcal{L}}\left(0,0,\dfrac{a}{c}\right)$ are $0,0,-a$. The axis p is the eigenspace associated to the eigenvalue $-a\neq 0$.

By Invariant Manifold Theory, see [3, 13], the center manifold is two-dimensional and is parametrized as

$$\begin{aligned}A(x,y)=&\quad(x,y,P(x,y))\\P(x,y)=&\quad\tfrac{a}{c}-\frac{\left(\delta\,a\sigma\,c^3+2\left(a^2-c^2\right)\left(a^2b_1+c^2a_1\right)\right)x}{c^4a}+2\,\frac{b_1\,\delta\,\left(a^2-c^2\right)y}{c^4}+O(2)\end{aligned}$$

Consider $Y=(Y_1,Y_2)$ defined by the conjugation equation

$$DA(x,y)Y(x,y)=X_{\mathcal{L}}(A(x,y)).$$

It follows that $Y=X_{\mathcal{L}}$ restricted to the center manifold W^c is given by:

$$\begin{aligned}Y_1(x,y)=&\quad\frac{\left(a^2+c^2\right)\left(a^2b_1+c^2a_1\right)x^2}{ac^3}-\frac{2b_1\,\delta\,\left(a^2+c^2\right)xy}{c^3}+\frac{\left(a^2+c^2\right)\left(\delta^2b_1+c^2c_1\right)y^2}{ac^3}+O(3)\\Y_2(x,y)=&\quad Y_1(x,y)(\sigma u+O(2))\end{aligned}$$

The singular set of Y is a pair of curves crossing transversally at the origin (resp. the origin) when $\zeta>0$ (resp. $\zeta<0$).

Therefore the phase portrait of $Y=X_{\mathcal{L}}\Big|_{W^c}$ is as shown in Fig. 6 left (resp. right) if $\mathbf{p}$ is of type $D^1_{23\times}$ (resp. $D^1_{23\bullet}$).

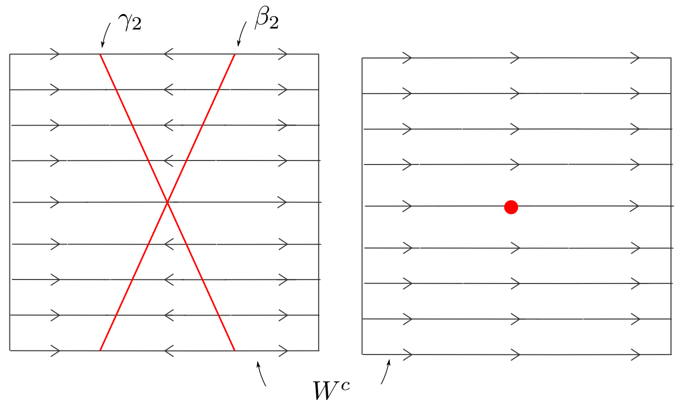

FIGURE 6. Behavior of $X_{\mathcal{L}}\Big|_{W^c}$ when $\zeta < 0$ (left) and $\zeta > 0$ (right)

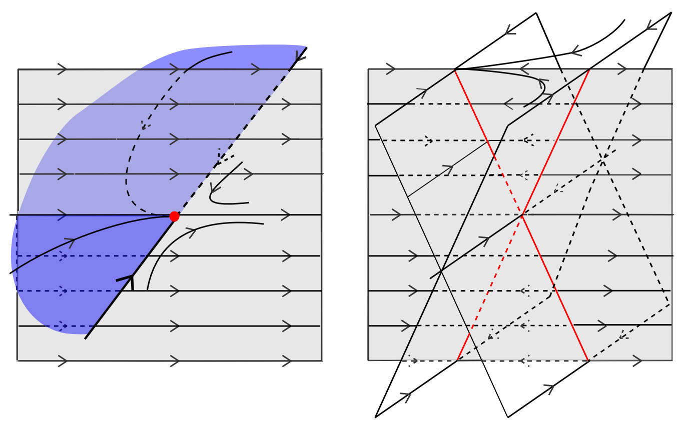

FIGURE 7. Behavior of $X_{\mathcal{L}}$ near the point $(0, a/c)$ when $\zeta > 0$ (left) and $\zeta < 0$ (right).

Figure 6 illustrates the phase portrait of the Lie Cartan vector field in the neighborhood $\gamma_2 \cup \beta_2$ (for the $D^1_{23\times}$ type) and $\mathcal{A}_2$ (for the $D^1_{23\bullet}$ type).

For the points $\mathcal{A}_1$ $(u = v = w = p = 0)$ and $\mathcal{A}_3$ $(u = v = w = 0, p = \infty (q = 0))$ it follows that

- the eigenvalues of $DX_{\mathcal{L}}(0)$ are $0, 0, a, -a$
- the eigenvalues of $DY_{\mathcal{G}}(0)$ are $0, 0, c, -c$.

The two-dimensional eigenspace space E_c associated to the double zero eigenvalue is spanned by

$$\{V_1 = (ac, a^2, 0, -2ca_1), \quad V_2 = (a\delta, 0, a^2, -2\delta a_1)\}$$

The two-dimensional center manifold $W^c(0)$ is tangent to E_c at 0 and, after long calculations, it is obtained that $W^c(0)$ is parametrized by:

$$
\begin{aligned}
A(x,y) = &\quad (x, y, W(x,y), P(x,y))\\
W(x,y) = &\quad \tfrac{ax}{\delta} - \tfrac{cy}{\delta} + O(3)\\
P(x,y) = &\quad -\tfrac{2a_1\,x}{a} - \tfrac{\left(2\,a^3c_1\,\sigma+2\,aa_1\,\delta^2\sigma-4\,c\delta a_1^2\right)x^2}{a^3\delta} + \tfrac{(2\,acc_1\,\sigma+4\,a_1\,b_1\,\delta)xy}{a^2\delta} + O(3).
\end{aligned}
$$

The vector field $X_{\mathcal{L}}$ restricted to $W^c(0)$, written in the (x,y) coordinates, is given by:

$$
\begin{aligned}
Y_1(x,y) = &\quad -\tfrac{4a_1\,\left(a^2c_1+a_1\,\delta^2\right)x^2}{a\delta^2} + \tfrac{8cc_1\,a_1\,xy}{\delta^2} - \tfrac{4\left(\delta^2b_1+c^2c_1\right)a_1\,y^2}{a\delta^2} + O(3)\\
Y_2(x,y) = &\quad Y_1(x,y)P(x,y).
\end{aligned}
$$

The phase portrait of Y is as shown in Figure 6 left (resp. right) when $\zeta > 0$ (resp. $\zeta < 0$).

