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APPARENT CONTOURS OF STABLE MAPS INTO THE SPHERE

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Abstract. For a stable map $\varphi\colon M \to S^2$ of a closed and connected surface into the sphere, let $c(\varphi)$ and $n(\varphi)$ denote the numbers of cusps and nodes respectively. In this paper, for each integer $i \geq 1$, in the given homotopy class with i fold curve components, we will determine the minimal number $c + n$.

1. Introduction

Let M be a closed and connected surface and N a connected surface. Let $\varphi\colon M \to N$ be a C^∞ map. Define the set of singular points of φ as

$$S(\varphi) = \{p \in M \mid \operatorname{rank} d\varphi_p < 2\}.$$

We call $\varphi(S(\varphi))$ the *apparent contour* (or *contour* for short) of φ and denote it by $\gamma(\varphi)$.

A C^∞ map $\varphi\colon M \to N$ is said to be *stable* if it satisfies the following two properties.

(1) The map germ at each $p \in M$ is C^∞ right-left equivalent to one of the map germs at $0 \in \mathbb{R}^2$ below;
 $(a, x) \mapsto (a, x)$: p is a regular point,
 $(a, x) \mapsto (a, x^2)$: p is a fold point,
 $(a, x) \mapsto (a, x^3 + ax)$: p is a cusp point.
 Hence, $S(\varphi)$ is a finite disjoint union of circles.
(2) For each $q \in \gamma(\varphi)$, the map germ $(\varphi|_{S(\varphi)}, \varphi^{-1}(q) \cap S(\varphi))$ is right-left equivalent to one of the three multi-germs as depicted in Figure 1.

According to a classical result of Whitney [8], stable maps form an open everywhere dense set in the space of all C^∞ maps $M \to N$. Thus, for a C^∞ map $M \to N$, there is a stable map $M \to N$ homotopic to the C^∞ map.

In this paper, we consider stable maps with singular points. When φ is stable, $S(\varphi)$ is called the *fold curve* of φ, and the numbers of cusps, fold curve components and nodes on $\gamma(\varphi)$ are denoted by $c(\varphi)$, $i(\varphi)$ and $n(\varphi)$ respectively.

An oriented closed surface of genus g is denoted by Σ_g. The 2-dimensional sphere and the plane are denoted by S^2 and $\mathbb{R}^2$ respectively.

Let $\varphi_0\colon M \to S^2$ be a C^∞ map and $\varphi\colon M \to S^2$ be a stable map which is homotopic to φ_0 and whose contour consists of i components. Then, call $\gamma(\varphi)$ an *i-minimal contour* of φ_0 if the number $c+n$ for $\gamma(\varphi)$ is the smallest among the contours of stable maps which are homotopic to φ_0 and whose contours consist of i components. A 1-minimal contour, which is called a *minimal contour* in [4], of a C^∞ map $M \to \mathbb{R}^2$ was studied by Pignoni [4]. A 1-minimal contour of a C^∞ map $M \to S^2$ was studied by Demoto [1], Kamenosono and the second author [2]. They obtained the following result:

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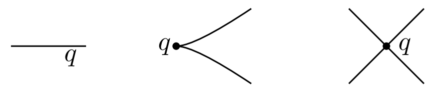

FIGURE 1. The multi-germs of $\varphi|_{S(\varphi)}$

Theorem 1.1 ([1], [2])**.** Let $d \geq 0$ and $f\colon \Sigma_g \to S^2$ be a degree d stable map whose contour consists of one component. The contour $\gamma(f)$ is 1-minimal if and only if the pair (c, n) for $\gamma(f)$ is one of the items below:

$$(c,n) = \begin{cases} (2d, 0) & \text{if } g = 0, \\ (2(d-1), 4) \text{ or } (2d+2, 0) & \text{if } g = 1 \text{ and for each } d \geq 1, \\ (2,4) \text{ or } (6,0) & \text{if } (d,g) = (1,2), \\ (2(d-g), 2g+2) & \text{if } d \geq g > 1, \\ (2, d+g+1) & \text{if } d \leq g \text{ and } g \not\equiv d \pmod 2,\ (d,g) \neq (1,2), \\ (0, d+g+2) & \text{if } d \leq g \text{ and } g \equiv d \pmod 2,\ (d,g) \neq (1,1). \end{cases}$$

On the other hand, the second author [9] introduced and studied a (c, i, n)-minimal contour of a C^∞ map $\Sigma_g \to S^2$: The apparent contour of a stable map $\varphi\colon M \to S^2$ is a (c, i, n)-minimal contour of a C^∞ map $\varphi_0\colon M \to S^2$ if the triple $(c(\varphi), i(\varphi), n(\varphi))$ is the smallest with respect to the lexicographic order among the stable maps homotopic to φ_0. Furthermore, he introduced some lemmas concerning apparent contours of stable maps $M \to S^2$ whose contours consist of some components.

In this paper, we will study an i-minimal contour of a C^∞ map $\Sigma_g \to S^2$ for each $i \geq 2$. Note that, for each number $i \geq 1$, there is a C^∞ map $\Sigma_g \to S^2$ whose contour consists of i components.

Recall that by virtue of Hopf's theorem (see [3] for example), two C^∞ maps $\Sigma_g \to S^2$ are homotopic if and only if their degrees coincide. Thus, the homotopy class of stable maps $\Sigma_g \to S^2$ of degree d is represented by the pair (d, g).

