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A GENERALIZATION OF ZAKALYUKIN'S LEMMA, AND SYMMETRIES OF SURFACE SINGULARITIES

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ABSTRACT. Zakalyukin's lemma asserts that the coincidence of the images of two wave front germs implies the right equivalence of corresponding map germs under a certain genericity assumption. The purpose of this paper is to give an improvement of this lemma for frontals. Moreover, we give several applications for singularities on surfaces.

INTRODUCTION

Let p be a fixed point on the n-dimensional Euclidean space $\boldsymbol{R}^n$ $(n \geq 1)$ and U a connected neighborhood of p in $\boldsymbol{R}^n$. In this paper, we set $r = \infty$ or $r = \omega$ and "C^r" means smoothness if $r = \infty$ and real analyticity if $r = \omega$.

A C^r-map $f : U \to \boldsymbol{R}^{n+1}$ is called a *frontal* or a *frontal map* if f admits a unit normal C^r-vector field ν defined on U. By parallel transport in $\boldsymbol{R}^{n+1}$, the vector field ν can be identified with its induced Gauss map $\nu : U \to S^n$, where S^n is the unit sphere centered at the origin of $\boldsymbol{R}^{n+1}$. In this setting, the pair of f and ν induces a C^r-map (called the *Legendrian lift* of f)

$$L_f := (f, \nu) : U \to \boldsymbol{R}^{n+1} \times S^n.$$

If L_f is an immersion, then f is called a *wave front*. In the case of $n = 2$, cuspidal edges and swallowtails are singular points appearing on wave fronts. Germs of cuspidal cross caps are not wave fronts, but are frontals. On the other hand, germs of cross caps are not frontals.

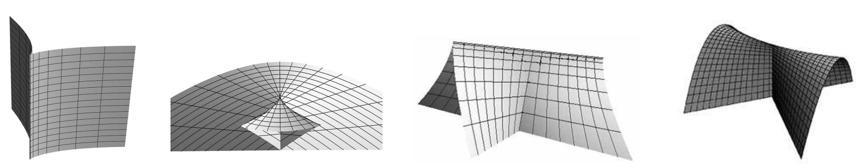

FIGURE 1. A cuspidal edge, swallowtail, cuspidal cross cap, cross cap, from the left.

Zakalyukin [16] pointed out that the coincidence of the images of two wave front germs induces the right equivalence of corresponding wave front germs under a certain properness of the map germs. It is then natural to ask under what possible weaker conditions the conclusion of Zakalyukin's lemma is still true. In this paper, we try to give such a condition as follows:

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Let $f : U \to \boldsymbol{R}^{n+1}$ be a continuous map and V an open neighborhood of $p \in U$. Then f is called *V-proper* at p if, for $\varepsilon(> 0)$, there exists $r \in (0, \varepsilon)$ such that $(f|_V)^{-1}(\overline{B(f(p), r)})$ is a compact subset of V (cf. Definition 1.1 in Section 1), where $B(f(p), r)$ is the open ball centered at $f(p)$ of radius r, $\overline{B(f(p), r)}$ is its closure in $\boldsymbol{R}^{n+1}$ and $f|_V$ is the restriction of f to the subset V. The following assertion gives a Zakalyukin-type lemma:

Theorem A. *Let U_i $(i = 1, 2)$ be a neighborhood of $p_i \in \boldsymbol{R}^n$ and let $f_i : U_i \to \boldsymbol{R}^{n+1}$ $(i = 1, 2)$ be two C^r-frontal maps with unit normal vector fields ν_i along f_i satisfying*

- (a1) *$f_1(U_1)$ is a subset of $f_2(U_2)$ and $(P :=)f_1(p_1) = f_2(p_2)$,*
- (a2) *f_2 is U_2-proper*[1] *at p_2 and $f_2^{-1}(P) = \{p_2\}$,*
- (a3) *the regular set of f_i $(i = 1, 2)$ is open dense in U_i,*
- (a4) *each Legendrian lift L_{f_i} $(i = 1, 2)$ is injective on a certain neighborhood of p_i (if f_i is a wave front, this condition is satisfied).*

Then there exists a homeomorphism $\psi : V_1 \to V_2$ between certain connected neighborhoods V_i $(i = 1, 2)$ of p_i satisfying the following properties:

- (1) *$\overline{V_i} \subset U_i$,*
- (2) *$f_1 = f_2 \circ \psi$ and $\nu_1 = \pm\nu_2 \circ \psi$ hold on V_1.*

Moreover, if f_1 and f_2 are wave fronts, then ψ can be taken as a C^r-diffeomorphism.

A Zakalyukin-type lemma for wave fronts was given in [10] (see also [11]), which was applied to prove criteria for cuspidal edges and swallowtails under the assumption that $f_1^{-1}(f_1(p_1))$ is finite as well as $f_2^{-1}(f_2(p_2))$. In the above theorem, the conclusion is obtained without any additional assumption for f_1. (The standard cuspidal edge and the standard swallowtail satisfy the condition (a2) for any choice of an open neighborhood U of the singular point $(0, 0)$ (see Proposition 1.14). So, to prove the criterion for swallowtails, Claim 1 in [11] is not needed.) The map ψ is called *the connecting map* between f_1 and f_2. In the statement of Theorem A, one cannot expect that ψ is smooth. In fact, if

$$f_1(t) := (t^2, t^3), \qquad f_2(t) = (t^6, t^9) \qquad (t \in \boldsymbol{R}),$$

then the connecting map is given by $\psi(t) = t^{1/3}$ which is not a diffeomorphism at $t = 0$. The authors are mainly interested in the case $n = 2$. In fact, a real analytic frontal in $\boldsymbol{R}^3$ usually admits a non-trivial isometric deformation at singular points (cf. [14, 8, 3]), and such isometric deformations of the surfaces are closely related to the properties of isomers (cf. Definition 5.1) as seen in the authors' previous work [3], and we shall discuss isomers of generalized cuspidal edges in the final section (Section 5) in this paper. Here, we consider generalized cuspidal edges as follows: We let I be a closed interval and fix a C^r-embedded curve $\mathbf{c} : I \to \boldsymbol{R}^3$, denoting by $C(:= \mathbf{c}(I))$ its image.

Definition 0.1. Let U be a domain in the uv-plane $(\boldsymbol{R}^2; u, v)$ containing the interval $I \times \{0\}$ on the u-axis. A C^r-map $f : U \to \boldsymbol{R}^3$ defined on a domain U of $\boldsymbol{R}^2$ is called a *C^r-differentiable generalized cuspidal edge along C* if $f(I \times \{0\})$ contains C and the singular set of f contains $I \times \{0\}$, and there exist

- a diffeomorphism φ from a tubular neighborhood V of $I \times \{0\}$ to the st-plane $(\boldsymbol{R}^2; s, t)$ satisfying $\varphi(I \times \{0\}) = [-1, 1] \times \{0\}$, and
- a diffeomorphism Φ from a tubular neighborhood of C to $\boldsymbol{R}^3$

[1] We cannot drop the condition that f_2 is U_2-proper. In fact, the condition $f_2^{-1}(P) = \{p_2\}$ implies only the existence of a neighborhood $V(\subset U_2)$ of p_2 so that f_2 is V-proper (see Theorem 1.12), but V may not coincide with U_2 in general.

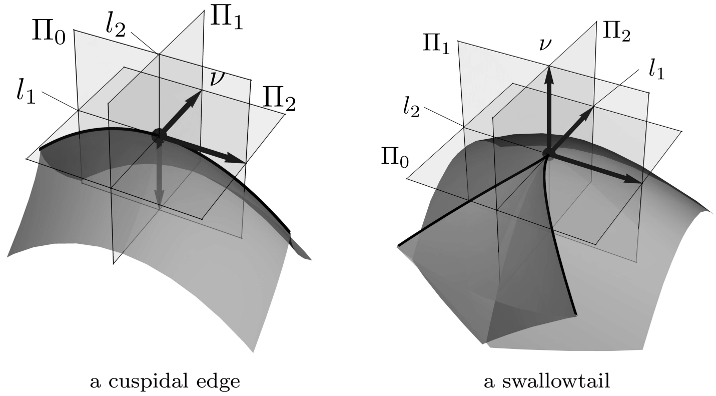

FIGURE 2. The limiting tangent plane Π_0, the normal plane Π_1 and the co-normal plane Π_2 for a cuspidal edge (left) and a swallowtail (right)

such that $\Phi \circ f \circ \varphi^{-1}(s,t) = (t^2, t^3\alpha(s,t), s)$ holds on $\varphi(V)$.

On the other hand, a point $p \in U$ is called *C^r-differentiable generalized cuspidal edge point* of a C^r-map $f : U \to \boldsymbol{R}^3$ defined on a domain U of $\boldsymbol{R}^2$ if there exist a local diffeomorphism ψ satisfying $\psi(p) = (0,0)$ and a local diffeomorphism Ψ in $\boldsymbol{R}^3$ such that

$$\Psi \circ f \circ \psi^{-1}(s,t) = (t^2, t^3\alpha(s,t), s)$$

holds on a neighborhood of the origin in the st-plane.

Since $f_0(s,t) := (t^2, t^3\alpha(s,t), s)$ has the non-vanishing normal vector field

$$\tilde{\nu}_0(s,t) := \left(-3t\alpha(s,t) - t^2\alpha_t(s,t),\ 2,\ -2t^3\alpha_s(s,t)\right),$$

generalized cuspidal edges are all frontals. Cuspidal edges and cuspidal cross caps are typical examples of generalized cuspidal edges. However, since generalized cuspidal edges are not wave fronts in general (e.g. cuspidal cross caps), the smoothness of connecting maps ψ does not follow from Theorem A directly. We let I be a closed interval and fix a C^r-embedded curve $\mathbf{c} : I \to \boldsymbol{R}^3$, denoting by $C(:= \mathbf{c}(I))$ its image. As an improvement of the statement of Theorem A for such singular points, we show the following:

Theorem B. *Let U_i $(i = 1,2)$ be an open subset containing a closed interval $I_i \times \{0\}$ on the u-axis in $(\boldsymbol{R}^2; u, v)$, and let $f_i : U_i \to \boldsymbol{R}^3$ $(i = 1,2)$ be a C^r-differentiable generalized cuspidal edge along the same embedded space curve C. If $f_1(U_1) \subset f_2(U_2)$ then there exist*

- *an open subset $V_i(\subset U_i)$ containing the interval $I_i \times \{0\}$, and*
- *a C^r-diffeomorphism $\psi : V_1 \to V_2$*

such that $f_1 = f_2 \circ \psi$ and $\nu_1 = \pm\nu_2 \circ \psi$ hold on V_1, where ν_i $(i = 1,2)$ is a unit normal vector field along f_i. As a consequence, under the assumption of Theorem A, if p_1 and p_2 are both generalized cuspidal edge points, then the connecting map ψ can be taken as a diffeomorphism.

The corresponding assertion for cross caps is given in the authors' previous work [7]. To give an application of Theorem B, we prepare several terminologies which are useful for investigating the symmetries of surfaces at singular points:

Definition 0.2. Let $p \in U$ be a co-rank one singular point of a frontal map $f : U \to \boldsymbol{R}^3$. Then there exists a local coordinate system (u,v) centered at p such that $f_v(p) = \mathbf{0} = (0,0,0)$, and the line

$$l_1 := \{f(p) + t f_u(p)\,;\, t \in \boldsymbol{R}\}$$

is defined, which is called the *tangent line* of f at p (see Figure 2). The plane Π_0 passing through $f(p)$ which is perpendicular to the unit normal vector ν of f at $f(p)$ is called the *limiting tangent plane* of f at p. The line l_2 passing through $f(p)$ lying in Π_0 which is perpendicular to l_1 is called the *co-normal line* of f at p.

On the other hand, the plane Π_1 passing through $f(p)$ which is perpendicular to the tangent line l_1 is called the *normal plane* of f at p. Finally, the plane Π_2 passing through $f(p)$ spanned by the vectors $\nu(p)$ and $f_u(p)$ is called the *co-normal plane* at p.