In a neighborhood of the origin, the linearizations $DX_{\mathcal{L}}$ and $DY_{\mathcal{G}}$ along the curves β_i and γ_i $(i = 1, 3)$ have a zero eigenvalue and two non-zero ones with opposite signs. So the curves γ_i, β_i $(i = 1, 3)$ are normally hyperbolic of saddle type of the Lie Cartan vector field restricted to the Lie Cartan hypersurface $\mathcal{L} = 0$.

The quasi transversality condition of partially umbilic points is equivalent to the fact that the implicit surfaces $\mathcal{L} = 0$ and $\mathcal{G} = 0$ have at the origin a Morse critical point. In fact,

$$\nabla\mathcal{L}(0) = \nabla\mathcal{G}(0) = 0,\ \det(\text{Hessian } \mathcal{L}) = \det(\text{Hessian } \mathcal{G}) = -4\zeta.$$

A direct analysis shows that the intersection of $W^c(0)$ with the Lie Cartan hypersurface is the origin (resp. the pair of curves γ_1, β_1) when $\zeta > 0$ (resp. $\zeta < 0$).

The analysis near the point $q = 0$ is analogous to that performed for $p = 0$. There exists also a two-dimensional center manifold which is normally hyperbolic. Restricted to the two-dimensional invariant center manifold, $X_{\mathcal{L}}$ is topologically equivalent to $(x^2 \pm y^2)(1, 0)$. □

4.1. **Proofs of Theorems 1 and 2.** Let $\mathbf{p} \in \mathcal{S}_{12}$ of type $D^1_{23\bullet}$. Proposition 5 the Lie Cartan vector field $X_{\mathcal{L}}$ in the chart (u, v, w, p) and $Y_{\mathcal{G}}$ the Lie Cartan vector field in the chart (u, v, w, q) has three singular points $\mathcal{A}_i$ $(i = 1, 2, 3)$.

Recall that the projection of the integral curves of the Lie Cartan vector field are the leaves of $\mathcal{F}_1(\alpha)$ and $\mathcal{F}_2(\alpha)$.

Th center manifold shown in Figure 7 (left) intersects the Lie Cartan hypersurface $\mathcal{L} = 0$ only at $(0, 0)$. In Figure 8 is shown the behavior of $X_{\mathcal{L}}$ near the origin. There are two separatrices (one is the projective axis p) asymptotic to $(0, 0)$.

The projections of the integral curves shown in Figure 8 left (resp. right) are the leaves of $\mathcal{F}_1(\alpha)$ (resp. $\mathcal{F}_2(\alpha)$).

From the analysis near $p = 0$ and $q = 0$, it follows that there exists a unique separatrix (leaf of $\mathcal{F}_1(\alpha)$) asymptotic to $\mathbf{p}$. The same conclusion holds for $\mathcal{F}_2(\alpha)$.

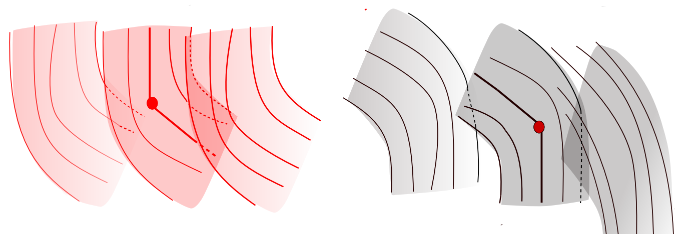

FIGURE 8. Behavior of $X_{\mathcal{L}}$ near the point $(0,0)$ when $\zeta > 0$.

Near $p = a/c$, from the projections of the integral curves of $X_{\mathcal{L}}$ it follows that there exists a two-dimensional funnel for, say, $\mathcal{F}_1(\alpha)$, with the leaves asymptotic to $\mathbf{p}$. The boundary of this funnel are the separatrices tangent to directions $p = 0$ and $q = 0$. The other principal foliation has no leaves asymptotic to $\mathbf{p}$ near the direction $p = a/c$, see Figure 7. The invariant surface $W_1(\mathbf{p})$ is the projection of the center manifold associated to the singular point $(0, a/c)$.

Gluing together, we obtain the description given in Figure 4.

Let $\mathbf{p} \in \mathcal{S}_{12}$ of type $D^1_{23\times}$. By Lemma 2 the Lie Cartan vector field $X_{\mathcal{L}}$ and $X_{\mathcal{G}}$ has six curves of singularities β_i and γ_i $(i = 1, 2, 3)$.
The curves γ_i, β_i $(i = 1, 3)$ are normally hyperbolic of saddle type of the Lie Cartan vector field restricted to the Lie Cartan hypersurface. See Figure 7 (right). The projections of the integral curves show that the partially umbilic set is formed by four open arcs of partially umbilic points, two of type D_2 and two of type D_3 as shown in Figure 5. The invariant surface $W_1(\mathcal{S}_{12})$ is the projection of the center manifold associated to the singular point $(0, a/c)$.