The main theorem of this paper is the following.

Theorem 1.2. Let $f\colon \Sigma_g \to S^2$ be a degree d stable map whose contour consists of i components. Then, the contour $\gamma(f)$ is i-minimal if and only if the pair (c, n) for $\gamma(f)$ is one of the items below:

$g = 0$**:**

$$(c,n) = \begin{cases} \text{(0-i)} \quad (2(|d|-i+1), 0) & \text{if } 1 \leq i \leq |d|+1, \\ \text{(0-ii)} \quad (2,0) & \text{if } i \geq |d|+2,\ i \equiv d \pmod 2, \\ \text{(0-iii)} \quad (0,0) & \text{if } i \geq |d|+2,\ i \not\equiv d \pmod 2, \end{cases}$$

$g = 1$**:**

$$(c,n) = \begin{cases} \text{(1-i)} \quad (2(|d|-i), 4) \text{ or } (2(|d|-i)+4, 0) & \text{if } 1 \leq i \leq |d|, \\ \text{(1-ii)} \quad (2,2) & \text{if } (d,i) = (0,1), \\ \text{(1-iii)} \quad (2,0) & \text{if } i \geq |d|+1,\ i \not\equiv d \pmod 2 \text{ except } (d,i) = (0,1), \\ \text{(1-iv)} \quad (0,0) & \text{if } i \geq |d|+1,\ i \equiv d \pmod 2, \end{cases}$$

$g = 2$**:**

$$(c,n) = \begin{cases} \text{(2-i)} \ \ (2(|d|-i-1), 6) & \text{if } 1 \leq i \leq |d|-1, \\ \text{(2-ii)} \ \ (2,4) \text{ or } (6,0) & \text{if } i = |d|, \\ \text{(2-iii)} \ \ (0,4) & \text{if } i = |d|+1, \\ \text{(2-iv)} \ \ (2,2) & \text{if } (d,i) = (0,2), \\ \text{(2-v)} \ \ (2,0) & \text{if } i \geq |d|+2,\ i \equiv d \pmod 2 \text{ except } (d,i) = (0,2), \\ \text{(2-vi)} \ \ (0,0) & \text{if } i \geq |d|+2,\ i \not\equiv d \pmod 2, \end{cases}$$

$g \geq 3$**:**

$$(c,n) = \begin{cases} \text{(g-i)} \ \ (2(|d|-g-i+1), 2+2g) & \text{if } 1 \leq i \leq |d|-g+1, \\ \text{(g-ii)} \ \ (2, |d|+g-i+2) & \text{if } |d|-g+2 \leq i < |d|+g-1 \text{ and } d+g \equiv i \pmod 2, \\ \text{(g-iii)} \ \ (0, |d|+g-i+3) & \text{if } |d|-g+2 \leq i \leq |d|+g-1 \text{ and } d+g \not\equiv i \pmod 2, \\ \text{(g-iv)} \ \ (2,2) & \text{if } (d,i) = (0,g), \\ \text{(g-v)} \ \ (2,0) & \text{if } i \geq |d|+g,\ i \equiv d+g \pmod 2 \text{ except } (d,i) = (0,g), \\ \text{(g-vi)} \ \ (0,0) & \text{if } i \geq |d|+g,\ i \not\equiv d+g \pmod 2. \end{cases}$$

Theorem 1.2 yields the following corollaries.

Corollary 1.3. Let $f\colon \Sigma_g \to S^2$ be a degree d stable map whose contour consists of i components. Then, the contour $\gamma(f)$ is i-minimal if and only if the number $c+n$ for $\gamma(f)$ is one of the items below:

$g = 0$**:**

$$c+n = \begin{cases} 2(|d|-i+1) & \text{if } 1 \leq i \leq |d|+1, \\ 2 & \text{if } i \geq |d|+2,\ i \equiv d \pmod 2, \\ 0 & \text{if } i \geq |d|+2,\ i \not\equiv d \pmod 2. \end{cases}$$

$g \geq 1$**:**

$$c+n = \begin{cases} 2(|d|-i+2) & \text{if } 1 \leq i \leq |d|-g+1, \\ |d|+g-i+4 & \text{if } |d|-g+2 \leq i < |d|+g-1 \text{ and } d+g \equiv i \pmod 2, \\ |d|+g-i+3 & \text{if } |d|-g+2 \leq i \leq |d|+g-1 \text{ and } d+g \not\equiv i \pmod 2, \\ 4 & \text{if } (d,i) = (0,g), \\ 2 & \text{if } i \geq |d|+g,\ i \equiv d+g \pmod 2 \text{ except } (d,i) = (0,g), \\ 0 & \text{if } i \geq |d|+g,\ i \not\equiv d+g \pmod 2, \end{cases}$$

Corollary 1.4. (1) For each i, any i-minimal contour of a C^∞ between S^2 has no node.
(2) For each i, the number of nodes on any i-minimal contour of a C^∞ map $\Sigma_g \to S^2$ is an even number.

We remark that the number of cusps on each stable map $\Sigma_g \to S^2$ is an even number, see [6] for details.

Note that for each d and i, there is a degree d stable map $\Sigma_g \to S^2$ whose contour consists of i components and whose contour has odd number of nodes.