By definition, the intersection of the two planes Π_0 and Π_1 is the co-normal line l_2. When p is a cuspidal edge, then one of the two half-lines in the co-normal line l_2 emanating from $f(p)$ points in the direction where the image of f lies, which is called the *cuspidal direction*. The section of the image of f by Π_1 at $f(p)$ gives a cusp (cf. [1] and [3]), which is called the *sectional cusp* at $f(p)$. The cuspidal direction is the line in the normal plane Π_1 at $f(p)$ which bisects the cusp. On the other hand, if p is a swallowtail, then the projection of the singular set image of f to Π_0 forms a cusp in Π_0 (cf. [13]). We show the following:

Theorem C. *Let $f : U \to \boldsymbol{R}^3$ be a C^r-map defined on a non-empty open subset $U\,(\subset \boldsymbol{R}^2)$, and let $p \in U$ be a cuspidal edge, a swallowtail, or a cuspidal cross cap. Suppose that f is U-proper at p, $f^{-1}(f(p)) = \{p\}$ and there exist an isometry T of $\boldsymbol{R}^3$ fixing $f(p)$ and an open neighborhood V of p such that $T \circ f(V) \subset f(U)$. Then T is an involution. If T is not the identity map, then it is*

(i) *the reflection with respect to the limiting tangent plane Π_0,*
(ii) *the reflection with respect to the normal plane Π_1,*
(iii) *the reflection with respect to the co-normal plane Π_2, or*
(iv) *the $180°$-rotation with respect to the co-normal line l_2.*

Moreover, there exist a connected open neighborhood $W(\subset V)$ of p and a C^r-involution $\psi : W \to W$ such that $f \circ \psi = T \circ f$ on W. Furthermore, the following assertions hold:

(c1) *If p is a cuspidal edge or a cuspidal cross cap singular point, then* (iii) *never happens. Moreover, if p is a point at which the limiting normal curvature does not vanish, then only* (ii) *happens.*
(c2) *If p is a swallowtail or a cuspidal cross cap singular point, then each point of the image $f(S)$ of the self-intersection set $S(\subset W)$ of f is fixed by T.*
(c3) *If p is a swallowtail, only* (iii) *happens.*

The assumption that f is U-proper at p and $f^{-1}(f(p)) = \{p\}$ is not artificial because if a smooth map has a singular point giving a cuspidal edge, a swallowtail, or a cuspidal cross cap singularity, then the restriction of the map to a sufficiently small neighborhood satisfies such a property (cf. Proposition 1.14). The corresponding assertions for cross cap singular points have been shown in [7]. The assertion (c1) contains a symmetric property of cuspidal edges with non-zero limiting normal curvature, which has been shown in [3, Theorem 5.1] as a special case.

Also, in the authors' previous work [3] (see also [4]), "isomers" of a given real analytic cuspidal edge f were introduced, which are cuspidal edges with the same first fundamental form as f whose singular set image coincides with that of a given cuspidal edge f but their images are not congruent to that of f. By Theorem B, we can use the fact that image equivalence (cf. Definition 1.18) of admissible generalized cuspidal edges is the same as right-left equivalence of them, like as in the case of cuspidal edges. As a consequence, almost all assertions on isomers of real analytic cuspidal edges in [3] and [4] can be generalized for real analytic admissible generalized cuspidal edges. We will prove this fact at the end of this paper. Moreover, in the authors' previous works [5] and [6], isomers of curved foldings are also discussed, which can be considered as analogues of isomers of real analytic cuspidal edges. We also point out the existence of a canonical map from the class of real analytic admissible generalized cuspidal edges to the class of real analytic curved foldings by which the isomers of them are obtained as the image of those of the generalized cuspidal edge.

The paper is organized as follows: In Section 1, we discuss properness of continuous maps at a given point. In Section 2, we prove Theorem A. In Section 3, we prove Theorem B, and Theorem C is proved in Section 4. Finally, in Section 5, we discuss isomers of generalized cuspidal edges and also the connection to curved foldings.

1. Pointwise properness for continuous maps

There seems to be no explicit definition of local properness of maps, not only in [16] but also in other references as far as the authors know. So, in this section, we discuss the pointwise properness mentioned in the introduction.

Let X, X_1 and X_2 be locally connected and locally compact Hausdorff spaces. We also fix a locally compact Hausdorff space Y satisfying the axiom of second countability. By Urysohn's metrization theorem, we can fix a distance function d_Y on Y which is compatible with the topology of Y. We fix it and also a point $p \in X$ with its open neighborhood U. For each $r(>0)$, we denote by $B_Y(P,r)$ the open ball of radius r centered at $P(\in Y)$ and by $\overline{B_Y(P,r)}$ its closure.

Definition 1.1. A continuous map $f : X \to Y$ is said to be *U-proper at a point* $p \in X$ if there exists $r > 0$ such that $(f|_U)^{-1}(\overline{B_Y(P,r)})$ is a compact subset of U. Moreover, f is said to be *strongly U-proper at* p, if for each neighborhood $V(\subset U)$ of p, there exists $r > 0$ such that $(f|_U)^{-1}(\overline{B_Y(P,r)})$ is a compact subset of V.

We then give the following definition:

Definition 1.2. A continuous map $f : X \to Y$ is said to be *proper at a point* $p \in X$ if there exists a neighborhood U of p such that f is strongly U-proper at p.

The following assertion implies that our pointwise properness defined in Definition 1.2 can be considered as a property of map germs:

Proposition 1.3. *Suppose that $f : X \to Y$ is a strongly U-proper map at $p \in U$. Then for each neighborhood $V(\subset U)$ of p, f is strongly V-proper at p.*

It is sufficient to show the following assertion:

Lemma 1.4. *Suppose that $f : X \to Y$ is a strongly U-proper map at $p \in U$. Then, for each neighborhood $V(\subset U)$ of p, there exists $r_V(>0)$ such that*

$$(f|_U)^{-1}(B_Y(f(p),r)) = (f|_V)^{-1}(B_Y(f(p),r)),$$
$$(f|_U)^{-1}(\overline{B_Y(f(p),r)}) = (f|_V)^{-1}(\overline{B_Y(f(p),r)}) \qquad (r \in (0,r_V]).$$

Proof. We fix a neighborhood $V(\subset U)$ of p. Since f is strongly U-proper at p, there exists $r_V(>0)$ such that

$$K := (f|_U)^{-1}(\overline{B_Y(f(p),r)}) \qquad (r \in (0,r_V])$$

is a compact subset of V. In particular, $O := (f|_U)^{-1}(B_Y(f(p),r))$ is also a subset of V for each $r \in (0,r_V]$. So we have

$$O \subset (f|_V)^{-1}(B_Y(f(p),r)), \qquad K \subset (f|_V)^{-1}(\overline{B_Y(f(p),r)}). \tag{1.1}$$

On the other hand, the opposite inclusions

$$(f|_V)^{-1}(B_Y(f(p),r)) \subset (f|_U)^{-1}(B_Y(f(p),r)) = O,$$
$$(f|_V)^{-1}(\overline{B_Y(f(p),r)}) \subset (f|_U)^{-1}(\overline{B_Y(f(p),r)}) = K$$

are clear. □

Recall that a continuous map $f : X \to Y$ between two topological spaces X and Y is said to be *proper* if for each compact subset $K\,(\subset Y)$, the inverse image $f^{-1}(K)$ is compact.

Example 1.5. We consider a function $f : \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) := xe^{-x^2}$. This function itself is not a proper map, but for each $\varepsilon(>0)$, the restriction of f to $U := (-\varepsilon, \varepsilon)$ is strongly U-proper at $x = 0$.

Example 1.6. Define a continuous function $f : \boldsymbol{R} \to \boldsymbol{R}$ so that $f(x) = x(1 - |x|^{-1})$ if $|x| > 1$ and $f(x) = 0$ if $|x| \le 1$. Obviously, f is a proper map, but not proper at $x = 0$. In fact, if we set $U := (-\varepsilon, \varepsilon)$ $(0 < \varepsilon < 1)$, then for each $r \in (0, \varepsilon)$

$$(f|_U)^{-1}([-r,r]) = (f|_U)^{-1}(\{0\}) = U$$

and so f cannot be U-proper at $x = 0$. This implies that the properness of a continuous map does not imply the properness of the map at a given point, in general.

Example 1.7. Consider a continuous function given by

$$f(x) := x \sin \frac{1}{x} \qquad (x \in [-1,1]).$$

Since $[-1, 1]$ is compact, f is a proper map. Moreover, it is easy to check that f is U-proper at $x = 0$ for each choice of an open interval $U := (-\varepsilon, \varepsilon)$ $(0 < \varepsilon < 1)$. However, f is not strongly U-proper at $x = 0$. In fact, we set

$$V_k := (-a_k, a_k), \qquad a_k := \frac{1}{k\pi},$$

where k is a positive integer satisfying $\pm a_k \in U$. Then we have $V_k \subset U$. We fix such a V_k and set $K_r := [-r, r]$ $(r > 0)$. Since 0 is an interior point of $f^{-1}(K_r)$ and $f(\pm a_k) = 0$, there exists $\delta(>0)$ depending on r satisfying $f((a_k - \delta, a_k)) \subset K_r$, which implies that $(f|_{V_k})^{-1}(K_r)$ cannot be a compact subset of V_k.

We prepare the following:

Lemma 1.8. *Let $f : X \to Y$ be a continuous map and U a neighborhood of $p \in X$. Suppose that f is U-proper at p. If $(f|_U)^{-1}(f(p)) = \{p\}$, then f is strongly U-proper at p.*

Proof. We fix a neighborhood $V(\subset U)$ of p. Since f is U-proper at p, there exists $r_0(>0)$ such that

$$K_r := (f|_U)^{-1}(\overline{B_Y(f(p),r)})$$

is a compact subset of U for each $r \in (0, r_0]$. It is sufficient to show that K_r is contained in V for sufficiently small r. If this fails, then, for each positive integer k satisfying $1/k < r_0$, there exists

$$q_k \in (f|_U)^{-1}(\overline{B_Y(f(p),1/k)})(= K_{r_0})$$

which does not belong to V. By our construction of the sequence $\{q_k\}_{k=1}^{\infty}$, it consists of infinitely many points. Since K_{r_0} is compact, the sequence $\{q_k\}_{k=1}^{\infty}$ has an accumulation point $q_\infty \in K_{r_0}$. Since $f(q_k) \in \overline{B_Y(f(p),1/k)}$, we have $f(q_\infty) = f(p)$. Since $(f|_U)^{-1}(f(p))$ is the single point set $\{p\}$, we can conclude $q_\infty = p$. On the other hand, since $q_k \in K_{r_0} \setminus V$, we have $q_\infty \in K_{r_0} \setminus V$, contradicting the fact $q_\infty = p$. □

Corollary 1.9. *Let $f : X \to Y$ be a proper map. If $f^{-1}(f(p)) = \{p\}$ holds, then f is strongly U-proper for each open neighborhood U of p.*

Proof. In the setting of Lemma 1.8, we put $U := X$. Since f is a proper map, f is strongly X-proper at p. Since $f^{-1}(f(p)) = \{p\}$ holds, f is strongly X-proper at p. By Proposition 1.3, f is strongly U-proper for any open neighborhood U of p. □

We next prove the following:

Lemma 1.10. *Let $f : X \to Y$ be a continuous map and U a neighborhood of a point $p \in X$. If f is strongly U-proper at p, then $(f|_U)^{-1}(f(p))$ coincides with $\{p\}$.*

Proof. We set $g := f|_U$ and suppose that $g^{-1}(f(p))$ contains a point $q \in U$ other than p. Since X is a Hausdorff space, there exists a pair (V_1, V_2) of disjoint open subsets such that $p \in V_1$ and $q \in V_2$.

Since f is strongly U-proper at p, there exists $\varepsilon(> 0)$ such that $g^{-1}(\overline{B_Y(f(p), \varepsilon)}) \subset V_1$. Then we have that

$$q \in g^{-1}(f(p)) \subset g^{-1}(\overline{B_Y(f(p), \varepsilon)}) \subset V_1,$$

contradicting the fact that $q \in V_2$. □

We next prepare the following assertion, which is a generalization of the proposition given in [10].