Near $p = 0$ and $q = 0$ it follows that there exist two normally hyperbolic invariant separatrix surfaces fibered by integral curves of $X_{\mathcal{L}}$. See Figure 9. Also we remark that the center manifold shown in Figure 7 (right) intersects the Lie Cartan hypersurface only along the two lines of singularities crossing at $(0, 0)$.

Gluing together the phase portraits of $X_{\mathcal{L}}$ and $Y_{\mathcal{G}}$ in the charts (u, v, w, p) and (u, v, w, q), ends the proof. In order to see this global picture is useful to review the analysis in [15, Sections 7, 8 and 9] and [7, Section 2].

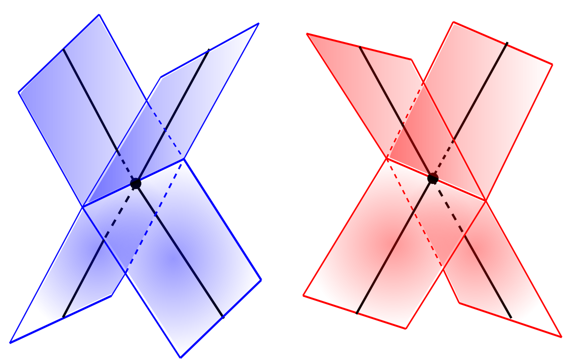

FIGURE 9. Illustration of the invariant manifolds of $X_{\mathcal{L}}$ near $p = 0$, left, and of $Y_{\mathcal{G}}$ near $q = 0$, right.

The principal lines near the partially umbilic points $D^1_{23\bullet}$ and $D^1_{23\times}$ are as shown in Figs. 4 and 5.

5. Unfolding of partially umbilic points

In this section will be considered the unfoldings, in one parameter families of immersions, of the partially umbilic points $D^1_{23\bullet}$ and $D^1_{23\times}$ considered above.

An unfolding of a smooth immersion $\alpha_0 : \mathbb{M} \to \mathbb{R}^4$ is a smooth map $\alpha : \mathbb{M} \times \mathbb{R}^k \to \mathbb{R}^4$ such that $\alpha(p, 0) = \alpha_0(p)$ and $\alpha_\lambda : \mathbb{M} \to \mathbb{R}^4$ is a smooth immersion. Here $\alpha_\lambda(p) = \alpha(p, \lambda)$.

Concerning the partially umbilic points considered in this work we define $\mathcal{S}_{12}(\lambda)$ formed by the partially umbilic points of α_λ.

It will be shown that the set $\mathcal{S}(\alpha) = \{\cup_\lambda(\mathcal{S}_{12}(\lambda), \lambda),\ \lambda \in \mathbb{R}\}$ is generically a regular surface, which will be called the *partially umbilic surface* of the unfolding α.

Two unfolding immersions α_λ and β_μ are equivalent if and only if there is a homeomorphism $h : \mathcal{S}(\alpha) \to \mathcal{S}(\beta)$ preserving the types of partially umbilic points.

When this is the case we say that we have a *transversal unfolding.*

For basic theory of unfolding of maps see [17, Chapter XIII].

5.1. Unfolding of critical partially umbilic points.

In this subsection an explicit deformation of α_0 will be analyzed. The main goal is to show that in a generic one parameter family of immersions α_λ the partially umbilic sets of α_λ, for $\lambda \neq 0$, are regular curves and the principal foliations behave as described in Theorems 3 and 4.

Proposition 6. *Consider a one parameter family of immersions given by*

$$\alpha_\lambda(u_1, u_2, u_3) = (u_1, u_2, u_3, h(u_1, u_2, u_3, \lambda) + \lambda u_1 u_2)). \tag{15}$$

such that α_0 has a partially umbilic point $D^1_{23\times}$ or $D^1_{23\bullet}$. Here h is given by Equation (1) with the coefficients being functions of λ. Then there is a change of coordinates

$$(u_1, u_2, u_3) = (u, v, w) + h_2(u, v, w, \lambda)$$

such that the differential equation of principal lines of α_λ is given by

$$\begin{aligned} \mathcal{L} =& (\lambda + a_1(\lambda)u^2 + b_1(\lambda)v^2 + c_1(\lambda)w^2 + O_1(3))dv^2 \\ &+ (-a(\lambda)u + c(\lambda)v + \delta(\lambda)w + O_2(3))\ dudv \\ &-(\lambda + a_1(\lambda)u^2 + a_{11}(\lambda)u_1v_1 + b_1(\lambda)v^2 + c_1(\lambda)w^2 \\ &r_1(\lambda)u_1 + r_2(\lambda)v_1 + r_3(\lambda)w_1 + O_3(3))du^2 = 0 \\ dw =& (\sigma(\lambda)u + O_4(2))du + O_5(2)dv. \end{aligned} \tag{16}$$

All coefficients $a_1(\lambda), b_1(\lambda), c_1(\lambda), a(\lambda), c(\lambda), \delta(\lambda), \sigma(\lambda)$ in Equation (16) are functions of the second jet of the functions L_r, M_r and N_r. Moreover for $\lambda = 0$ all the functions are as given in Equation (11).

Proof. The proof follows the same approach developed in the proof of Proposition 2. As the normal form depends on a parameter λ and the partially umbilic point is not preserved at the origin to obtain the coefficients stated it is necessary to apply the Implicit Function Theorem to solve the homological equation involved. □

Proposition 7. *Let* **p** *be a partially umbilic point of types $D^1_{23\bullet}$ and $D^1_{23\times}$ of α_0 given in the Monge form (1). Consider the one parameter family defined by*

$$\alpha_\lambda(u_1, u_2, u_3) = (u_1, u_2, u_3, h(u_1, u_2, u_3) + \lambda u_1 u_2)). \tag{17}$$

Then α_λ is an unfolding of α_0. The partially umbilic surface is stratified in curves of D_{23} partially umbilic points and open strata D_2 and D_3 of partially umbilic points as shown in Figure 10.