This paper is organized as follows: In §2, we introduce some notions concerning the apparent contour of a stable map between surfaces. In §3, some stable maps $\Sigma_g \to S^2$ are described. In §4, Theorem 1.2 is proved. In §5, we consider the case of a stable map which has no cusps. In §6, some problems are posed.

Throughout this paper, all surfaces are connected and of class C^∞, and all maps are of class C^∞. The symbols $d, g \geq 0, i \geq 1$ denote integers unless stated otherwise.

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2. Preliminaries

In the following, we describe some notions concerning the apparent contour of a stable map $M \to S^2$ of a closed surface which is not necessary orientable.

Let M be a closed surface and $\varphi\colon M \to S^2$ a stable map with singular points. Let $S(\varphi) = S_1 \cup \cdots \cup S_\ell$ be the decomposition of $S(\varphi)$ into the connected components and set $\gamma_i = \varphi(S_i)$ $(i = 1, \ldots, \ell)$. Then, $\gamma(\varphi) = \gamma_1 \cup \cdots \cup \gamma_\ell$. Denote by $n_1(\varphi)$ the total number of self-intersection points of γ_i $(i = 1, \ldots, \ell)$ and $n_2(\varphi)$ the total of the number of points $\gamma_i \cap \gamma_j$ for all i and j with $i \neq j$. Note that $n_2(\varphi)$ is an even number and that $n(\varphi) = n_1(\varphi) + n_2(\varphi)$. Let $m(\varphi)$ be the smallest number of elements in the set $\varphi^{-1}(y)$, where $y \in S^2$ runs over all regular values of φ. Fix a regular value ∞ such that $\varphi^{-1}(\infty)$ consists of $m(\varphi)$ points. For each γ_i, denote by U_i the component of $S^2 \setminus \gamma_i$ which contains ∞. Note that $\partial U_i \subset \gamma_i$.

Orient γ_i so that at each fold point image, the surface is "folded to the left". More precisely, for a point $y \in \gamma_i$ which is not a cusp or a node of γ_i, choose a normal vector v of γ_i at y such that $\varphi^{-1}(y')$ contains more elements than $\varphi^{-1}(y)$, where y' is a regular value of φ close to y in the direction of v. Let τ be a tangent vector of γ_i at y with respect to the above orientation of γ_i. Then, orient S^2 by the ordered pair (τ, v). It is easy to see that this gives a well-defined orientation of S^2.

Definition 2.1. A point $y \in \partial U_i \setminus \{\text{cusps, nodes}\}$ is said to be *positive* if the normal orientation v at y points toward U_i. Otherwise, it is said to be *negative*.

A component γ_i is said to be *positive* if all points of $\partial U_i \setminus \{\text{cusps, nodes}\}$ are positive; otherwise, γ_i is said to be *negative*. The numbers of positive and negative components are denoted by i^+ and i^- respectively. Note that there is at least one negative component unless $S(\varphi) = \emptyset$.

Definition 2.2. A point $y \in \partial U_i \setminus \{\text{cusps, nodes}\}$ is called an *admissible starting point* if

(1) y is a positive point of a positive component γ_i or
(2) y is a negative point of a negative component γ_i.

Note that for each i, there always exists an admissible starting point in γ_i.

Definition 2.3. Let $y \in \gamma_i$ be an admissible starting point. Suppose that $Q \in \gamma_i$ is a node, and let $\alpha\colon [0, 1] \to \gamma_i$ be a parameterization consistent with the orientation which is singular only when the image is a cusp such that $\alpha^{-1}(y) = \{0, 1\}$. Then, there are two numbers $t_1 < t_2$ satisfying $\alpha(t_1) = \alpha(t_2) = Q$.

We say that Q is *positive* if the orientation of S^2 at Q defined by the ordered pair $(\alpha'(t_1), \alpha'(t_2))$ coincides with that of S^2 at Q; *negative*, otherwise. See Figure 2 for details.

The numbers of positive and negative nodes on γ_i are denoted by N_i^+ and N_i^- respectively. The definition of a positive (or negative) node of γ_i depends on the choice of an admissible starting point y. However, it is known that the algebraic number $N_i^+ - N_i^-$ does not depend on the choice of y, see [7] for details. Thus, the algebraic number $N^+ - N^- = \sum_{i=1}^{k}(N_i^+ - N_i^-)$ is well defined. Note that nodes arising from $\gamma_i \cap \gamma_j$ $(i \neq j)$ play no role in the computation.

Then, the following formula was obtained in [2].

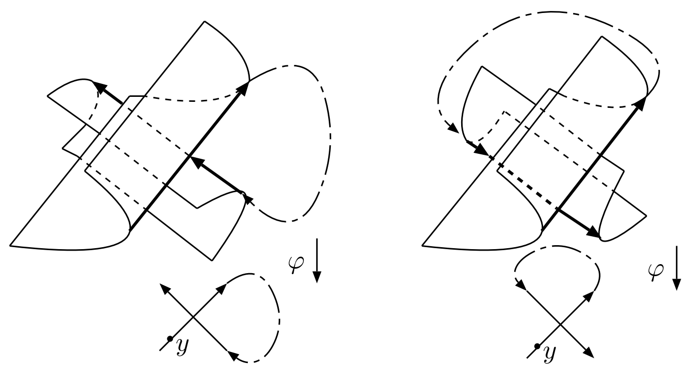

FIGURE 2. A positive node and a negative node.