Proposition 1.11. *Let $f : (X, p) \to (Y, P)$ be a continuous map such that $f^{-1}(P)$ is a finite point set. Let U be a neighborhood of p. Then there exists $\delta(> 0)$ such that the connected component V_r of $f^{-1}(B_Y(P, r))$ containing p satisfies $\overline{V_r} \subset U$ for each $r \in (0, \delta]$. Moreover, V_r is a relatively compact open neighborhood of p and f is V_r-proper at p.*

Proof. The case that $X := U(\subset \boldsymbol{R}^n)$ is discussed in [10]. We need a few modification to prove this assertion: Since X is locally connected, V_r is an open neighborhood of p. Moreover, since X is a locally compact Hausdorff space, we can take a relatively compact open neighborhood W of p such that $\overline{W}$ is contained in U. Since $f^{-1}(P)$ is a finite point set, we may assume that

$$f^{-1}(P) \cap W = \{p\} \tag{1.2}$$

holds. As in the statement of the proposition, we let V_r be the connected component of $f^{-1}(B_Y(P, r))$ containing p for each $r > 0$. Since $V_s \subset V_r$ for $s < r$, it is sufficient to show that $\overline{V_{1/k}} \subset W$ for a sufficiently large integer $k > 0$. If not, we have the following decomposition

$$\overline{V_{1/k}} = \left(W \cap \overline{V_{1/k}}\right) \cup \left((X \setminus \overline{W}) \cap \overline{V_{1/k}}\right).$$

The right-hand side is the union of two non-empty open subsets of $\overline{V_{1/k}}$, which is a contradiction, because $\overline{V_{1/k}}$ is connected. Thus, we can find a point $p_k \in \overline{V_{1/k}} \cap \partial W$ for each k. Since f is continuous, we have

$$f(\overline{V_{1/k}}) \subset \overline{f(V_{1/k})} \subset \overline{B_Y(P, 1/k)}.$$

By our construction, the sequence $\{p_k\}_{k=1}^{\infty}$ consists of infinitely many points, and has an accumulation point $p_\infty \in \partial W$ because ∂W is compact. Since $p_k \in \overline{V_{1/k}}$, $f(p_k)$ belongs to $B_Y(P, 1/k)$. In particular, $\{f(p_k)\}_{k=1}^{\infty}$ converges to the point P, and so $f(p_\infty) = P$ holds, which contradicts (1.2). So we have shown that $\overline{V_{1/k}} \subset W$ for a sufficiently large integer $k > 0$. Since $\overline{W}$ is compact, $\overline{V_{1/k}}$ is also compact.

We fix such an integer k and set $r_0 := 1/k$ and now prove that f is V_r-proper at p under the assumption that $r < r_0$. Since V_r is a subset of V_{r_0}, its closure $\overline{V_r}(\subset U)$ is compact. We set $g := f|_{V_r}$ and let K be a compact subset of $B_Y(P, r)$. Suppose that $g^{-1}(K)$ is not compact.

Then there exists a sequence $\{x_k\}_{k=1}^{\infty}$ in $g^{-1}(K)$ which does not accumulate to any point in V_r. Since $\overline{V_r}$ is compact, $\{x_k\}_{k=1}^{\infty}$ must have an accumulation point $x_\infty \in \partial V_r$. Since $f(x_k) \in K$, we have

$$f(x_\infty) \in K \subset B_Y(P, r).$$

In particular, there exists a neighborhood O of x_∞ such that $f(O) \subset B_Y(P, r)$, which implies

$$f(V_r \cup O) \subset B_Y(P, r).$$

Since $x_\infty \in V_r \cap O$, the union $V_r \cup O$ is connected. Since V_r is a connected component of $(f|_U)^{-1}(B_Y(P, r))$, we have $V_r \cup O = V_r$, contradicting the fact that $x_\infty \in \partial V_r$. Thus, $g^{-1}(K)$ is a compact subset of V_r. □

Theorem 1.12. *Let $f : X \to Y$ be a continuous map. Then the following three conditions are equivalent:*

(1) *The map f is proper at p.*
(2) *There exists a neighborhood U of p such that $(f|_U)^{-1}(f(p))$ is a finite point set.*
(3) *There exists a neighborhood U of p such that $(f|_U)^{-1}(f(p)) = \{p\}$, and f is U-proper at p.*

In particular, (2) *can be considered as a useful criterion for the pointwise properness of continuous maps.*

Proof. By Lemma 1.10, (1) implies (2). We set $g := f|_U$. Since $g^{-1}(f(p))$ is a finite point set, we may assume that $g^{-1}(g(p)) = \{p\}$. On the other hand, by Proposition 1.11, (2) implies that f is U-proper at p for a sufficiently small neighborhood U of p. So (3) is obtained. Finally, (3) implies (1) by Lemma 1.10. □

Corollary 1.13. *Let U be a non-empty open subset of $\boldsymbol{R}^n$ and $f : U(\subset \boldsymbol{R}^n) \to \boldsymbol{R}^N$ $(n \leq N)$ a C^r-immersion. Then f is proper at each point of U.*

Proof. Since f is an immersion, it is locally injective. So we obtain the assertion. □

The standard cuspidal edge, the standard swallowtail, the standard cuspidal cross cap and the standard cross cap (see Figure 1) are defined by

$$\begin{aligned} &f_C(u, v) = (v^2, v^3, u), \qquad f_S(u, v) = (3v^4 + uv^2, 4v^3 + 2uv, u), \\ &f_{CW}(u, v) = (v^2, uv^3, u), \qquad f_W(u, v) = (uv, v^2, u) \end{aligned} \tag{1.3}$$

as maps from $\boldsymbol{R}^2$ into $\boldsymbol{R}^3$, respectively. Using these expressions, we can prove the following:

Proposition 1.14. *The standard cuspidal edge f_C, the standard swallowtail f_S, the standard cuspidal cross cap f_{CW} and the standard cross cap f_{CW} are U-proper at their singular point* $(0, 0)$ (which is the origin of the domain) *for any choice of an open neighborhood U of* $(0, 0)$*. Moreover, $f^{-1}(f(0, 0)) = \{(0, 0)\}$ holds.*

Proof. The property that $f^{-1}(f(0, 0)) = \{(0, 0)\}$ is obvious. By Corollary 1.9, it is sufficient to show that f_C, f_S, f_{CW} and f_C are proper maps on $\boldsymbol{R}^2$. Here we only show that f_S is a proper map. (The properness of the other maps can be proved using the same argument.) We let K be a compact subset of $\boldsymbol{R}^3$, and let $\{(u_k, v_k)\}_{k=1}^{\infty}$ be a sequence in $f_S^{-1}(K)$. We set $f_S(u_k, v_k) = (a_k, b_k, c_k)$, then $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ and $\{c_k\}_{k=1}^{\infty}$ are bounded sequence in $\boldsymbol{R}$ because of the compactness of K. Then $\{v_k\}_{k=1}^{\infty}$ is bounded, because $v_k = c_k$. Moreover, the first component of $f_S(u_k, v_k)$ satisfies $3u_k^4 + u_k^2 v_k = a_k$, that is u_k is a solution of the equation $3t^4 + c_k t^2 = a_k$. Since c_k, a_k are bounded, we can conclude that $\{u_k\}_{k=1}^{\infty}$ is also bounded. Thus $\{(u_k, v_k)\}_{k=1}^{\infty}$ contains a convergent subsequence. So $f_S^{-1}(K)$ is compact. □

Corollary 1.15. *A C^r-map $f : U \to \boldsymbol{R}^3$ which has a cuspidal edge, a swallowtail, a cuspidal cross cap or a cross cap singularity at p is proper at p.*

Proof. It is sufficient to show that $(f|_V)^{-1}(f(p)) = \{p\}$ for a sufficiently small neighborhood $V(\subset U)$ of p. However, this is obvious because the standard maps for cuspidal edges, swallowtails, cuspidal cross caps and cross caps have such a property. $\square$

We next prove the following, which will be applied to prove Theorem A:

Theorem 1.16. *Let $f_i : (X_i, p_i) \to (Y, P)$ $(i = 1, 2)$ be continuous maps (in particular, $P := f_1(p_1) = f_2(p_2)$). Suppose that f_2 is U_2-proper at p_2 and $f_2^{-1}(f_2(p_2)) = \{p_2\}$. Then the following three conditions are equivalent:*

(1) *There exists a neighborhood $V_i(\subset U_i)$ of p_i for each $i = 1, 2$ such that $f_1(V_1) \subset f_2(V_2)$.*
(2) *There exist $r > 0$ and a neighborhood $V_i(\subset U_i)$ of p_i for each $i = 1, 2$ such that*
$$f_1(V_1) \cap B_Y(P, r) \subset f_2(V_2) \cap B_Y(P, r).$$
(3) *For each neighborhood $V_i(\subset U_i)$ of p_i $(i = 1, 2)$, there exists a relatively compact neighborhood W_i of p_i such that $f_1(W_1) \subset f_2(W_2)$ and $\overline{W_i} \subset V_i$.*

Proof. Obviously (1) implies (2), and also (3) implies (1). So it is sufficient to show that (2) implies (3). So we assume (2). We set $g_i := f_i|_{V_i}$ $(i = 1, 2)$. Since f_2 is U_2-proper at p_2 and $g_2^{-1}(P) = \{p_2\}$, Lemma 1.8 implies that f_2 is strongly U_2-proper at p_2. Hence, f_2 is strongly V_2-proper at p_2 by Proposition 1.3. As a consequence,
$$K_2 := g_2^{-1}(\overline{B_Y(P, r)})$$
is a compact subset of V_2 for sufficiently small $r(> 0)$. We fix such an r, and set
$$W_2 := g_2^{-1}(B_Y(P, r)).$$
Then we have $\overline{W_2} \subset K_2(\subset V_2)$. Since X is a locally compact Hausdorff space, there exists a relatively compact neighborhood W_1 of p_1 satisfying $\overline{W_1} \subset V_1$. Moreover, since f_1 is continuous, we may assume $W_1 \subset g_1^{-1}(B_Y(P, r))$, and so
$$\begin{aligned} f_1(W_1) = g_1(W_1) &\subset B_Y(P, r) \cap g_1(V_1) \\ &\subset B_Y(P, r) \cap g_2(V_2) = g_2(W_2) = f_2(W_2). \end{aligned}$$
Since $\overline{W_i} \subset V_i$ $(i = 1, 2)$, we obtain (3). $\square$

Example 1.17. In Theorem 1.16, the assumption that $f_2 : U_2 \to Y$ is proper at p_2 cannot be removed. In fact, we set $f_1(x) := x$ $(x \in \boldsymbol{R})$ and let $f_2(x)$ be the function on $\boldsymbol{R}$ defined in Example 1.6. Then $f_1(\boldsymbol{R}) = f_2(\boldsymbol{R}) = \boldsymbol{R}$ holds. However, if we choose $V_1 = V_2 = (-1, 1)$, then $f_1(\overline{W_1}) \subset f_2(\overline{W_2})$ never holds for any choice of a pair of open intervals (W_1, W_2) containing the origin in $(-1, 1)$. In this case, $f_2(x)$ is not proper at $x = 0$ as shown in Example 1.6.

Here, we give the following terminology:

Definition 1.18. Let $f_i : (X_i, p_i) \to (Y, P)$ $(i = 1, 2)$ be two continuous maps. Then f_1 is said to be *image equivalent to* f_2 with respect to the pair of points (p_1, p_2) if for any choice of a neighborhood $U_i(\subset X_i)$ $(i = 1, 2)$ of p_i, there exists a neighborhood $V_i(\subset U_i)$ $(i = 1, 2)$ of p_i such that $f_1(V_1) \subset f_2(U_2)$ and $f_2(V_2) \subset f_1(U_1)$ hold simultaneously.

Related to this definition, we also give the following.

Definition 1.19. Let $f_i : (X_i, p_i) \to (Y, P)$ $(i = 1, 2)$ be two continuous maps. Then f_1 is said to be *equi-image equivalent to* f_2 *as a map germ* if for any choice of a neighborhood $U_i(\subset X_i)$ $(i = 1, 2)$ of p_i, there exists a neighborhood $V_i(\subset U_i)$ $(i = 1, 2)$ of p_i such that $f_1(V_1) = f_2(V_2)$.