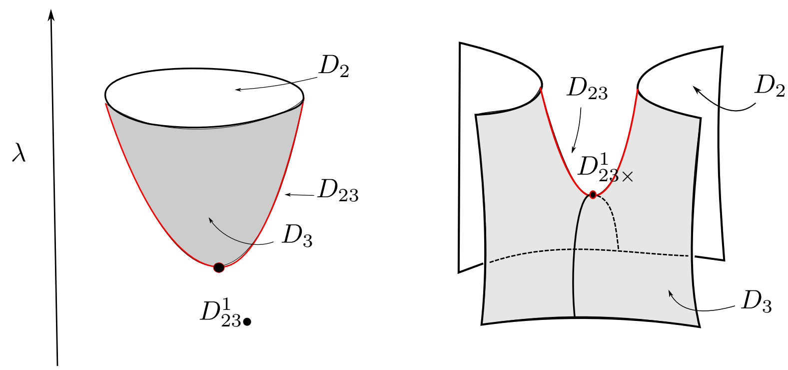

FIGURE 10. Partially umbilic surfaces of the points $D^1_{23\bullet}$ and $D^1_{23\times}$.The intersection with the plane $\lambda=\lambda_0$ is the critical partially umbilic set.

Proof. Consider the differential equation of principal lines of α_λ given by Equation (16). The partially umbilic set is given by

$$\begin{aligned} L(u_1,v_1,w_1,\lambda) =& \lambda + a_1(\lambda)u^2+b_1(\lambda)v^2+c_1(\lambda)w^2+O_1(3)\\ M(u_1,v_1,w_1,\lambda) =& (-a(\lambda)u+c(\lambda)v+\delta(\lambda)w+O_2(3))\end{aligned}$$

The set defined by $L=M=0$ is a local regular surface of $\mathbb{R}^4$ in a neighborhood of the origin. Explicitly it is given by

$$\begin{aligned} S(u,v) =& (u,v,w(u,v),\lambda(u,v))\\ w(u,v) =& \frac{au}{\delta}-\frac{cv}{\delta}+O_1(3)\\ \lambda(u,v) =& -\frac{\left(a^2c_1+a_1\,\delta^2\right)u^2}{\delta^2}+\frac{2acc_1\,uv}{\delta^2}-\frac{\left(\delta^2b_1+c^2c_1\right)v^2}{\delta^2}+O_2(3)\end{aligned}$$

In order to determine the partially umbilic points of type D^1_{23} consider the principal plane field (orthogonal to the regular principal direction e_3) defined by the kernel of the one form ω_3. See Equation (6) and Equation (16). Let $H(u_1,v_1,w_1)=\omega_3(T)$, where the vector $T=\nabla L\wedge\nabla M$ and $\nabla f=(f_{u_1},f_{v_1},f_{w_1})$. The set given by $L=M=H=0$ is a regular curve in a neighborhood of the origin and is formed by the partially umbilic points D^1_{23} of α_λ for $\lambda\neq 0$. Applying the Implicit Function Theorem it follows that this curve is parametrized by

$$\begin{aligned} P(u) =& (u,v(u),w(u),\lambda(u))\\ v(u) =& -\frac{a_1\,cu}{b_1\,a}-\frac{c\sigma\zeta}{\delta\,a^2b_1^2}u^2+O_1(u^3)\\ w(u) =& \frac{\left(a^2b_1+c^2a_1\right)u}{b_1\,a\delta}+\frac{c^2\sigma\,\zeta}{a^2b_1^2\delta^2}u^2+O_2(u^3)\\ \lambda(u) =& -\frac{(a^2b_1+c^2a_1)\zeta}{a^2b_1^2\delta^2}u^2+O_3(u^3)\\ \zeta =& a^2b_1\,c_1+\delta^2a_1\,b_1\,+c^2a_1\,c_1\end{aligned}$$

For $\lambda > 0$ the partially umbilic set is a closed curve formed of two arcs of D_2 and D_3 points separated by two partially umbilic points D^1_{23}. See Figure 10 left. For $\lambda < 0$ the partially umbilic set is the union of two arcs of regular curves formed by D_2 and D_3 partially umbilic points separated by two partially umbilic points D^1_{23} or two arcs of partially umbilic points D_2 and D_3, see Figure 10 right. □

Theorem 3. *Let $p \in \mathcal{S}_{12}$ be a partially umbilic point of type $D^1_{23\bullet}$. Then, the principal foliations $\mathcal{F}_1(\alpha)$ and $\mathcal{F}_2(\alpha)$ of the unfolding α_λ given by Equation (15) are as shown in Figure 11 (center).*

i) *For $\lambda < 0$ the partially umbilic set is empty, all leaves are regular curves and,*
ii) *For $\lambda > 0$ the partially umbilic set consists of a closed curve, having two partially umbilic points D_{23} and two open arcs, one of Darbouxian partially umbilic points of type D_2 and the other is of type D_3.*

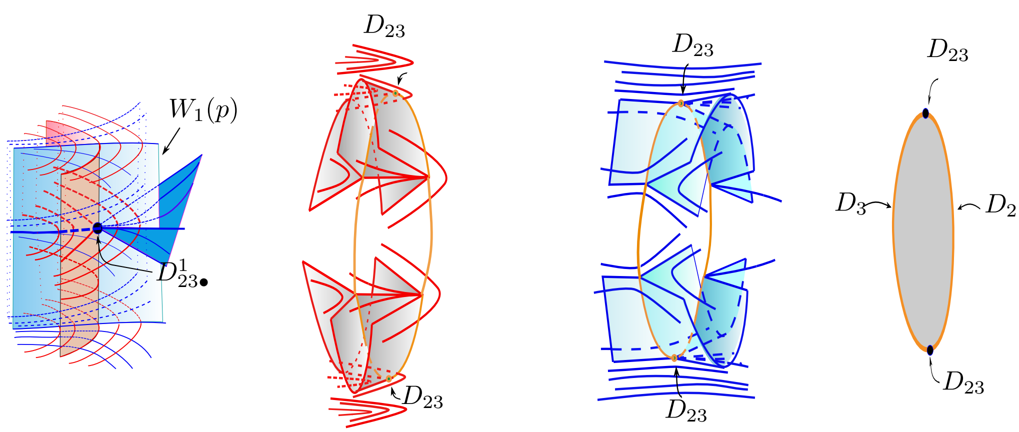

FIGURE 11. Principal foliations $\mathcal{F}_1(\alpha_\lambda)$ and $\mathcal{F}_2(\alpha_\lambda)$ near a partially umbilic point $D^1_{23\bullet}$ (left for $\lambda = 0$) and near a partially umbilic curve which unfolds of a partially umbilic point $D^1_{23\bullet}$ (center for $\lambda > 0$).