Proposition 2.4 ([2])**.** *For a stable map $\varphi\colon M \to S^2$ of a closed surface of genus g, we have*

$$g = \varepsilon(M)\left[(N^+ - N^-) + \frac{c(\varphi)}{2} + (1 + i^+ - i^-) - m(\varphi)\right] \tag{2.1}$$

where $\varepsilon(M)$ is equal to 1 if M is orientable and 2 if M is not orientable.

The second author has obtained an extension of the formula (2.1) to a stable map $M \to \Sigma_h$ ($h \geq 1$) whose contour consists of one component that will be published in the forthcoming paper [10].

In the following, we assume $\gamma_i \cap \gamma_j = \emptyset$ for all $i \neq j$. Denote by $U_\infty \subset S^2 \setminus \gamma(\varphi)$ the component which contains ∞. Denote by γ_1 the component of $\gamma(\varphi)$ which contains ∂U_∞. Note that γ_1 is a negative component of φ. Then, the following lemmas and corollary were obtained in [9].

Lemma 2.5. If γ_1 has a node, then it has a negative node.

Lemma 2.6. If a positive component γ_i has a node, then it has a positive node.

Corollary 2.7. If the number of negative components of $\gamma(\varphi)$ is equal to one and $\gamma(\varphi)$ has a node, then it has a negative node.

3. Stable maps $\Sigma_g \to S^2$

In this section, we introduce some stable maps $\Sigma_g \to S^2$ which we employ the following sections. In the following, the symbol $f_{a,b,c}$ denote the degree a stable map of Σ_b into S^2 having c connected components of singular set.

For each $g \geq 0$, define a degree zero stable map $f_{0,g,g+1}\colon \Sigma_g \to S^2$ by $f_{0,g,g+1} = \iota \circ p_g$, where $p_g\colon \Sigma_g \to \mathbb{R}^2$ is defined by Figure 3 and ι is the inclusion $\iota\colon \mathbb{R}^2 \hookrightarrow \mathbb{R}^2 \cup \{\infty\} = S^2$. Then, the triple (c, n, i) for $\gamma(f_{0,g,g+1})$ is equal to $(0, 0, g + 1)$.

The following lemma can be easily proven as illustrated in Figure 4.

Lemma 3.1. Let $f\colon \Sigma_g \to S^2$ be a degree d stable map. Then, there is a degree d stable map $\widetilde{f}\colon \Sigma_g \to S^2$ whose triple (c, n, i) is equal to $(c(f), n(f), i(f) + 2)$ such that $\gamma(\widetilde{f}) = \gamma(f) \coprod S^1 \coprod S^1$.

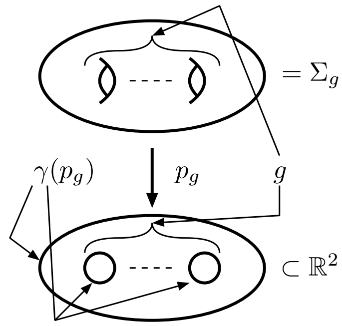

Figure 3. The contour $\gamma(p_g)$

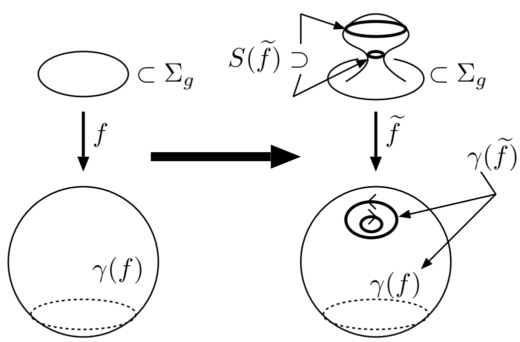

Figure 4. Proof of Lemma 3.1.

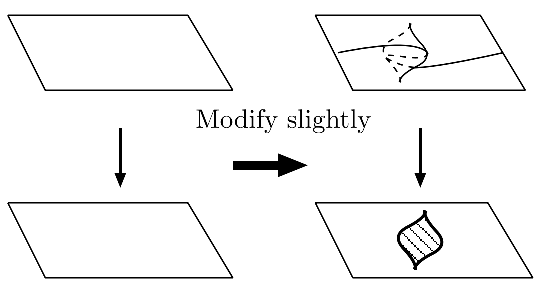

Figure 5. Making a pleat

By applying Lemma 3.1 inductively to $f_{0,g,g+1}$, we obtain the degree zero stable map $f_{0,g,i} \colon \Sigma_g \to S^2$ whose triple (c, n, i) is equal to $(0, 0, i)$ for each pair (g, i) which satisfies $i \geq g+1$ and $i \equiv g + 1 \pmod 2$.

By making a pleat to $f_{0,g,i}$ (see Figure 5 for details), we obtain a degree zero stable map $f_{0,g,i+1} \colon \Sigma_g \to S^2$ whose triple (c, n, i) is equal to $(2, 0, i + 1)$.