Proposition 1.20. *Let $f_i : (X_i, p_i) \to (Y, P)$ $(i = 1, 2)$ be two continuous maps which are proper at p_i. Then, the following assertions are equivalent each other:*

(1) *The map f_1 is image equivalent to f_2 with respect to (p_1, p_2).*
(2) *The map f_1 is equi-image equivalent to f_2 with respect to (p_1, p_2).*

Proof. Since the assertions (1) and (2) are local, we may assume that $f_i^{-1}(P) = \{p_i\}$, without loss of generality. It is obvious that (2) implies (1). So it is sufficient to show that (1) implies (2). We let U_i $(i = 1, 2)$ be a neighborhood of p_i. Since X_i is a locally compact Hausdorff space, we can take a neighborhood W_i of p_i $(i = 1, 2)$ so that $\overline{W_i}(\subset U_i)$ is compact and f_i is W_i-proper at p_i (cf. Proposition 1.11). By Theorem 1.12, we may assume that f_i $(i = 1, 2)$ is strongly W_i-proper at p_i. By (1), we may assume that both $f_2(V_2) \subset f_1(W_1)$ and $f_1(V_1) \subset f_2(W_2)$ hold for some $V_i \subset W_i$ $(i = 1, 2)$, and there exists $r_0(> 0)$ such that

$$(f_2|_{W_2})^{-1}(B_Y(P, r)) \subset V_2 \quad \text{and} \quad (f_1|_{W_1})^{-1}(B_Y(P, r)) \subset V_1$$

for $r \in (0, r_0]$. Then we have

$$\begin{aligned} f_2\Big((f_2|_{W_2})^{-1}(B_Y(P, r))\Big) &= B_Y(P, r) \cap f_2(V_2) \\ &\subset B_Y(P, r) \cap f_1(W_1) = f_1\Big((f_1|_{W_1})^{-1}(B_Y(P, r))\Big) \end{aligned}$$

and

$$\begin{aligned} f_1\Big((f_1|_{W_1})^{-1}(B_Y(P, r))\Big) &= B_Y(P, r) \cap f_1(V_1) \\ &\subset B_Y(P, r) \cap f_2(W_2) = f_2\Big((f_2|_{W_2})^{-1}(B_Y(P, r))\Big). \end{aligned}$$

So, we have

$$f_1\Big((f_1|_{W_1})^{-1}(B_Y(P, r))\Big) = f_2\Big((f_2|_{W_2})^{-1}(B_Y(P, r))\Big) \quad (r \in (0, r_0]).$$

□

It should be remarked that Theorem A in the introduction gives a sufficient condition for equi-image equivalency without assuming the image equivalency between two maps (see Corollary 2.4).

Remark 1.21. We consider the condition that for any choice of a neighborhood $U_2(\subset X_2)$ of p_2, there exists a neighborhood $V_1(\subset U_1)$ of p_1 such that $f_1(V_1) \subset f_2(U_2)$. This condition does not imply the existence of a neighborhood V_2 such that $f_2(V_2) \subset f_1(V_1)$. In fact, if we set

$$f_1(x) = x^2, \qquad f_2(x) = x,$$

then $f_1((-r, r)) \subset f_2(\boldsymbol{R})$, but the opposite inclusion $f_2((-r, r)) \subset f_1(\boldsymbol{R})$ never holds for any $r > 0$. So, to show equi-image equivalency, we need to assume the image equivalency of f_1 and f_2 in the statement of Proposition 1.20.

2. Proof of Theorem A

To prove Theorem A, we prepare several propositions and lemmas: Let U_i $(i = 1, 2)$ be two domains in $\boldsymbol{R}^n$, and let $f_i : U_i \to \boldsymbol{R}^{n+1}$ be two C^r-differentiable frontal maps with C^r-differentiable unit normal vector fields ν_i defined on U_i. Since p_2 satisfies (a2),

$$f_2^{-1}(f_2(p_2)) = \{p_2\} \tag{2.1}$$

holds. We take the 'Legendrian lift'

$$L_{f_i} := (f_i, \nu_i) \colon U_i \longrightarrow \boldsymbol{R}^{n+1} \times S^n$$

of f_i $(i=1,2)$. We also consider the map

$$L'_{f_2} := (f_2, -\nu_2)\colon U_2 \longrightarrow \boldsymbol{R}^{n+1} \times S^n.$$

By (a4), there exists an open neighborhood $U'_i(\subset U_i)$ $(i=1,2)$ such that L_{f_i} is injective on U'_i. Since $(P, \nu_2(p_2)) \neq (P, -\nu_2(p_2))$ $(P := f_1(p_1) = f_2(p_2))$, there exists a relatively compact open subset V_2 of U'_2 satisfying $\overline{V_2} \subset U'_2$ and

$$L_{f_2}(\overline{V_2}) \cap L'_{f_2}(\overline{V_2}) = \emptyset. \tag{2.2}$$

By Theorem 1.16, there exists a relatively compact open subset V_1 of U'_1 satisfying $\overline{V_1} \subset U'_1$ and $f(\overline{V_1}) \subset f_2(V_2)$. We then set

$$L_1 := L_{f_1}|_{\overline{V_1}}, \quad L_2 := L_{f_2}|_{\overline{V_2}}, \quad L'_2 := L_{f_2}|_{\overline{V_2}},$$

and

$$B_+ := \{p \in \overline{V_1}\,;\, L_1(p) \in L_2(\overline{V_2})\}, \quad B_- := \{p \in \overline{V_1}\,;\, L_1(p) \in L'_2(\overline{V_2})\}.$$

Then we can rewrite

$$B_+ = L_1^{-1}(L_2(\overline{V_2})), \qquad B_- = L_1^{-1}(L'_2(\overline{V_2}))$$

and so $B_\pm$ are closed subsets of $\overline{V_1}$. We set $g_i := f_i|_{\overline{V_i}}$ $(i=1,2)$ and let $\mathcal{R}_i$ be the set of regular values of the map g_i. We set $\mathcal{R} := \mathcal{R}_1 \cap \mathcal{R}_2$ and

$$A_1 := g_1^{-1}(\mathcal{R}), \qquad S_1 := \overline{V_1} \setminus A_1.$$

Proposition 2.1. *The relation $\overline{V_1} = B_+ \cup B_-$ holds.*

To prove this, we prepare the following lemma:

Lemma 2.2. *A_1 is a dense subset of $\overline{V_1}$.*

Proof. We suppose that A_1 is not dense in $\overline{V_1}$. Then S_1 has an interior point q. Since the boundary ∂V_1 of $V_1(\subset U_1)$ has no interior point and g_1 gives an immersion on an open dense set of V_1, we may assume that there exists an open neighborhood $W(\subset V_1)$ of q such that $W \subset S_1$ and $g_1|_W$ is an immersion. By the Sard theorem, $f(S_1)$ is of Hausdorff dimension less than n, contradicting the fact that $f_1|_W$ is an immersion. ☐

Proof of Proposition 2.1. We fix an arbitrary $\boldsymbol{a} \in \mathcal{R}$, and show the inverse image $g_i^{-1}(\boldsymbol{a})$ $(i=1,2)$ are finite point sets. It is enough to show this for $i=1$, namely, showing it for g_1. We assume $g_1^{-1}(\boldsymbol{a})$ is an infinite point set. Since $\overline{V_1}$ is compact, taking a sequence $\{q_k\}_{k=1}^\infty \subset g_1^{-1}(\boldsymbol{a})$ consisting of distinct points, it has an accumulation point $q \in \overline{V_1}$. Replacing $\{q_k\}_{k=1}^\infty$ by a suitable subsequence if necessary, we may assume that it converges to q. Since $f_1(q_k) = \boldsymbol{a}$, by the continuity of f_1, it holds that $f_1(q) = \boldsymbol{a}$. By the definition of $\mathcal{R}$ and because $\boldsymbol{a} \in \mathcal{R}$, q is a regular point of f_1. Thus there exists a neighborhood W of q such that $f_1|_W$ is an embedding. Since $\{q_k\}_{k=1}^\infty$ converges to q, we have

$$f_1(q_k) = \boldsymbol{a} = f_1(q),$$

contradicting the fact that $f_1|_W$ is injective.

Hence, there exist positive integers l and m such that

$$g_1^{-1}(\boldsymbol{a}) = \{x_1, \dots, x_l\}, \quad g_2^{-1}(\boldsymbol{a}) = \{y_1, \dots, y_m\}.$$

Since L_i are injective on $\overline{V_i}$ $(i=1,2)$, $\nu_1(x_a) \in S^n$ $(a = 1, \dots, l)$ are mutually distinct, and $\nu_2(y_b) \in S^n$ $(b = 1, \dots, m)$ are also mutually distinct. Thus, the images of g_i $(i=1,2)$ at $\boldsymbol{a}$ are finitely many hypersurfaces that intersect transversally to each other. In particular, the fact $f_1(\overline{V_1}) \subset f_2(\overline{V_2})$ implies $l \leq m$. Thus, changing the order appropriately, we may assume

$$L_1(x_j) = L_2(y_j) \quad \text{or} \quad L'_2(y_j) \quad (j = 1, \dots, l). \tag{2.3}$$

Namely, $g_1^{-1}(\boldsymbol{a}) \subset B_+ \cup B_-$ holds. In particular, we have

$$A_1 = \bigcup_{\boldsymbol{a}\in\mathcal{R}} g_1^{-1}(\boldsymbol{a}) \subset B_+ \cup B_-. \tag{2.4}$$

Since B_+ and B_- are closed subsets of $\overline{V_1}$, by taking the closure of (2.4), Lemma 2.2 yields the conclusion. □

We next prepare the following:

Lemma 2.3. *$L_1(p_1)$ coincides with $L_2(p_2)$ or $L_2'(p_2)$.*

Proof. Take a sequence $\{q_k\}_{k=1}^{\infty} \subset A_1$ which converges to p_1. We set $Q_k := f_1(q_k)$. Noticing that f_1 is regular at q_k, let T_k be the tangent hyperplane of f_1 at $f_1(q_k)$. Since $f_2^{-1}(Q_k)$ is a finite point set, there exists a point $q_k' \in f_2^{-1}(Q_k)(\subset A_2)$ such that the tangent hyperplane of f_2 at $f_2(q_k')$ coincides with T_k. Then $L_1(q_k) = L_2(q_k')$ or $L_1(q_k) = L_2'(q_k')$ holds. Since $\{q_k'\}_{n=1}^{\infty}$ is a sequence in $\overline{V_2}$, there is an accumulation point $q' \in \overline{V_2}$. Then $L_1(p_1) = L_2(q')$ or $L_1(p_1) = L_2'(q')$ holds. In particular, $f_1(p_1) = f_2(q')$ holds. If $L_1(p_1) = L_2(p_2)$, then the injectivity of L_2 and (2.2) imply $q' = p_2$. On the other hand, when $f_2^{-1}(f_2(p))$ is a finite point set, then (2.1) yields that $q' = p_2$. So we obtain the conclusion. □

Proof of Theorem A. Replacing ν_2 by $-\nu_2$ if necessary, we may assume $L_1(p_1) = L_2(p_2)$. By Lemma 2.2, we have $B_+ \cup B_- = \overline{V_1}$. By (2.2), $\overline{V_1} = B_+ \cup B_-$. Since $\overline{V_1}$ is connected, either $\overline{V_1} = B_+$ or $\overline{V_1} = B_-$ holds. Since $L_1(p_1) = L_2(p_2)$, B_+ is non-empty. Thus $\overline{V_1} = B_+$ holds and so $L_1(\overline{V_1}) \subset L_2(\overline{V_2})$. Since L_2 is an injective continuous map from the compact space $\overline{V_2}$ to a Hausdorff space, $L_2^{-1} : L_2(\overline{V_2}) \to \overline{V_2}$ is also a continuous map. Thus, we can define a continuous map $\psi : \overline{V_1} \to \overline{V_2}$ by

$$\psi := L_2^{-1} \circ L_1 : \overline{V_1} \to \overline{V_2}.$$

By definition, it satisfies $L_1 = L_2 \circ \psi$, that is $f_1 = f_2 \circ \psi$ and $\nu_1 = \nu_2 \circ \psi$ on $\overline{V_1}$.