Proof. The proof follows from Proposition 7 and [15, Theorem 3]. □

Theorem 4. *Let* **p** *be a partially umbilic point of type $D^1_{23\times}$. Then. the principal foliations $\mathcal{F}_1(\alpha)$ and $\mathcal{F}_2(\alpha)$ of the unfolding α_λ given by Equation (17) are as shown in Figure 12.*

i) *For $\lambda < 0$ there are two arcs of partially umbilic points, one of Darbouxian type D_2 and other D_3.*
ii) *For $\lambda > 0$ there are two arcs of partially umbilic points having a point D_{23} and two open arcs, one of Darbouxian type D_2 and other of type D_3.*

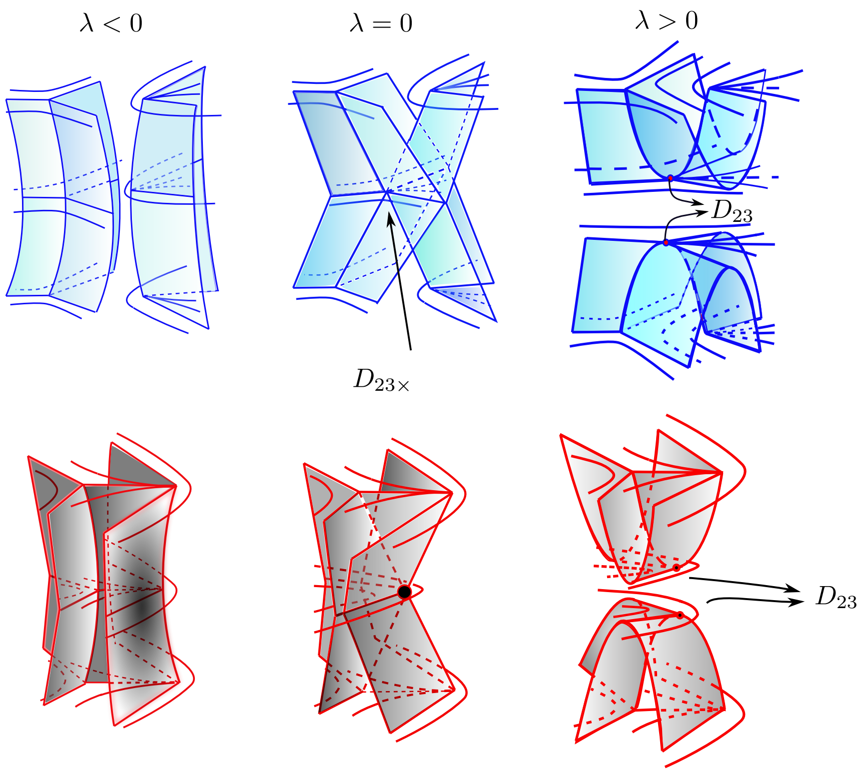

FIGURE 12. Principal foliations $\mathcal{F}_1(\alpha_\lambda)$ and $\mathcal{F}_2(\alpha_\lambda)$ near a partially umbilic point $D^1_{23\times}$ (center for $\lambda = 0$) and near arcs of partially umbilic curves of the unfolding of a partially umbilic point $D^1_{23\times}$. For $\lambda < 0$ (left) two arcs of partially umbilic points D_2 and D_3 and for $\lambda > 0$ two arcs D_2 and D_3 of partially umbilic points separated by a D_{23} point.

Proof. It follows from Proposition 7 and [15, Theorem 3]. □

6. Coordinate expressions for geometric functions appearing in this work

Consider a chart (u_1, u_2, u_3) and an isometry R so that the immersion α composed with R has the Monge form: $(u_1, u_2, u_3) \to (u_1, u_2, u_3, h(u_1, u_2, u_3))$, where h is given by Equation (1). Here will be obtained the coordinate expressions of functions that are essential for the calculations carried out in this work.

6.1. First Fundamental Form.

First Fundamental Form. The coefficients of the first fundamental form (g_{ij}) of α in the Monge chart (u_1, u_2, u_3) are:

$$g_{11} = \langle \alpha_{u_1}, \alpha_{u_1} \rangle = 1 + k^2 u_1^2 + O(3)$$
$$g_{12} = \langle \alpha_{u_1}, \alpha_{u_2} \rangle = k^2 u_1 u_2 + O(3)$$
$$g_{13} = \langle \alpha_{u_1}, \alpha_{u_3} \rangle = k_3 u_1 u_3 + O(3)$$

$$\begin{aligned}
g_{22} &= \langle \alpha_{u_2}, \alpha_{u_2} \rangle = 1 + k^2 u_2^2 + O(3) \\
g_{23} &= \langle \alpha_{u_2}, \alpha_{u_3} \rangle = k k_3 u_2 u_3 + O(3) \\
g_{33} &= \langle \alpha_{u_3}, \alpha_{u_3} \rangle = 1 + k_3^2 u_3^2 + O(3)
\end{aligned}$$