For each odd number g, by attaching $(g - 1)$ handles vertically (see Figure 6 for details) to a degree zero stable map $T^2 \to S^2$ whose contour is in Figure 7(a) with $\ell_1 = 0$, we obtain a degree zero stable map $f_{0,g,g} \colon \Sigma_g \to S^2$ whose contour is in Figure 7(a) with $\ell_1 = (g - 1)$. Similarly, for each even number $g \geq 2$, by attaching $(g - 2)$ handles vertically to a degree zero stable

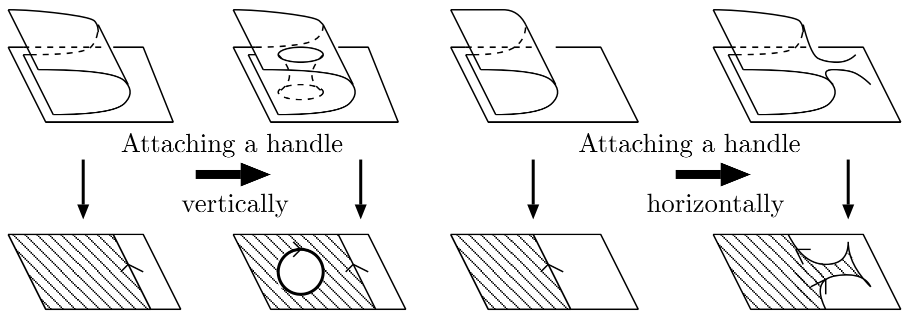

FIGURE 6. Attaching a handle

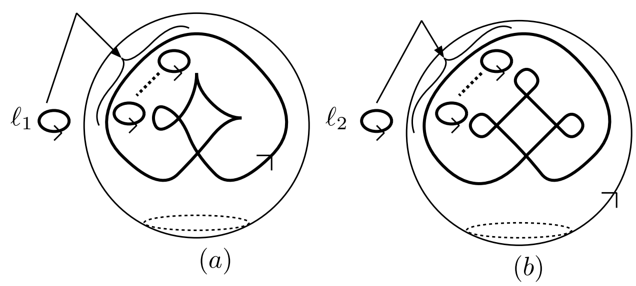

FIGURE 7. The contours $\gamma(f_{0,g,g})$ (g is odd), and $\gamma(f_{0,g,g-1})$ (g is even)

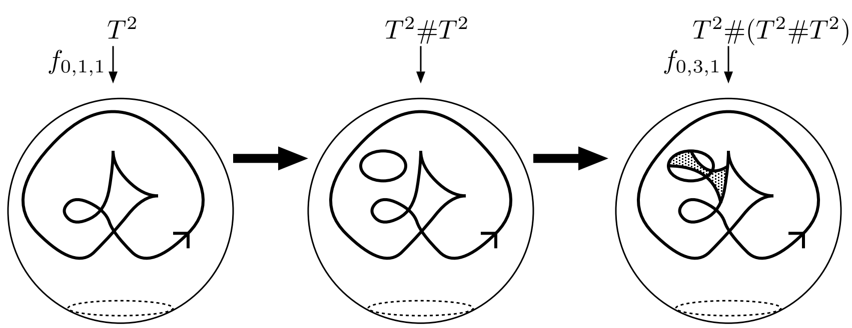

FIGURE 8. Attaching a pair of handles to $f_{0,1,1}$

map $\Sigma_2 \to S^2$ whose contour is in Figure 7(b) with $\ell_2 = 0$, we obtain a degree zero stable map $f_{0,g,g-1}\colon \Sigma_g \to S^2$ whose contour is in Figure 7(b) with $\ell_2 = (g-2)$. Remark that the degree zero stable maps $f_{0,1,1}$ and $f_{0,2,1}$ were obtained in [2].

For each $g \geq 1$, by attaching a pair of handles, attaching a handle vertically first and attaching a handle horizontally, see Figure 6 for details, second, see Figure 8 for example, or by attaching a handle vertically inductively to the degree zero stable map $\Sigma_g \to S^2$ whose contour is 1-minimal, the degree zero stable map is in Theorem 1.1, we obtain a degree zero stable map $f_{0,g,i}\colon \Sigma_g \to S^2$

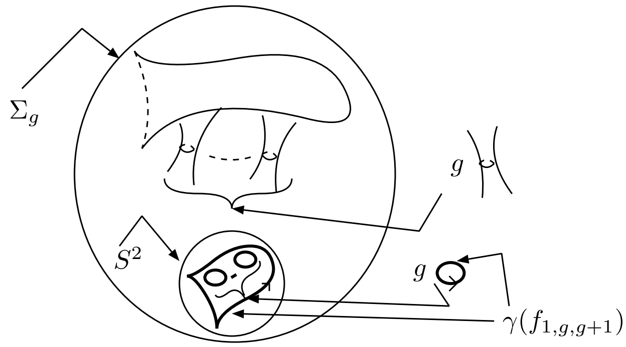

Figure 9. The stable map $f_{1,g,g+1}$

whose contour consists of i components and whose pair (c,n) is equal to

$$(c,n) = \begin{cases} (2, g-i+2) & \text{if } 1 \leq i \leq g \text{ and } i \equiv g \pmod 2, \\ (0, g-i+3) & \text{if } 1 \leq i \leq g \text{ and } i \not\equiv g \pmod 2. \end{cases}$$

Thus, we obtain the following maps.