Finally, since L_1 is injective, ψ is an injective continuous map from the compact space $\overline{V_1}$ to the Hausdorff space $\overline{V_2}$. So it gives a homeomorphism between $\overline{V_1}$ and $\psi(\overline{V_1})(\subset \boldsymbol{R}^n)$. By the invariance of domain (cf. [2]), $V_2' := \psi(V_1)$ is a connected open subset of $\boldsymbol{R}^n$. Thus, we have $L_1(V_1) = L_2(V_2')$ and ψ gives a homeomorphism between V_1 and V_2'. Replacing V_2 by V_2', we obtain the relation $f_1(V_1) = f_2(V_2)$. □

Corollary 2.4. *Let f_1 and f_2 are as in Theorem A, then these two maps are equi-image equivalent.*

Proof. Since we have shown that $f_1 = f_2 \circ \psi$, which implies that $f_1|_{V_1}$ is V_1-proper at p_1 and $f_1^{-1}(f_1(p_1)) = \{p_1\}$ as well as $f_2|_{V_2}$. Since $f_1(V_1) = f_2(V_2)$, Theorem 1.16 implies that f_1 and f_2 are image equivalent. Then Proposition 1.20 implies that f_1 and f_2 are equi-image equivalent. □

3. Proof of Theorem B

3.1. **The half-arc-length parameter of generalized cusps.** Let $\sigma : (a, b) \to \boldsymbol{R}^2$ be a C^r-curve defined on an open interval $(a, b)(\subset \boldsymbol{R})$ where $a < b$. A point $t = c$ on (a, b) is called a *generalized cusp* if $\sigma'(c)(= d\sigma(c)/dt) = (0, 0)$ and $\sigma''(c) \neq (0, 0)$. In this situation, we can take the inverse function $t = t(w)$ of the function $w : (a, b) \to \boldsymbol{R}$ which is C^r-differentiable and satisfies

$$w(t)^2 := \left| \int_c^t |\sigma'(u)| du \right|, \qquad \frac{dw}{dt} > 0.$$

Then w gives the half-arc-length parameter of the curve σ at $u = c$. Using the half-arc-length parameter w, the curve σ has the following expression (cf. [15] and [3])

$$\sigma(w) = 2\int_0^w u(\cos\lambda(u), \sin\lambda(u))du, \quad \lambda(w) := \frac{1}{\sqrt{2}}\int_0^w \mu(u)du, \tag{3.1}$$

where $\mu(u)$ is a C^r-function. Regarding this geometric meaning of w, the following assertion is obvious:

Proposition 3.1. *Let $\sigma_i(w)$ $(w \in J)$ $(i = 1, 2)$ be two C^r-differentiable generalized cusps at $w = 0$, where $J := (-a, a)$ $(a > 0)$. Suppose that w is the half-arc-length parametrization of σ_i for each $i = 1, 2$ at $w = 0$. If $\sigma_1(J)$ coincides with $\sigma_2(J)$, then either $\sigma_1(w) = \sigma_2(w)$ or $\sigma_1(w) = \sigma_2(-w)$ holds.*

3.2. **Smoothness of ψ for generalized cuspidal edges.** Let C be a curve C^r-embedded in $\boldsymbol{R}^3$ which is not closed. To prove Theorem B, we consider the following situation:

Let $f : (U; u, v) \to \boldsymbol{R}^3$ be a C^r-map such that U contains a closed interval $I \times \{0\}$ on the u-axis in the uv-plane $\boldsymbol{R}^2$. We assume that $I \times \{0\}\,(\subset U)$ consists of generalized cuspidal edge points. Without loss of generality, we may assume that $\gamma(u) = (u, 0)$ $(u \in I)$ giving the arc-length parametrization of $\hat\gamma := f \circ \gamma$ such that $\hat\gamma(I) \subset C$. We let $\hat\Pi_{\hat\gamma(u)}$ be the normal plane of f for each $u \in I$. We first prove the following:

Lemma 3.2. *There exist $\varepsilon(> 0)$ and an embedding*

$$\psi : V \to U \qquad (V := I \times [-\varepsilon, \varepsilon])$$

satisfying the following properties:

(1) *$\gamma(s) := \psi(s, 0)$ parametrizes the singular set of f such that $\hat\gamma(s) = f \circ \gamma(s)$ gives an arc-length parametrization of C.*
(2) *for each fixed $(s, 0) \in V$, the curve $\sigma^s : t \mapsto f \circ \psi(s, t)$ parametrizes the section of the image of f by $\hat\Pi_{\hat\gamma(s)}$ such that t is the half-arc-length parameter of the curve σ^s.*

The plane curve σ^s lying in the plane $\hat\Pi_{\hat\gamma(s)}$ is called the *sectional cusp* at $\hat\gamma(s)$.

Proof. By the definition of generalized cuspidal edge (cf. Definition 0.1), there exist

- an open subset $U_1(\subset U)$ in the uv-plane containing $I \times \{0\}$,
- a tubular neighborhood $\Omega(\subset \boldsymbol{R}^3)$ of the curve C containing the image $f(U_1)$,
- a C^r-diffeomorphism $\varphi : U_1 \to (\boldsymbol{R}^2; x, y)$ giving a diffeomorphism between U_1 and $\varphi(U_1)$ and
- a C^r-diffeomorphism $\Phi : \Omega \to \Phi(\Omega)$

such that

$$\Phi \circ f \circ \varphi^{-1}(x, y) = (y^2, y^3\alpha(x, y), x)\,(=: f_0(x, y)), \tag{3.2}$$

where α is a C^r-function. Then $x \mapsto f_0(x, 0)$ gives a parametrization of the image of the singular curve of f_0, and so we can write

$$f_0(x(s), 0) = (0, 0, x(s)) \qquad (s \in I)$$

so that $|d\hat\gamma(s)/ds| = 1$, where $x(s)$ is a C^r-function satisfying $x' = dx/ds > 0$. By choosing a sufficiently small Ω, we may assume that the image $\Phi(\hat\Pi_{\hat\gamma(s)})$ $(s \in I)$ of the normal plane of f is a surface embedded in $\Phi(\Omega)$. Thus, there exists a family of functions $\{g^s(x, y)\}_{s\in I}$ such that

$$g^s(0, 0) = x(s)$$

and the graph of the function $z = g^s(x, y)$ gives a local parametrization of $\Phi(\hat\Pi_{\hat\gamma(s)})$. Then, the section of the image of f_0 by the graph $z = g^s(x, y)$ corresponds to the image of the sectional cusp of f, which can be characterized by the implicit function $F^s(x,y)=0$, in the xy-plane (as the domain of definition of g^s) by setting

$$F^s(x,y) := x - g^s(y^2, y^3\alpha(x,y)).$$

Since $g^s(0,0) = x(s)$, we have $F^s(0,0) = 0$. Since the derivative $\partial F^s(0,0)/\partial x$ is equal to 1, the implicit function theorem yields that there exists a C^r-function $x = A^s(y)$ of y which parametrizes the set $F^s = 0$, that is,

$$A^s(y) = g^u\Big(y^2, y^3\alpha(A^s(y), y)\Big), \qquad A^0(0) = 0$$

hold, and

$$\hat{\sigma}^s(y) := \Big(y^2, y^3\alpha(A^s(y), y), A^s(y)\Big)$$

gives a parametrization of the slice of f_0 by $\Phi(\hat{\Pi}_{\hat{\gamma}(s)})$. Since

$$A^s(0) = g^s(0,0) = x(s),$$

the fact $dx/ds > 0$ implies that

$$\varphi_1 : I \times (-\varepsilon_1, \varepsilon_1) \ni (s,y) \mapsto (A^s(y), y) \in \boldsymbol{R}^2$$

is a C^r-diffeomorphism into the xy-plane for sufficiently small $\varepsilon_1 > 0$, and the parameters s, y give a new local coordinate system of the xy-plane at $(0,0)$.

Computing the derivatives of the curves $y \mapsto \hat{\sigma}^s(y)$, we have ($' := d/dy$)

$$(\hat{\sigma}^s)'(0) = \mathbf{0}, \qquad (\hat{\sigma}^s)''(0) = (2, 0, *), \tag{3.3}$$

where $*$ means a certain value, which is not required in the later discussions.

By setting $\sigma^s(y) := \Phi^{-1} \circ \hat{\sigma}^s(y)$, the formula (3.2) implies that σ^s parametrizes the section of f by the normal plane $\hat{\Pi}_{\hat{\gamma}(s)}$. By (3.3), $y \mapsto \sigma^s(y)$ gives the generalized cusp at $y = 0$ as the section of f by the normal plane $\hat{\Pi}_{\hat{\gamma}(s)}$. If we set

$$w^s(y) := \operatorname{sgn}(y)\sqrt{|B^s(y)|}, \qquad B^s(y) := \int_0^y |(\sigma^s)'(t)| dt,$$

then

$$\varphi_2 : I \times (-\varepsilon_2, \varepsilon_2) \ni (s,y) \mapsto (s, w^s(y)) \in \boldsymbol{R}^2$$

is a C^r-diffeomorphism into the st-plane for sufficiently small $\varepsilon_2 \in (0, \varepsilon_1)$, and $t := w^s(y)$ is the half-arc-length parametrization of σ^s. So if we consider the C^r-map given by

$$\psi(s,t) := \varphi^{-1} \circ \varphi_1 \circ \varphi_2^{-1}(s,t),$$

then ψ gives the desired parametrization of f defined on $I \times [-\varepsilon, \varepsilon]$ for sufficiently small $\varepsilon > 0$. □

Proof of Theorem B. Without loss of generality, we may assume that there exists a closed interval I_i $(i = 1, 2)$ in $\boldsymbol{R}$ and

$$I_i \ni u_i \mapsto f_i(u_i, 0) \in \boldsymbol{R}^3$$

gives the arc-length parametrization of C. By replacing u_2 by $-u_2$, we may assume that these two parametrizations of C give the same orientation. Then we may also assume that $I_1 = I_2 (= I)$ and

$$\gamma(s) := f_1(s, 0) = f_2(s, 0).$$

By Lemma 3.2, for each $i = 1, 2$, there exist a positive number ε_i and an embedding

$$\psi_i : I \times [-\varepsilon_i, \varepsilon_i] \ni (s_i, t_i) \mapsto \psi_i(s_i, t_i) \in U_i$$

such that (s_i, t_i) satisfies (1) and (2) of Lemma 3.2. We set $g_i := f_i \circ \psi_i$ for $i = 1, 2$. Since $f_1(U_1) \subset f_2(U_2)$, we may assume that $\varepsilon_1 \leq \varepsilon_2$. We denote by $\hat{\Pi}_s$ the normal plane of the curve

γ at $\gamma(s)$. Since t is the half-arc-length parameter of each section of g_i $(i = 1, 2)$ by the plane $\hat{\Pi}_s$, Proposition 3.1 yields that

$$g_1(s,t) = g_2(s, e(s)t) \qquad (e(s) \in \{+,-\})$$

holds at each point $(s,t) \in I \times [-\varepsilon_1, \varepsilon_1]$, where $e(s)$ is a sign depending on s. By the continuity of g_1 and g_2, we can conclude that $e := e(s)$ does not depend on s. In particular, the unit normal vector field of g_1 coincides with that of g_2 up to a sign. If we set $\varphi(s,t) := (s, et)$, then $f_1 \circ \psi_1 = f_2 \circ \psi_2 \circ \varphi$ holds.

Let U_i $(i = 1, 2)$ be a neighborhood of $p_i \in \boldsymbol{R}^2$ and $f_i : U_i \to \boldsymbol{R}^3$ a C^r-frontal map so that p_i is a generalized cuspidal edge point satisfying the conditions (a1)–(a4) in Theorem A. By Theorem A, there exists a homeomorphism $\psi : V_1 \to V_2$ between certain connected neighborhoods $V_i (\subset U_i)$ $(i = 1, 2)$ of p_i satisfying $f_1 = f_2 \circ \psi$ and $\nu_1 = \pm\nu_2 \circ \psi$ on V_1. Comparing these with $f_1 \circ \psi_1 = f_2 \circ \psi_2 \circ \varphi$, we have $\psi = \psi_2 \circ \varphi \circ \psi_1^{-1}$, proving the smoothness of ψ. $\square$

4. Proof of Theorem C

In this section, we prove Theorem C.

4.1. **Proof of the first part of Theorem C.** Let $f : (U; u, v) \to \boldsymbol{R}^3$ be a C^r-frontal map and ν a unit normal vector field of f. A singular point $p \in U$ of f is said to be *non-degenerate* if the exterior derivative $d\lambda$ does not vanish at p, where

$$\lambda := \det(f_u, f_v, \nu).$$

Cuspidal edges, swallowtails and cuspidal cross caps are non-degenerate singular points on frontal maps.

We denote by $\Sigma(f)$ the singular set of f. We consider the case that p is a non-degenerate singular point. By the implicit function theorem, there exists a regular curve $\gamma(t)$ parametrizing $\Sigma(f)$ near p such that $\gamma(0) = p$. This curve γ is called the *singular curve* and $\gamma'(0) (\neq \mathbf{0})$ is called the *singular direction* at p. A non-zero tangent vector $\mathbf{v} \in T_pU$ is called a *null vector* of f at p if $df_p(\mathbf{v})$ vanishes. Then p is called *type I*, if the null-vector $\mathbf{v}$ at p is linearly independent of $\gamma'(0)$. Otherwise, p is called *type II*.

Generalized cuspidal edges are all non-degenerate singular points of type I. (In particular, cuspidal edges and cuspidal cross caps are of type I.) On the other hand, swallowtails are of type II. If p is of type I, then the *limiting normal curvature* at p is given by (cf. [13])

$$\kappa_\nu(p) := \frac{\hat{\gamma}''(0) \cdot \nu(p)}{\hat{\gamma}'(0) \cdot \hat{\gamma}'(0)}, \tag{4.1}$$

where $\hat{\gamma}(t) := f \circ \gamma(t)$. Here, we discuss symmetries of the standard cuspidal edge, swallowtail and cuspidal cross cap.