6.2. **Unit Normal Vector.** Taylor expansion of the components of the unit normal of α, $N = \mathcal{N}_\alpha = (\alpha_{u_1} \wedge \alpha_{u_2} \wedge \alpha_{u_3}) / |\alpha_{u_1} \wedge \alpha_{u_2} \wedge \alpha_{u_3}|$, gives the following expressions for $N = (n_1, n_2, n_3, n_4)$ in a neighborhood of $(0,0,0)$:

$$\begin{aligned}
n_1 &= -k u_1 - \frac{1}{2} a u_1^2 - q_{201} u_1 u_3 - \frac{1}{2} q_{102} u_3^2 + O(3), \\
n_2 &= -k u_2 - \frac{1}{2} c u_2^2 - q_{021} u_2 u_3 - \frac{1}{2} q_{012} u_3^2 + O(3) \\
n_3 &= -k_3 u_3 - \frac{1}{2} q_{201} u_1^2 - q_{111} u_1 u_2 - \frac{1}{2} q_{021} u_2^2 \\
&\quad - q_{102} u_1 u_3 - q_{012} u_2 u_3 - \frac{1}{2} q_{003} u_3^2 + O(3) \\
n_4 &= 1 - \frac{1}{2} k^2 (u_1^2 + u_2^2) - \frac{1}{2} k_3^2 u_3^2 + O(3)
\end{aligned}$$

6.3. **Second Fundamental Form.** The coefficients of the second fundamental form (λ_{ij}) are:

$$\begin{aligned}
\lambda_{11} = \langle \alpha_{u_1 u_1}, N \rangle &= k + a u_1 + q_{201} u_3 + \frac{1}{2}(A - k^3) u_1^2 + B u_1 u_2 + Q_{301} u_1 u_3 \\
&\quad + \frac{1}{2}(C - k^3) u_2^2 + Q_{211} u_2 u_3 + \frac{1}{2}(Q_{202} - k k_3^2) u_3^2 + O(3), \\
\lambda_{12} = \langle \alpha_{u_1 u_2}, N \rangle &= \frac{1}{2} B u_1^2 + C u_1 u_2 + \frac{1}{2} D u_2^2 + Q_{211} u_1 u_3 + Q_{121} u_2 u_3 + \frac{1}{2} Q_{112} u_3^2 + O(3), \\
\lambda_{13} = \langle \alpha_{u_1 u_3}, N \rangle &= q_{201} u_1 + q_{102} u_3 + \frac{1}{2} Q_{301} u_1^2 + + Q_{211} u_1 u_2 + \frac{1}{2} Q_{121} u_2^2 \\
&\quad + Q_{202} u_1 u_3 + Q_{112} u_2 u_3 + \frac{1}{2} Q_{103} u_3^2 + O(3), \\
\lambda_{22} = \langle \alpha_{u_2 u_2}, N \rangle &= k + c u_2 + q_{021} u_3 + \frac{1}{2}(C - k^3) u_1^2 + D u_1 u_2 + Q_{121} u_1 u_3 \\
&\quad + \frac{1}{2}(E - k^3) u_2^2 + Q_{031} u_2 u_3 + \frac{1}{2}(Q_{022} - k k_3^2) u_3^2 + O(3), \\
\lambda_{23} = \langle \alpha_{u_2 u_3}, N \rangle &= q_{021} u_2 + q_{012} u_3 + \frac{1}{2} Q_{211} u_1^2 + Q_{121} u_1 u_2 + \frac{1}{2} Q_{031} u_2^2 \\
&\quad + Q_{112} u_1 u_3 + Q_{022} u_2 u_3 + \frac{1}{2} Q_{013} u_3^2 + O(3), \\
\lambda_{33} = \langle \alpha_{u_3 u_3}, N \rangle &= k_3 + q_{102} u_1 + q_{012} u_2 + q_{003} u_3 + \frac{1}{2}(Q_{202} - k_3 k^2) u_1^2 + Q_{112} u_1 u_2 \\
&\quad + \frac{1}{2}(Q_{022} - k_3 k^2) u_2^2 + Q_{103} u_1 u_3 + Q_{013} u_2 u_3 + \frac{1}{2}(Q_{004} - k_3^3) u_3^2 + O(3).
\end{aligned}$$

6.4. **The principal curvature k_3.** The principal curvature k_3, which is smooth near the origin, is given by

$$k_3(u_1,u_2,u_3) = k_3 + q_{102}u_1 + q_{012}u_2 + q_{003}u_3 + \frac{1}{2}\left(Q_{202} - k^2k_3 + \frac{2q_{201}^2}{k_3-k}\right)u_1^2 + Q_{112}u_1u_2 + \frac{1}{2}\left(Q_{022} - k^2k_3 + \frac{2q_{021}^2}{k_3-k}\right)u_2^2 + \left(Q_{103} + \frac{2q_{201}q_{102}}{k_3-k}\right)u_1u_3 + \left(Q_{013} + \frac{2q_{021}q_{012}}{k_3-k}\right)u_2u_3 + \frac{1}{2}\left(Q_{004} - 3k_3^3 + \frac{2(q_{102}^2+q_{012}^2)}{k_3-k}\right)u_3^2 + O(3)$$

6.5. **The principal direction** e_3**.** The smooth principal direction $e_3(q) = (du_1, du_2, du_3)$ is defined by

$$\frac{du_1}{du_3} = \frac{U_1(u_1,u_2,u_3)}{W_1(u_1,u_2,u_3)}, \qquad \frac{du_2}{du_3} = \frac{V_1(u_1,u_2,u_3)}{W_1(u_1,u_2,u_3)}.$$

The functions U_1, V_1 and W_1 are given by solving the linear system (2), taking $i = 3$. Using the Equation (4) and the subsections 6.1, 6.3 and 6.4, it is obtained:

(18)
$$\begin{aligned} U_1 = &(k_3-k)q_{201}u_1 + (k_3-k)q_{102}u_3 \\ &+(\tfrac{1}{2}Q_{301}(k_3-k) + q_{201}q_{102})u_1^2 + \tfrac{1}{2}(k_3-k)Q_{121}u_2^2 \\ &+(Q_{211}(k_3-k) + q_{201}(q_{012}-c))u_1u_2 \\ &+[(Q_{202}-kk_3^2)(k_3-k) + q_{201}(q_{003}-q_{021}) + q_{102}^2]u_1u_3 \\ &+((k_3-k)Q_{112} + q_{102}(q_{012}-c))u_2u_3 \\ &+(\tfrac{1}{2}(k_3-k)Q_{103} + q_{102}(q_{003}-q_{021}))u_3^2 + O(3). \end{aligned}$$

(19)
$$\begin{aligned} V_1 = &(k_3-k)q_{021}u_2 + (k_3-k)q_{012}u_3 \\ &+\tfrac{1}{2}Q_{211}(k_3-k)u_1^2 + (\tfrac{1}{2}(k_3-k)Q_{031} + q_{012}q_{021})u_2^2 \\ &+[(k_3-k)Q_{121} + q_{021}(q_{102}-a)]u_1u_2 \\ &+((k_3-k)Q_{112} + q_{012}(q_{102}-a))u_1u_3 \\ &+[(Q_{022}-kk_3^2)(k_3-k) + q_{021}(q_{003}-q_{201}) + q_{012}^2]u_2u_3 \\ &+[\tfrac{1}{2}(k_3-k)Q_{013} + q_{012}(q_{003}-q_{201})]u_3^2 + O(3). \end{aligned}$$

(20)
$$\begin{aligned} W_1 = &(k_3-k)^2 + (k_3-k)(2q_{102}-a)u_1 + (k_3-k)(2q_{012}-c)u_2 \\ &+(k_3-k)(2q_{003}-q_{021}-q_{201})u_3 + O(2). \end{aligned}$$

6.6. **The field of planes** $\mathcal{P}_3$**.** The plane field $\mathcal{P}_3$ defined by $\omega = 0$ (see Equation (5)) can be written as $du_3 = \mathcal{U}du_1 + \mathcal{V}du_2$ where

$$\mathcal{U} = -\frac{[g_{11}U_1 + g_{12}V_1 + g_{13}W_1]}{[g_{13}U_1 + g_{23}V_1 + g_{33}W_1]} \quad \text{and} \quad \mathcal{V} = -\frac{[g_{12}U_1 + g_{22}V_1 + g_{23}W_1]}{[g_{13}U_1 + g_{23}V_1 + g_{33}W_1]}$$

The Taylor expansions of $\mathcal{U}$ and $\mathcal{V}$ in a neighborhood of zero, are given by:

(21)
$$\begin{aligned} \mathcal{U} = &\frac{1}{2(k-k_3)^2}[2q_{201}(k-k_3)u_1 + 2q_{102}(k-k_3)u_3 + (k-k_3)Q_{121}u_2^2 \\ &+ ((k-k_3)Q_{301} + 2q_{201}(q_{102}-a))\,u_1^2 + ((k-k_3)Q_{103} + 2\,(q_{003}-q_{201})q_{102})\,u_3^2 \\ &+2((k-k_3)(Q_{202}-kk_3^2) + q_{102}(q_{102}-a) + q_{201}(q_{003}-q_{201}))u_1u_3 \\ &+2((k-k_3)Q_{112} + q_{012}q_{102})u_2u_3 + 2\,((k-k_3)Q_{211} + q_{201}q_{012})\,u_1u_2] + O(3) \end{aligned}$$

$$
\begin{aligned}
\mathcal{V} \; = & \frac{1}{2(k-k_3)^2}[2q_{021}(k-k_3)u_2 + 2q_{012}(k-k_3)u_3 + Q_{211}(k-k_3)u_1^2 \\
& +(Q_{031}(k-k_3) + 2q_{021}(q_{012}-c))u_2^2 + (Q_{013}(k-k_3) + 2q_{012}(q_{003}-q_{021}))u_3^2 \\
& +2((k-k_3)(Q_{022}-k_3^2 k) + q_{012}(q_{012}-c) + q_{021}(q_{003}-q_{021}))u_2u_3 \\
& +2(Q_{121}(k-k_3) + q_{102}q_{021})u_1u_2 + 2(Q_{112}(k-k_3) + q_{012}q_{102})u_1u_3] + O(3)
\end{aligned} \tag{22}
$$

6.7. **The first fundamental form restricted to the plane $\mathcal{P}_3$.** The first fundamental form $I = \sum g_{ij} du_i du_j$ restricted to the plane field $\mathcal{P}_3$ is given by:

$$
I_r(du_1, du_2) = I_\alpha \Big|_{du_3 = \mathcal{U}du_1 + \mathcal{V}du_2} = E_r du_1^2 + 2F_r du_1 du_2 + G_r du_2^2,
$$

$$
E_r = 1 + k^2u_1^2 + \frac{1}{(k-k_3)^2}\left(q_{201}u_1 + q_{201}u_3\right)^2 + O(3); \tag{23}
$$

$$
F_r = k^2u_1u_2 + \frac{1}{(k-k_3)^2}(q_{201}u_1 + q_{102}u_3)(q_{021}u_2 + q_{012}u_3) + O(3); \tag{24}
$$

$$
G_r = 1 + k^2u_2^2 + \frac{1}{(k-k_3)^2}\left(q_{021}u_2 + q_{012}u_3\right)^2 + O(3); \tag{25}
$$