Proposition 3.2. For each $i \geq 1$ and $g \geq 0$, there is a degree zero stable map $f_{0,g,i} \colon \Sigma_g \to S^2$ whose contour consists of i components and whose pair (c,n) is one of the items below:

$$(c,n) = \begin{cases} \text{(a)} \ (2, g-i+2) & \text{if } 1 \leq i \leq g \text{ and } i \equiv g \pmod 2, \\ \text{(b)} \ (0, g-i+3) & \text{if } 1 \leq i \leq g \text{ and } i \not\equiv g \pmod 2, \\ \text{(c)} \ (2, 0) & \text{if } i \geq g+1 \text{ and } i \equiv g \pmod 2, \\ \text{(d)} \ (0, 0) & \text{if } i \geq g+1 \text{ and } i \not\equiv g \pmod 2. \end{cases}$$

For a sufficiently large sphere whose center is the origin of $\mathbb{R}^3$, make a pleat. Then, by attaching g handles to the sphere, we obtain a Σ_g as in Figure 9. Then, define the map $f_{1,g,g+1} \colon \Sigma_g \to S^2$ by $\pi|_{\Sigma_g}$, where $\pi \colon \mathbb{R}^3 \setminus \{0\} \to S^2$ defined by $\pi(x) = x/|x|$. Thus, we obtain the following Lemma.

Proposition 3.3. The map $f_{1,g,g+1} \colon \Sigma_g \to S^2$ is a degree one stable map whose triple (c,n,i) is equal to $(2, 0, g+1)$.

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Note that for a C^∞ map $\Sigma_g \to S^2$ of degree d, by changing the orientation of Σ_g, we obtain a C^∞ map $\Sigma_g \to S^2$ of degree $-d$. In the following, we assume $d \geq 0$.

Proof of Theorem 1.2. The contour $\gamma(f_{0,g,i})$, the degree zero stable map $f_{0,g,i}$ in Proposition 3.2(d), is trivially i-minimal.

The following lemma can be easily proven as illustrated in Figure 10 where $(\Sigma_g)_-$ denotes the closure of the set of regular points whose neighborhoods are orientation reversed by the map.

Lemma 4.1. Let $f \colon \Sigma_g \to S^2$ be a degree d stable map having a singular point. Then, there is a degree $d+1$ stable map $f' \colon \Sigma_g \to S^2$ such that $\gamma(f') = \gamma(f) \coprod S^1$. The triple (c,n,i) for $\gamma(f')$ is equal to $(c(f), n(f), i(f)+1)$.

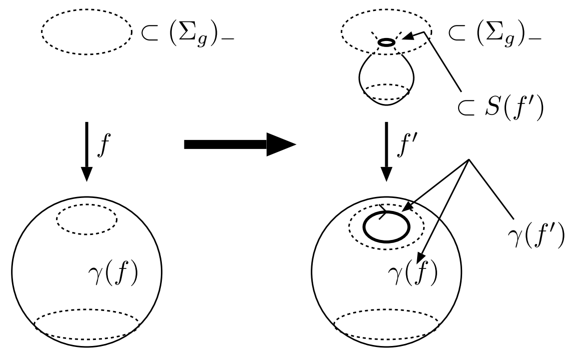

FIGURE 10. Proof of Lemma 4.1

Thus, the contour of the map $\Sigma_g \to S^2$ which is obtained by applying Lemma 4.1 inductively to the degree zero stable map $f_{0,g,i}$ in Proposition 3.2(d) is trivially i-minimal. The cases (0-iii), (1-iv), (2-vi) and (g-vi) of Theorem 1.2 are proved.

We introduce the following lemma.

Lemma 4.2. Let $f\colon \Sigma_g \to S^2$ be a degree d stable map whose contour consists of i components. If the number $d+g+i$ is even, then $\gamma(f)$ has at least two cusps.

Proof. To prove this Lemma, apply a result of Quine [5]: for a stable map $f\colon M \to N$ between oriented surfaces, we have

$$\chi(M) - 2\chi(M_-) + \sum_{q_k:\text{cusp}} \operatorname{sign}(q_k) = (\deg f)(\chi(N)),$$

where M_- denotes the closure of the set of regular points whose neighborhoods are orientation reversed by f, and $\operatorname{sign}(q_k) = \pm 1$ the sign of a cusp q_k, see [5] for definition.

Apply our situation to the Quine's formula:

$$\sum_{q_k:\text{cusp}} \operatorname{sign}(q_k) = 2(d+g-1+\chi((\Sigma_g)_-)). \tag{4.1}$$

Note that $\chi((\Sigma_g)_-) \equiv i \pmod 2$. Then, it follows immediately. □

Lemma 4.2 shows that the following:

Proposition 4.3.
(1) The contour of the degree zero stable map $f_{0,g,i}$ in Proposition 3.2(c) is i-minimal.
(2) The contour of the degree one stable map $f_{1,g,g+1}$ in Proposition 3.3, is $(g+1)$-minimal for each $g \geq 1$.

Thus, the contours of the maps $\Sigma_g \to S^2$ which are obtained by applying Lemma 4.1 inductively to $f_{0,g,i}$ in Proposition 3.2(c) and $f_{1,g,g+1}$ in Proposition 3.3 are i-minimal. The cases (0-ii), (1-iii), (2-v) and (g-v) of Theorem 1.2 are proved.