Example 4.1. The images of the standard cuspidal edge f_C and the standard cuspidal cross cap f_{CW} (cf. (1.3)) are both invariant under two orthogonal transformations fixing the origin corresponding to the following orthogonal matrices:

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad T_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here

- T_1 is the reflection with respect to the normal plane Π_1,
- T_2 is the reflection with respect to the limiting tangent plane Π_0, and
- $T_3 := T_1 \circ T_2$ is the $180°$-rotation with respect to the co-normal line l_2.

Example 4.2. The image of the standard swallowtail f_S (cf. (1.3)) is invariant under an orthogonal transformation fixing the origin corresponding the orthogonal matrix T_2, which is the reflection with respect to the co-normal plane Π_2.

Proof of the first part of Theorem C. Let $f : U \to \boldsymbol{R}^3$ be as in Theorem C. Then we can apply Theorem A to the maps f and $T \circ f$, and there exists a local homeomorphism ψ satisfying

$$f \circ \psi = T \circ f, \qquad \nu \circ \psi = eT \circ \nu, \qquad \psi(p) = p, \qquad T \circ f(p) = f(p), \tag{4.2}$$

where ν is the unit normal vector field of f and $e \in \{+,-\}$. If $p \in U$ is a cuspidal edge or a swallowtail, then, by Theorem A, ψ is a local C^r-diffeomorphism, because cuspdial edges and swallowtails are wave fronts. On the other hand, if p is a cuspidal cross cap, then, by Theorem B, we can conclude that ψ is also a local C^r-diffeomorphism.

Without loss of generality, we may set $f(p) = (0,0,0)$ and T is an orthogonal matrix. So we denote $T \circ \nu$ by $T\nu$. We can take a local coordinate system (u,v) centered at p such that $f_v(p) = \mathbf{0}$ and $f_u(p) \neq \mathbf{0}$. Since $f \circ \psi = T \circ f$, the vector $f_u(p)$ is an eigenvector of T. Since $T \circ f(p) = f(p) = (0,0,0)$ and $\psi(p) = p$, the second formula of (4.2) implies that $T\nu(p) = \pm\nu(p)$, that is, $\nu(p)$ is also an eigenvector of T. We consider the vector

$$\boldsymbol{w} := f_u(p) \times \nu(p),$$

which points in the co-normal direction. Since $f_u(p)$ and $\nu(p)$ are eigenvectors, $\boldsymbol{w}$ is also an eigenvector of T. Thus, we can write

$$Tf_u(p) = \lambda_1 f_u(p), \quad T\nu(p) = \lambda_2 \nu(p), \quad T\boldsymbol{w} = \lambda_3 \boldsymbol{w},$$

where $\lambda_i \in \{1,-1\}$ $(i = 1,2,3)$. Thus, all eigenvalues of T^2 are equal to 1. Since T^2 is an orthogonal matrix, it must be the identity matrix, that is, T is an involution.

By (4.2), we have

$$f \circ \psi \circ \psi = T \circ f \circ \psi = T^2 \circ f = f.$$

Since cuspidal edge has no self-intersections and the self-intersection set of cuspidal cross caps and swallowtails have no interior points, we can conclude that ψ is a C^r-involution. Moreover, if ψ is an identity map, then the fact that T is not the identity map implies that the image of f lies in a plane, which is a contradiction. So ψ is a non-trivial involution, that is, it is not the identity map. □

Lemma 4.3. *Let $f : U \to \boldsymbol{R}^3$ be a C^r-differentiable map with a generalized cuspidal edge singular point p. Suppose that T is an isometry of $\boldsymbol{R}^3$ fixing $f(p)$ such that $T \circ f(V) \subset f(U)$ for a neighborhood $V (\subset U)$ of p. Then the co-normal vector of f at p is a 1-eigenvector of T. In particular, the case* (iii) *of Theorem C never happens.*

Proof. Without loss of generality, we may set $f(p) = (0,0,0)$ and T is an orthogonal matrix. The above proof of the first part of Theorem C can apply for generalized cuspidal edges (cf. Theorem B) and can conclude that the co-normal vector $\mathbf{v}$ at p is a (± 1)-eigenvector of T. Without loss of generality, we may assume that $f(s,t)$ is the parametrization of f given in Lemma 3.2. Then $\mathbf{v} := \partial^2 f(0,0)/\partial t^2$ points in the co-normal direction that the generalized cusp $\sigma_0(t)$ lies in. So we can conclude that $\mathbf{v}$ is a 1-eigenvector. □

Proposition 4.4. *Let $p \in U$ be a generalized cuspidal edge singular point of a C^r-map $f : U \to \boldsymbol{R}^3$. Suppose that there exist an isometry T of $\boldsymbol{R}^3$ fixing $f(p)$ and a neighborhood V of p such that $T \circ f(V) \subset f(U)$. If T is not the identity map, then one of the following two cases occurs:*

(1) *T is the reflection with respect to the limiting tangent plane Π_0 at p, and the singular set image of f lies in Π_0. Moreover, the limiting normal curvature of f vanishes at p.*

(2) *T is the reflection with respect to the normal plane Π_1 or the $180°$-rotation with respect to the co-normal line $l_2 := \Pi_0 \cap \Pi_1$ at p. In addition, the connecting map ψ is a C^r-involution interchanging the orientation of the singular curve.*

Proof. We may assume that $f(s,t)$ is the parametrization of f as in Lemma 3.2. Since s is the arc-length parametrization of C and t is the half-arc-length parametrization of each sectional cusp, as seen in the proof of Theorem B, there exists a local C^r-diffeomorphism ψ on a neighborhood of p such that $T \circ f \circ \psi = f$ and

$$\psi(s,t) = (e_1 s, e_2 t),$$

where $e_1,\ e_2 \in \{+,-\}$. Then $\gamma(s) := (s,0)$ parametrizes the singular set of f. By Proposition 4.4, the co-normal direction of f at p is a 1-eigenvector of T. Since p is of type I, $\gamma'(0)$ is linearly independent of the null-direction of f at p. We set $\hat{\gamma}(s) := f \circ \gamma(s)$.

We first consider the case that $e_1 = +$. In this case, we have

$$T \circ f \circ \gamma(s) = f \circ \psi \circ \gamma(s) = f \circ \gamma(s),$$

that is, the singular points of f are fixed by T. In particular, the tangential direction $\hat{\gamma}'(0)$ is the 1-eigenvector of T. Since the co-normal direction at p is also a 1-eigenvector of T (cf. Lemma 4.3), the limiting tangent plane Π_0 is contained in the fixed point set of T. Since T is not the identity map, T must be the reflection with respect to the limiting tangent plane Π_0. In this situation, if the limiting normal curvature at $\gamma(s)$ does not vanish, then the Gaussian curvature takes opposite sign on the two sides of γ (cf. [13] or [8, Proposition 4]), which contradicts that T is the reflection with respect to Π_0. So the limiting normal curvature vanishes identically along γ. This is the case (1).

We next consider the case that $e_1 = -$, that is, the case that the local C^r-diffeomorphism ψ is reversing the orientation of the singular curve. This is the case (2). If we set $\hat{\gamma} := f \circ \gamma$, then we have $T\hat{\gamma}'(0) = -\hat{\gamma}'(0)$. Since the co-normal direction at p is a 1-eigenvector of T, if $\nu(p)$ is a (-1)-eigenvector of T, then T is the $180°$-rotation with respect to the co-normal line. On the other hand, if $\nu(p)$ is a 1-eigenvector of T, then T is the reflection with respect to the normal plane Π_1. □

4.2. **Symmetries of cuspidal edges and cuspidal cross caps.** For cuspidal edges and cuspidal cross caps, we can prove the following:

Proposition 4.5. *Let $f : U \to \boldsymbol{R}^3$ be a C^r-map defined on a non-empty open subset of $\boldsymbol{R}^2$, and let $p \in U$ be a cuspidal edge or a cuspidal cross cap singular point. Suppose that*

- *the limiting normal curvature κ_ν does not vanish at p, and*
- *there exists an isometry T of $\boldsymbol{R}^3$ fixing $f(p)$ such that $T \circ f(V) \subset f(U)$ and T is not the identity map, where V is an open neighborhood $V(\subset U)$ of p.*

Then T must be the reflection with respect to the normal plane Π_1, and there exists a local C^r-diffeomorphism ψ (determined by Theorems A and B) satisfy the following:

(1) *If p is a cuspidal edge singular point, then ψ is an orientation reversing C^r-involution which reverses the orientation of the singular curve.*
(2) *If p is a cuspidal cross cap, then ψ is an orientation preserving C^r-involution which reverses the orientation of the singular curve at p. Moreover, each point of the image of the set of self-intersections is fixed by T and is lying in the normal plane Π_1 near $f(p)$.*

Before proving the proposition, we prepare the following:

Lemma 4.6. *Let $f : U \to \boldsymbol{R}^3$ be a C^r-frontal satisfying $T \circ f \circ \psi = f$ on U. Suppose that $\tau(t) : (-\varepsilon, \varepsilon) \to U$ is a C^r-regular curve in U such that*

(a) *for each* $t \in (0, \varepsilon)$*, there exists* $t_1 \in (0, \varepsilon)$ *such that* $f \circ \tau(t) = f \circ \tau(-t_1)$*,*
(b) $\tau(t)$ *meets the singular set of* f *only at* $t = 0$*, and*
(c) $f(\psi \circ \tau(t)) = f \circ \tau(t)$*.*

Then $f \circ \tau(t)$ *is a fixed point of* T *for sufficiently small* $|t|$*.*

Proof. Without loss of generality, we may assume that $f(p) = (0, 0, 0)$ and T is an orthogonal matrix. We let ds^2 be the first fundamental form of f and set

$$s(t) := \int_0^t |\tau'(t)| dt \qquad \left(|\tau'(t)| := \sqrt{ds^2(\tau'(t), \tau'(t))}\right),$$

which is the arc-length of the arc $\tau([0, t])$. By the condition (b), $\tau'(t)$ does not vanish for each $t \neq 0$ sufficiently close to $t = 0$. In particular, $t \mapsto s(t)$ is monotone increasing, and we may consider s as a continuous parametrization of the curve τ. By (a), we have $f \circ \tau(s) = f \circ \tau(-s)$. Then (c) implies that

$$\psi \circ \tau(s) = \tau(s) \text{ or } \psi \circ \tau(s) = \tau(-s). \tag{4.3}$$

Thus, we have

$$T \circ f \circ \tau(s) = f \circ \psi \circ \tau(s) = f \circ \tau(\pm s) = f \circ \tau(s), \tag{4.4}$$

that is, $f \circ \tau(s)$ is a fixed point of T. □

Proof of Proposition 4.5. Without loss of generality, we may assume that $f(p) = (0, 0, 0)$ and T is an orthogonal matrix. We let (u, w) be the local coordinate system centered at p as in Lemma 3.2. Since p is a cuspidal edge or a cuspidal cross cap, we can take a vector $\mathbf{v}(\neq \mathbf{0})$ at $f(p)$ pointing in co-normal direction and in the image of sectional cusp of f at the same time. So $\mathbf{v}$ is a $(+1)$-eigenvector of T (cf. Lemma 4.3). Since the limiting normal curvature does not vanish at p, the involution ψ reverses the orientation of the singular curve (cf. Proposition 4.4), $\psi(u, w) = (-u, w)$ or $\psi(u, w) = (-u, -w)$ happens.

We first consider the case that p is a cuspidal edge. Then, the second case never occurs, because the Gaussian curvature changes sign along the u-axis (cf. [8, Proposition 4]). So we obtain $\psi(u, w) = (-u, w)$. In this case, the section of the image of f by the normal plane Π_1 at $f(p)$ is a cusp. If $T\nu(p) = -\nu(p)$ holds, then T maps a side of the cusp into the opposite side in the plane Π_1. However, it contradicts the fact that the Gaussian curvature changes sign along the u-axis. Thus, we can conclude that $T\nu(p) = \nu(p)$, and so T is a reflection with respect to the normal plane Π_1. Hence (1) is obtained.

We next consider the case that p is a cuspidal cross cap. As in the case of cuspidal edges, we must have $\psi(u, w) = (-u, w)$ or $\psi(u, w) = (-u, -w)$. However, by the behavior of Gaussian curvature of f (cf. [8, Corollary 1]), $\psi(u, w) = (-u, w)$ can never occur, and so $\psi(u, w) = (-u, -w)$ must hold.