6.8. **The second fundamental form restricted to the plane field $\mathcal{P}_3$.** The second fundamental form $II = \sum \lambda_{ij} du_i du_j$ restricted to the plane field $\mathcal{P}_3$ is given by:

$$
II_r(du_1, du_2) = II_\alpha \Big|_{du_3 = \mathcal{U}du_1 + \mathcal{V}du_2} = e_r du_1^2 + 2f_r du_1 du_2 + g_r du_2^2,
$$

$$
\begin{aligned}
e_r \; & = k + au_1 + q_{201}u_3 + \frac{1}{2}\left(A - k^3 + \frac{2(2k-k_3)q_{201}^2}{(k-k_3)^2}\right)u_1^2 \\
& + \frac{1}{2}\left(C - k^3\right)u_2^2 + \frac{1}{2}\left(Q_{202} - kk_3^2 + \frac{2\,(2k-k_3)\,q_{102}^2}{(k-k_3)^2}\right)u_3^2 \\
& + Bu_1u_2 + \left(Q_{301} + \frac{2\,(2k-k_3)\,q_{102}q_{201}}{(k-k_3)^2}\right)u_1u_3 + Q_{211}u_2u_3 + O(3);
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
f_r \; & = \frac{B}{2}u_1^2 + \frac{D}{2}u_2^2 + \frac{1}{2}\left(Q_{112} + \frac{2\,(2k-k_3)\,q_{012}q_{102}}{(k-k_3)^2}\right)u_3^2 \\
& + \left(C + \frac{(2k-k_3)\,q_{021}q_{201}}{(k-k_3)^2}\right)u_1u_2 + \left(Q_{211} + \frac{(2k-k_3)\,q_{012}q_{201}}{(k-k_3)^2}\right)u_1u_3 \\
& + \left(Q_{121} + \frac{(2k-k_3)\,q_{021}q_{102}}{(k-k_3)^2}\right)u_2u_3 + O(3);
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
g_r \; & = k + cu_2 + q_{021}u_3 + \frac{1}{2}(C - k^3)u_1^2 + \frac{1}{2}\left(E - k^3 + \frac{2(2k-k_3)q_{021}^2}{(k-k_3)^2}\right)u_2^2 \\
& + \frac{1}{2}\left(Q_{022} - kk_3^2 + \frac{2(2k-k_3)q_{012}^2}{(k-k_3)^2}\right)u_3^2 + Du_1u_2 + Q_{121}u_1u_3 \\
& + \left(Q_{031} + \frac{2(2k-k_3)q_{012}q_{021}}{(k-k_3)^2}\right)u_2u_3 + O(3)
\end{aligned} \tag{28}
$$

7. Geometric coefficients defining the partially umbilic points

The coefficient $\zeta = \zeta_2 + \zeta_1 + \zeta_0$ in the Equation (29) below was obtained supposing

$$ac(q_{021} - q_{201}) = ac\delta \neq 0$$

and $b = q_{111} = 0$.

$$\begin{aligned}
\zeta_2 =& BD(k_3-k)^2q^2 + BQ_{112}c^2(k_3-k)^2 - 2BQ_{121}c(k_3-k)^2q \\
&-C^2(k_3-k)^2q^2 + 2CQ_{112}ac(k_3-k)^2 - 2CQ_{121}a(k_3-k)^2q \\
&+2CQ_{211}c(k_3-k)^2q + DQ_{112}a^2(k_3-k)^2 + 2DQ_{211}a(k_3-k)^2q \\
&-Q_{121}^2a^2(k_3-k)^2 - 2Q_{121}Q_{211}ac(k_3-k)^2 - Q_{211}^2c^2(k_3-k)^2 \\
\zeta_1 =& (k_3-k)(-2c^2q_{012}q_{102} + 2cq^2q_{102} + 2cqq_{102}q_{201})B \\
&-(k_3-k)(2a^2q_{012}q_{102} + 2aqq_{012}q_{201})D \\
&+(k_3-k)(2q_{201}q^3 + (2k^3(k_3-k) + (2aq_{102} + 2q_{201}^2))q^2 \\
&+2q_{201}(aq_{102} - cq_{012})q - 4acq_{012}q_{102})C \\
&-(k_3-k)(2ack^3(k_3-k) + 2acqq_{201} + 2acq_{201}^2)Q_{112} \\
&+(k_3-k)(2aq_{201}q^2 + (2ak^3(k_3-k) + 2a(aq_{102} + q_{201}^2))q \\
&+2aq_{201}(aq_{102} + cq_{012}))Q_{121} + 2q_{201}c(aq_{102} + cq_{012}))Q_{211} \\
&+(k_3-k)(-2cq_{201}q^2 + (-2ck^3(k_3-k) + 2c(aq_{102} - q_{201}^2))q \\
\zeta_0 =& -q^4q_{201}^2 + (-2k^3q_{201}(k_3-k) - 2q_{201}(aq_{102} + q_{201}^2))q^3 \\
&+(-k^6(k_3-k)^2 - 2k^3(aq_{102} + q_{201}^2)(k_3-k) - a^2q_{102}^2)q^2 \\
&\;(-4aq_{102}q_{201}^2 + 2cq_{012}q_{201}^2 - q_{201}^4)q^2 \\
&+(-2k^3q_{201}(aq_{102} - cq_{012})(k_3-k) - 2q_{201}(aq_{102} + q_{201}^2)(aq_{102} - cq_{012}))q \\
&+4ack^3q_{012}q_{102}(k_3-k) - q_{201}^2(aq_{102} - cq_{012})^2
\end{aligned} \tag{29}$$

Lemma 3. *In the space of 4-jets the equation $\zeta = 0$ is a cone of type $4\beta\gamma - \delta^2 = 0$.*

Proof. In relation to the variables $(B, C, D, Q_{112}, Q_{121}, Q_{211})$, ζ is a quadric with coefficients in the space of 3-jets. Defining $z_1 = \zeta_B = \partial\zeta/\partial B$, $z_2 = \zeta_C$, $z_3 = \zeta_D$, and solving the linear system, it follows that

$$\zeta = \frac{1}{4}\frac{4\,z_1\,z_3 - z_2^2}{(q_{021} - q_{201})^2\,(k - k_3)^2}.$$

□

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