We prove the remaining cases of Theorem 1.2.

4.1. **The case of $g = 0$.** Let us consider the case (0-i) of Theorem 1.2. For a fixed $d \geq 0$ and each $i \leq d+1$, the formula (4.1) shows that the contour of a degree d stable map between S^2 whose contour consists of i components has at least $2(d-i+1)$ cusps. This shows that the contour of a degree $d+1$ stable map between S^2 which obtained by applying Lemma 4.1 to a degree d stable map between S^2 whose contour is 1-minimal is 2-minimal. By applying this inductively, the case (0-i) of Theorem 1.2 is proved.

4.2. **The case of $g = 1$.** Note that the case (1-ii) is contained in Thorem 1.1. Let us consider the case (1-i) of Theorem 1.2. The formula (2.1) for a degree d stable map $\Sigma_g \to S^2$ whose contour consists of i components induces the following equality:

$$m(f) + g + 2i^- = (N^+ - N^-) + \frac{c}{2} + (1+i)$$

Thus, by $i^- \geq 1$ and $m(f) \geq d$, we obtain the following inequality for the stable map

$$d + g + 1 \leq (N^+ - N^-) + \frac{c}{2} + i. \tag{4.2}$$

Note that the formula (2.1) for a degree $d+1$ stable map $\Sigma_g \to S^2$ whose contour consists of $i+1$ components induces the inequality (4.2).

Let us consider the case that $d = i = 1$. Then, the formula (4.2) shows

$$2 \leq (N^+ - N^-) + \frac{c}{2}. \tag{4.3}$$

If the contour has a node, by Lemma 2.5, then $c + n \geq 4$. Otherwise, then $c \geq 4$. On the other hand, in the case that $d = i = 2$, the formula (4.2) also induces inequality (4.3). Then, by the similarly argument as the above, the number $c+n$ of the contour of a degree two stable map $T^2 \to S^2$ whose contour consists of two components is greater than or equal to four. Thus, the contour of the degree two stable map $T^2 \to S^2$ which is obtained by applying Lemma 4.1 to by the degree one stable map $T^2 \to S^2$ whose contour is 1-minimal is 2-minimal.

In general, we obtain the following proposition.

Proposition 4.4. Let f be a degree d stable map $\Sigma_g \to S^2$ whose contour consists of i components and f' be a degree $d+1$ stable map obtained by applying Lemma 4.1 to f. If the contour $\gamma(f)$ is i-minimal and the number $c+n$ for $\gamma(f)$ is the smallest with respect to the inequality induced by (4.2), then $\gamma(f')$ is $(i+1)$-minimal.

***Remark* 4.5.** The degree one stable map $f' \colon T^2 \to S^2$ obtained by applying Lemma 4.1 to a degree zero $f \colon T^2 \to S^2$ whose contour is 1-minimal is not 2-minimal. The number $c+n$ of $\gamma(f)$ is equal to four. The number $c+n$ of a 2-minimal contour of a degree one C^∞ map $\Sigma_g \to S^2$ is two, see Proposition 4.3(2).

Note that for each $d \geq 1$, the number $c+n$ of a degree d stable map $T^2 \to S^2$ whose contour is 1-minimal is the minimal with respect to the inequality induced by (4.2), see [2] for details. Hence, the case (1-i) of Theorem 1.2 can be proven inductively by using Theorem 1.1 and Proposition 4.4.

4.3. **The case of $g \geq 2$.** Let us consider the cases (2-iv) and (g-iv). Let $f \colon \Sigma_g \to S^2$ be a degree zero stable map whose contour consists of g components. Note that Lemma 4.2 shows the contour $\gamma(f)$ has at least two cusps. We divide this case into the following cases (i) and (ii).

(i) $n_2(f) = 0$: Assume (i^+, i^-) for $\gamma(f)$ is equal to $(g-1, 1)$. Then, by the formula (2.1), we have $1 + m(f) - c/2 = (N^+ - N^-)$. Thus, we have

$$n_1(f) = 1 + m(f) + 2N^- - \frac{c}{2}. \tag{4.4}$$

If $\gamma(f)$ has a node, then by the inequality (4.4) and Corollary 2.7,

$$c+n = c+n_1(f) \geq c + \left(1+m(f)+2N^- - \frac{c}{2}\right) \geq 1+2+1 = 4. \tag{4.5}$$

Note that there is no degree zero stable map $f\colon \Sigma_g \to S^2$ with $m(f)=0$ whose pair (c,n) is equal to $(2,0)$ by the geometrical meaning of cusps. Thus, if $\gamma(f)$ has no node, then $m(f) \geq 2$. Then, by (4.4), we have

$$c+n \geq 2(1+m(f)) \geq 6. \tag{4.6}$$

Assume (i^+, i^-) for $\gamma(f)$ is equal to $(g-\lambda, \lambda)$, where $\lambda = 2, \dots, g+d$. Then, by the formula (2.1), we have $3 - c/2 \leq (N^+ - N^-)$. Thus, we have

$$n_1(f) \geq 3 + 2N^- - \frac{c}{2} \geq 3 - \frac{c}{2}.$$

Therefore, we have

$$c+n = c+n_1(f) \geq c + \left(3 - \frac{c}{2}\right) \geq 3+1 = 4. \tag{4.7}$$

(ii) $n_2(f) \neq 0$: Put (i^+, i^-) for $\gamma(f)$ is equal to $(g-\lambda, \lambda)$, where $\lambda = 1, \dots, g$. Then, by the formula (2.1), we have $1 - c/2 \leq (N^+ - N^-)$. Thus,

$$n_1(f) \geq 1 - \frac{c}{2}.$$

Therefore, we have

$$c+n = c+n_1(f)+n_2(f) \geq c + \left(1 - \frac{c}{2}\right) + 2 \geq 1+1+2 = 4. \tag{4.8}$$