We suppose $T\nu(p) = -\nu(p)$. Then T must be the 180°-rotation about the co-normal line. However, in this case, T maps a point $f(u, w)$ satisfying $u, w > 0$ to the point $f(u', w')$ satisfying $u' < 0$ and $w' > 0$ (because cuspidal cross caps have self-intersections), but it never happens since the sign of the Gaussian curvature changes sign along the w-axis (cf. [8, Proposition 4]). So we have $T\nu(p) = \nu(p)$, and T is the reflection with respect to the normal plane.

Finally, we discuss the self-intersections of f. For the case of standard cuspidal cross cap f_{CW} (cf. (1.3)), the map $\tau : v \mapsto f_{CW}(0, v)$ parametrizes the set of self-intersections. Since f is right-left equivalent to f_{CW}, f has a parametrization of the set of its self-intersections satisfying the assumption of Lemma 4.6. So each point of the image of the set of self-intersections is fixed by T, and (2) for cuspidal cross caps is obtained. □

Proof of the second part of Theorem C. The remaining assertions in Theorem C, except for swallowtails, follow from Propositions 4.4 and 4.5. □

Example 4.7. As shown in [12], any germ of a cuspidal edge is congruent to

$$f(u,v) = \Big(u, a_0(u) + v^2, b_0(u)u^2 + b_2(u)uv^2 + b_3(u,v)v^3\Big), \tag{4.5}$$

where $b_3(0,0) \neq 0$. In the normal form for germs of a cuspidal edges, the limiting normal curvature of f at $(0,0)$ is non-zero if and only if $b_0(0) \neq 0$. Moreover, f admits a non-trivial symmetry at $(0,0)$ if

$$a_0(u) = a_0(-u), \quad b_0(u) = b_0(-u), \quad b_2(u) = -b_2(-u), \quad b_3(u,v) = b_3(-u,v).$$

The normal plane is the yz-plane.

Example 4.8. The map $f(u,v) := (u, v^2, u^2 + uv^3)$ has a cuspidal cross cap at $(0,0)$ whose limiting normal curvature does not vanish. This map has a symmetry satisfying

$$f(-u,-v) = T \circ f(u,v), \qquad T := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Other examples of cuspidal edges and cuspidal cross caps with symmetries are in Example 6.3 of [3].

4.3. **Symmetries of swallowtails.** We have proved Theorem C except for swallowtails. In this subsection, we will discuss symmetries of swallowtails mainly, and complete the proof of Theorem C. We first prove the following:

Lemma 4.9. *Let $f : U \to \boldsymbol{R}^3$ be a C^r-frontal map, and let p be a non-degenerate singular point satisfying $T \circ f \circ \psi = f$ on U for an isometry T and a C^r-involution ψ on U. If ψ is not the identity map, then there exists a local coordinate system (x,y) centered at p satisfying the following properties:*

1. *The x-axis is the singular curve of f.*
2. *If ψ is an orientation preserving local C^r-diffeomorphism, then $\psi(x,y) = (-x,-y)$.*
3. *If ψ reverses the orientation of the singular curve, then either $\psi(x,y) = (x,-y)$ or $\psi(-x,y) = (x,y)$ holds.*
4. *If $p = (0,0)$ is of type II, then $\partial/\partial x$ points in the null-direction.*

Proof. Without loss of generality, we may assume that $f(p) = (0,0,0)$ and T is an orthogonal matrix. We fix a C^r-differentiable Riemannian metric ds_0^2 defined on U and set

$$ds_1^2 := \frac{ds_0^2 + \psi^* ds_0^2}{2}.$$

Then we have $\psi^* ds_1^2 = ds_1^2$. Since p is a non-degenerate singular point, we can take a regular curve $\gamma(s)$ parametrizing the singular curve such that $\gamma(0) = p$, that is, γ is the singular curve. Without loss of generality, we may assume that s is the arc-length parameter of γ with respect to the metric ds_1^2. Let $\xi(t)$ be the vector field of unit length with respect to ds_1^2 along the curve $\gamma(t)$ so that $\{\xi(t), \gamma'(t)\}$ is linearly independent in $T_{\gamma(t)}U$ for each t. We let $\operatorname{Exp}_p : T_pU \to U$ be the exponential map of ds_1^2 at p. We set

$$\Gamma(x,y) := \operatorname{Exp}_{\gamma(x)}(y\xi(x)) \in U.$$

Then (x, y) gives a local C^r-coordinate system centered at p such that the x-axis corresponds to the singular curve. Since $\psi^* ds_1^2 = ds_1^2$ and ψ preserves the singular curve, we can write

$$\psi(x, y) = (e_1 x, e_2 y),$$

where $e_i \in \{+, -\}$ $(i = 1, 2)$. If ψ is an orientation preserving isometric involution, we have $e_1 = e_2 = -1$, because ψ is not the identity map. We next consider the case that ψ is orientation reversing. Then either $(e_1, e_2) = (1, -1)$ or $(e_1, e_2) = (-1, 1)$ holds. If $p = \gamma(0)$ is of type II, then the tangential direction $\partial/\partial x$ of the singular curve at the origin points in the null-direction. $\square$

We now prove the following:

Theorem 4.10. *Let $f : U \to \boldsymbol{R}^3$ be a C^r-map which is U-proper at p, $f^{-1}(f(p)) = \{p\}$ and has a swallowtail singularity at $p \in U$. If there exist an isometry T of $\boldsymbol{R}^3$ fixing $f(p)$ and a neighborhood $V(\subset U)$ of p such that $T \circ f(V) \subset f(U)$, and if T is not the identity map, then there exist a connected open neighborhood $W(\subset V)$ of p and a C^r-involution $\psi : W \to W$ such that $f \circ \psi = T \circ f$ on W. Moreover, T and ψ have the following properties:*

1. *T is the reflection with respect to the co-normal plane Π_2,*
2. *T fixes each point of the image of the set of self-intersections of f, and*
3. *ψ is the orientation reversing involution which reverses the orientation of the singular curve.*

Proof. Without loss of generality, we may assume that $f(p) = (0, 0, 0)$ and T is an orthogonal matrix. Since p is a swallowtail, we may also assume that f has a unit normal vector field ν along f. By Theorem A and the proof of the first part of Theorem C, T is an involution and there exists a connected open neighborhood $W(\subset V)$ of p and the associated non-trivial C^r-involution $\psi : W \to W$ satisfying $T \circ f \circ \psi = f$ on W. So it is sufficient to show the remaining assertions: The self-intersection set of the standard swallowtail f_S as in (1.3) is the parabola $\tau(v) := (-2v^2, v)$ in the uv-plane, and f_S satisfies the assumption of Lemma 4.6 for the set of self-intersection. Since f is right-left equivalent to f_S, the set of self-intersections of f also satisfies the assumption of Lemma 4.6. So the image of each point in the self-intersection set of f is fixed by T, proving (2).

We project the image of the singular curve into the limiting tangent plane, and then its image gives a cusp by [13, Corollary 4.10], and the line bisecting the cusp is just the limiting tangential direction. Thus the tangential direction of f is the 1-eigenvector of T. Since swallowtails cannot be symmetric with respect to the limiting tangent plane at p, we have $T\nu = \nu$, that is, ν is a 1-eigenvector of T. Since T is not the identity, the co-normal vector is a (-1)-eigenvector. Thus, T is a reflection with respect to the co-normal plane Π_1, proving (1).

We now prove (3): Let $\gamma(t)$ be a regular curve in W parametrizing the singular set of f satisfying $\gamma(0) = p$. We let ds^2 be the first fundamental form of f and set

$$s(t) := \int_0^t |\gamma'(t)| dt \qquad \left(|\gamma'(t)| := \sqrt{ds^2(\gamma'(t), \gamma'(t))} \right),$$

which gives the arc-length of the arc $\gamma([0, t])$, and $t \mapsto s(t)$ is monotone increasing. So we may take s as a (continuous) parametrization of γ. (Although $s(t)$ is not differentiable at $t = 0$, it does not affect the following discussion.) Since $\psi^* ds^2 = ds^2$, we have

$$\psi \circ \gamma(s) = \gamma(s) \quad \text{or} \quad \psi \circ \gamma(s) = \gamma(-s). \tag{4.6}$$

We let $\tau(t)$ be the regular curve in W parametrizing the self-intersection set of f. Since f is right-left equivalent to the standard swallowtail f_S, we may assume that $f \circ \tau(t) = f \circ \tau(-t)$ holds.

Let (x,y) be the local coordinate system as in Lemma 4.9. Since p is of type II, the x-axis is the null-direction at the origin. We then set $\sigma(y) := (0,y)$. Suppose that $\psi\circ\sigma(y) = \sigma(-y)$. If we set $\hat\sigma(y) := f\circ\sigma(y)$, then we have $T\hat\sigma(y) = \hat\sigma(-y)$. Since $\partial/\partial x$ gives the null-direction at p, $\partial/\partial y$ does not. Hence $\hat\sigma'(0)\neq 0$ and $T\hat\sigma'(0) = -\hat\sigma'(0)$ hold. However, this contradicts that the tangential direction is a 1-eigenvector. So we have

$$\psi\circ\sigma(y) = \sigma(y). \tag{4.7}$$

Suppose that $\psi\circ\gamma(s) = \gamma(s)$ happens. Since the tangential direction of $\gamma(t)$ at $t=0$ coincides with that of $\tau(t)$ at $t=0$, we have $\psi\circ\tau'(0) = \tau'(0)$. Moreover, since the image of τ is invariant by ψ, we have $\psi\circ\tau(t) = \tau(t)$. Then we have $d\psi(\tau'(0)) = \tau'(0)$. Similarly, (4.7) implies that $d\psi(\sigma'(0)) = \sigma'(0)$. Since $\tau'(0)$ and $\sigma'(0)$ are linearly independent for the standard swallowtail, it is so for f as well. Thus $d\psi_p$ must be the identity map on T_pU. Since the isometry of the Riemannian metric ds_1^2 is determined only by its differential $d\psi$ at p, we can conclude that ψ is the identity map, a contradiction. So we have $\psi\circ\gamma(t) = \gamma(-t)$. This implies that $\psi(x,y) = \psi(-x,y)$, proving the assertion (3). □

Proof of the remaining part of Theorem C. The remaining statements of Theorem C for swallowtails follow from Theorem 4.10. □

Example 4.11. We set

$$f(u,v) := \left(u + \frac{v^2}{2} - \frac{b^2uv^2}{2} - \frac{b^2v^4}{8}, \frac{bv^3}{3} + buv, \frac{cu^2}{2}\right) \qquad (b,\, c\in \boldsymbol{R}\setminus\{0\}),$$

which is an example of a swallowtail with non-zero limiting normal curvature given in [13]. This example admits a non-trivial symmetry. The singular set is the v-axis, whose image lies in the xy-plane.

5. Isomers of generalized cuspidal edges and curved foldings

In this section, we will generalize the results on cuspidal edges in the authors' previous work [3] to generalized cuspidal edges and construct a canonical map from the set of C^ω-differentiable generalized cuspidal edges to the set of curved foldings.

Isomers of generalized cuspidal edges. We let $I := (a,b)$ $(a<b)$ be a closed interval and fix a C^r-embedded curve $\mathbf{c} : I\to \boldsymbol{R}^3$, denoting by $C(:= \mathbf{c}(I))$ its image. Hereafter, we assume that the curvature function of C never vanishes.

We recall the following definition of "isomers" of generalized cuspidal edges given in [3].

Definition 5.1. Let $f_i : U_i \to \boldsymbol{R}^3$ $(i=1,2)$ be two C^r-differentiable generalized cuspidal edges along C. Then f_2 is called an *isomer* of f_1 if it satisfies

(1) f_2 is isometric to f_1, that is, there exists a local diffeomorphism germ φ such that $\varphi^* ds_1^2 = ds_2^2$ near C, where each ds_i^2 $(i=1,2)$ is the first fundamental form of f_i.
(2) f_2 is not right equivalent to f_1 as a map germ along C.

In this situation, we say that f_2 is a *faithful isomer* of f if

- the orientations of C induced by $u\mapsto f_1\circ\varphi(u,0)$ and $u\mapsto f_2(u,0)$ are compatible with respect to the one induced by $u\mapsto f_1(u,0)$.

When C is not a closed curve (i.e. $\mathbf{c}(a)\neq\mathbf{c}(b)$), as in Theorem I in [3], the following assertion was proved:

Fact 5.2. *Let f be a C^ω-differentiable generalized cuspidal edge along C whose limiting normal curvature function* (*cf.* (4.1)) *does not admit any zeros. Then there exists a faithful isomer $\check f$* (*called the dual*) *of f.*

By virtue of Theorem B, we can prove the following assertion, which is the generalization of [3, Proposition 5.1] for the case of cuspidal edges:

Proposition 5.3. *Let $f : U \to \boldsymbol{R}^3$ be a C^ω-differentiable generalized cuspidal edge along C, whose limiting normal curvature function does not admit any zeros. Then the image of $\check{f}$ is congruent to that of f (i.e. the image of $\check{f}$ coincides with that of f by a suitable isometry of $\boldsymbol{R}^3$) as a map germ along C if and only if it satisfies the following two conditions:*

1. *C lies in a plane, or*
2. *C has a positive symmetry and the first fundamental form ds^2_f has an effective symmetry,*

where the definition of positive symmetry and negative symmetry are given in [3, Definition 1.2] *and the definition of effective symmetry is given in* [3, Definition 0.4].

Proof. Applying our Theorem B in the introduction: We suppose that $\check{f}$ is congruent to f. By [3, Remark 4.5], it is sufficient to consider the case that C does not lie in any plane. By Theorem B, there exist an isometry T of $\boldsymbol{R}^3$ and a C^r-diffeomorphism φ defined on a neighborhood of the singular curve of f such that

$$T \circ f \circ \varphi = \check{f}. \tag{5.1}$$

Then the remaining argument is completely parallel to the case of cuspidal edge as of [3, Proposition 5.1]. □

Fukui's formula for generalized cuspidal edges. Let $\mathbf{c}(u)$ ($|u| \le I$) be a C^r-regular space curve with arc-length parameter whose curvature function $\kappa(u)$ is positive everywhere and has no self-intersections. We let $\theta(u)$ ($u \in I$) and $\mu(u,v)$ ($u \in I$, $|v| < \varepsilon$) are smooth functions, where ε is a positive number. We then consider a map given by (cf. (3.1))

$$f(u,t) := \mathbf{c}(u) + (A(u,t), B(u,t)) \begin{pmatrix} \cos\theta(u) & -\sin\theta(u) \\ \sin\theta(u) & \cos\theta(u) \end{pmatrix} \begin{pmatrix} \mathbf{n}(u) \\ \mathbf{b}(u) \end{pmatrix}, \tag{5.2}$$

where two functions A and B are given by

$$(A(u,t), B(u,t)) := 2\int_0^t v(\cos\lambda(u,v), \sin\lambda(u,v))dv, \tag{5.3}$$

$$\lambda(u,t) := \frac{1}{\sqrt{2}} \int_0^t \mu(u,v)dv.$$

We call such a map $f(u,v)$ a *normal form* of generalized cuspidal edge, which was introduced by Fukui [1] see also [3]. The following fact was known:

1. $f(u,v)$ is actually a generalized cuspidal edge along C,
2. θ gives the *cuspidal angle* of f along C,
3. f is a cuspidal edge along C if and only if $\mu(u,0) \ne 0$ for $u \in I$,
4. $\kappa_s := \kappa\cos\theta$ is called the *singular curvature function*, and $\kappa_\nu = \kappa\sin\theta$ coincides with the *limiting normal curvature function* of f along C.

The following assertion holds:

Proposition 5.4. *For a given C^r-differentiable generalized cuspidal edge f along C, there exists a normal form g of a generalized cuspidal edge along C such that g is right equivalent to f.*

Proof. This is a direct consequence of Lemma 3.2. In fact, the parametrization of a generalized cuspidal edge as in Lemma 3.2 just can be written in the normal form as shown in [1] and [3]. □

We denote by $\mathbb{E}^\omega(C)$ the set of C^ω-differentiable generalized cuspidal edges along C which may not be written in the normal form, but we assume that each $f \in \mathbb{E}^\omega(C)$ is defined on $I \times (-\varepsilon, \varepsilon)$ for some positive ε and $I \ni s \mapsto f(s, 0)$ gives the arc-length parametrization of C. Then its singular curvature function κ_s coincides with that of its normal form.

Definition 5.5 (cf. [3, (0.9)])**.** A C^r-differentiable generalized cuspidal edge f along C is said to be *admissible* if its singular curvature function κ_s satisfies

$$\max_{u\in I} |\kappa_s(u)| < \min_{u\in I} \kappa(u),$$

where κ is the curvature function of C.

We denote by $\mathbb{E}^r_*(C)$ the set of admissible C^ω-differentiable generalized cuspidal edges along C belonging to $\mathbb{E}^r(C)$

Moreover, we define the subset $\mathbb{E}^r_{**}(C)$ of $\mathbb{E}^r_*(C)$ consisting of generalized cuspidal edges with non-vanishing singular curvature functions, that is, $f \in \mathbb{E}^r_{**}(C)$ if and only if

$$0 < \min_{u\in I} |\kappa_s(u)| \le \max_{u\in I} |\kappa_s(u)| < \min_{u\in I} \kappa(u).$$

As in Theorem II in [3], the following assertion holds:

Fact 5.6. *For each $f \in \mathbb{E}^\omega_*(C)$, there exist non-faithful isomers f_* (the inverse), $\check{f}_*$ (the inverse dual) such that if g is a C^ω-differentiable admissible generalized cuspidal edge along C which is an isomer of f, then g is right-left equivalent to one of $\check{f}$, f_* and $\check{f}_*$.*

By Fact 5.6 with Proposition 5.3, all the arguments in Section 5 in [3] hold not only for admissible cuspidal edges but also for admissible generalized cuspidal edge without any changes of proofs. So we obtain the following theorem as a consequence:

Theorem 5.7 (A generalization of Theorems III and IV in [3])**.** *Let f be a C^ω-differentiable admissible generalized cuspidal edge along C. Then the number of the right equivalence classes of f, $\check{f}$, f_* and $\check{f}_*$ is four if and only if ds^2_f has no symmetries (the definition of symmetries of ds^2_f is given in* [3, Definition 0.4]*). Moreover, let N_f be the number of congruence classes of the images of the four maps f, $\check{f}$, f_* as map germs along C. Then*

(1) *if C has no non-trivial symmetries and also ds^2_f has no symmetries, then $N_f = 4$,*
(2) *if not the case in* (1)*, it holds that $N_f \le 2$, and*
(3) *$N_f = 1$ if and only if*
   (a) *C lies in a plane and has a non-trivial symmetry,*
   (b) *C lies in a plane and ds^2_f has a symmetry, or*
   (c) *C has a positive symmetry and ds^2_f also has a symmetry.*

Isomers of developable surfaces.

Definition 5.8. A C^r*-developable strip* along C is a C^r-embedding $F : U \to \boldsymbol{R}^3$ defined on a tubular neighborhood U of $I \times \{0\}$ in $I \times \boldsymbol{R}$ such that

- $I \ni u \mapsto \mathbf{c}(u) := F(u, 0) \in \boldsymbol{R}^3$ gives the arc-length parametrization of C,
- there exists a unit vector field $\xi_F(u)$ of F along C (called a *ruling vector field*) such that F can be expressed as

$$F(u, v) = \mathbf{c}(u) + v\xi_F(u) \qquad ((u, v) \in U), \text{ and}$$

- the Gaussian curvature of F vanishes on U identically.

A C^r-developable strip F represents a map germ along C. We identify this induced map germ with F itself if it creates no confusion. We denote by $\mathbf{e}(u)$, $\mathbf{n}(u)$ and $\mathbf{b}(u)$ the unit tangent, unit principal normal and unit bi-normal vector field, respectively. With these notations, we can express ξ_F as

$$\xi_F(u) = \cos\beta_F(u)\mathbf{e}(u) + \sin\beta_F(u)\Big(\cos\alpha_F(u)\mathbf{n}(u) + \sin\alpha_F(u)\mathbf{b}(u)\Big). \tag{5.4}$$

This $\alpha_F : I \to \boldsymbol{R}$ is called the *first angular function*, and $\beta_F : I \to \boldsymbol{R}$ is called the *second angular function* of F. Here, we consider the developable strips satisfying $0 < |\cos\alpha_F| < 1$. Then, we can choose the first angular function α_F so that

$$0 < |\alpha_F(u)| < \frac{\pi}{2} \qquad (u \in I). \tag{5.5}$$

The fact that the Gaussian curvature of F vanishes identically enable us to write

$$\cot\beta_F(u) = \frac{\alpha_F'(u) + \tau_F(u)}{\kappa_F(u)\sin\alpha_F(u)}, \tag{5.6}$$

where $\kappa_F(u)$ and $\tau_F(u)$ are the curvature and torsion functions of $\mathbf{c}(u)$. In particular, we may assume that

$$0 < \beta_F(u) < \pi \qquad (u \in I). \tag{5.7}$$

In particular, $\xi_F(u)$ satisfies $\xi_F \cdot \mathbf{n} > 0$. We call this $F(u, v)$ a *normal form* of a developable strip along C. (This definition of normal form is the same as that in [3].) We denote by $\mathbb{D}^r(C)$ the set of C^r-developable strips along C written in the normal form. Comparing the normal form of generalized cuspidal edges and the normal form of developable strips, the following map

$$\Phi : \mathbb{E}^{\omega}_{**}(C) \ni f \mapsto \Phi_f \in \mathbb{D}^{\omega}(C)$$

is uniquely induced so that the cuspidal angle of f coincides with the first angular function of Φ_f. The developable surface Φ_f was introduced in Izumiya-Saji-Takeuchi [9, Section 5.1] as the map producing the “osculating developable surface” associated with a given cuspidal edge. So we call Φ the *IST-map*.

In [5], for a given developable strip F along C, its isomers

$$\check{F}, \quad F_*, \quad \check{F}_*$$

are defined, which are developable strips along C whose generating curve corresponding to C is congruent to that of F in the Euclidean plane $\boldsymbol{R}^2$. We remark that the IST-maps have the following nice property:

Proposition 5.9. *Let $f \in \mathbb{E}^{\omega}_{**}(C)$. Then it holds that*

$$\Phi_{\check{f}} = \check{\Phi}_f, \quad \Phi_{f_*} = (\Phi_f)_*, \quad \Phi_{\check{f}_*} = (\check{\Phi}_f)_*.$$

Proof. As in [3, Corollary 3.13], the cuspidal angle of the dual $\check{f}$ is equal to $-\theta_f$. On the other hand, by [3, Page 79], the cuspidal angle θ_{f_*} of the dual f_* satisfies

$$\cos\theta_{f_*}(u) = \frac{\kappa(u)}{\kappa(-u)}\cos\theta_f(u), \qquad \theta_f(-u)\theta_{f_*}(u) > 0$$

on I. These two facts imply $\Phi_{\check{f}} = \check{\Phi}_f$ and $\Phi_{f_*} = (\Phi_f)_*$, respectively, since the IST-map Ψ is determined only by θ_f. Since $\check{f}_* = (\check{f})_*$ holds, the third formula is also obtained. $\square$

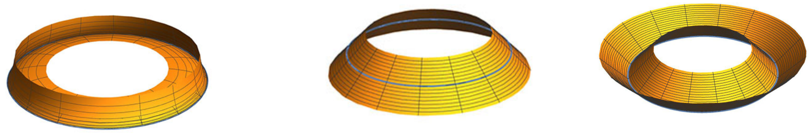

FIGURE 3. A cuspidal edge, a developable strip and a curved folding along a circle

A curved folding associated with a developable strip $F \in \mathbb{D}^r(C)$ is the image of the map Ψ_F defined by

$$\Psi_F(u,v) := \begin{cases} F(u,v) & \text{if } u > 0, \\ \check{F}(u,v) & \text{if } u < 0. \end{cases}$$

It is well-known that each C^r-curved folding along C whose absolute value of the geodesic curvature function (which is common in F and $\check{F}$) along C is less than κ can be realized as the image of a certain Ψ_F for a suitable choice of $F \in \mathbb{D}^r(C)$. In the authors' previous work, for a given C^r-curved folding along C, there are three other curved foldings along C whose crease pattern is the same as the given curved folding. By the above proposition, three curved foldings associated to the image of Ψ_F are $\check{\Psi}_F$, $(\Psi_F)_*$ and $(\check{\Psi}_F)_*$. Thus, the composition of the two maps Φ and Ψ connects the isomers of C^ω-differentiable generalized cuspidal edges to isomers of C^ω-curved foldings (see Figure 3).

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