The inequalities (4.5), (4.6), (4.7) and (4.8) shows that the pair (c,n) of a g-minimal contour of a degree zero stable map $\Sigma_g \to S^2$ is equal to $(2,2)$.

Thus, the contour $\gamma(f_{0,g,g})$, $f_{0,g,g}$ is in Proposition 3.2(a) with $i=g$, is g-minimal for each number $g \geq 2$.

By the similar argument as the cases (2-iv) and (g-iv), we can prove the contour $\gamma(f_{0,g,i})$, $f_{0,g,i}$ is in Proposition 3.2(a) and (b), is i-minimal. The contours of the stable maps $\Sigma_g \to S^2$ which are obtained by applying Lemma 4.1 inductively to the stable maps in Proposition 3.2(a), (b) and Theorem 1.1 with $(d,g) = (1,2)$ are also i-minimal. We omit the proof here. The cases (2-ii), (2-iii), (g-ii) and (g-iii) are proved.

Note that for each $d \geq 0$, the number $c+n$ of a degree d stable map $\Sigma_g \to S^2$ whose contour is 1-minimal is the minimal with respect to the inequality induced by (4.2), see [2] for details. Hence, the cases (2-i) and (g-i) of Theorem 1.2 can be proven inductively by using Theorem 1.1 and Proposition 4.4.

This completes the proof of Theorem 1.2. □

5. Fold map case

Let M be a connected and closed surface, and N be a connected surface. A stable map $f\colon M \to N$ which has no cusp is called a *fold map*.

Let $\varphi_0\colon M \to S^2$ be a C^∞ map and $\varphi\colon M \to S^2$ be a fold map which is homotopic to φ_0 and whose contour consists of i components. Then, call the contour $\gamma(\varphi)$ a *regular i-minimal contour* of φ_0 if the number $c+n$ for $\gamma(\varphi)$ is the smallest among the contours of fold maps which are homotopic to φ_0 and whose contours consist of i components.

Note that by Lemma 4.2 if $d+g+i$ is even, then there is no degree d fold map $\Sigma_g \to S^2$ whose contour consists of i components.

Then, as a corollary of Theorem 1.2, we obtain the following.

Theorem 5.1. Assume $d+g+i$ be an odd number. Let $f\colon \Sigma_g \to S^2$ be a degree d fold map whose contour consists of i components. Then, $\gamma(f)$ is a regular i-minimal contour if and only if the number of nodes n for $\gamma(f)$ is one of the items below:

$g = 0$**:**

$$n = 0 \quad \text{if } i \geq |d|+1 \text{ and } i \not\equiv d \pmod 2$$

$g \geq 1$**:**

$$n = \begin{cases} 2+2g & \text{if } i = |d|-g+1, \\ |d|+g-i+3 & \text{if } |d|-g+2 \leq i \leq |d|+g-1 \text{ and } i \not\equiv d+g \pmod 2, \\ 0 & \text{if } i \geq |d|+g,\ i \not\equiv |d|+g \pmod 2. \end{cases}$$

6. Problems

In this section, we pose some problems with respect to the apparent contour of a stable map $M \to N$ between surfaces.

Kamenosono and the second author studied a 1-minimal contour of a C^∞ map $F \to S^2$ of a non-orientable surface. Then, there are the following problems.

***Problem* 6.1.** Study an i-minimal contour and a regular i-minimal contour of a C^∞ map $F \to S^2$ of a non-orientable closed surface into the sphere for each $i \geq 2$.

Let $\varphi_0\colon M \to N$ be a C^∞ map between surfaces and $\varphi\colon M \to N$ a stable map which is homotopic to φ_0 and whose contour consists of i components. Then, the contour $\gamma(f)$ is an *i-essential contour* if the pair (c,n) is the smallest with respect to the lexicographic order, among the stable maps $M \to N$ which are homotopic to φ_0 and whose contour consists of i components. Then, Theorem 1.2 yields the following Theorem.

Theorem 6.2. Let $f\colon \Sigma_g \to S^2$ be a degree d stable map whose contour consists of i components. Then, $\gamma(f)$ is i-essential if and only if the pair (c,n) for $\gamma(f)$ is one of the items below:

$$(c,n) = \begin{cases} (2|d|-i, 4) & \text{if } g = 1 \text{ and } 1 \leq i \leq |d|, \\ (2,4) & \text{if } g = 2 \text{ and } i = |d|. \end{cases}$$

In the other case, the pair (c,n) is of an i-minimal contour.

Corollary 6.3. Let $f_0\colon \Sigma_g \to S^2$ be a C^∞ map whose contour consists of i components. An i-essential contour of f_0 is an i-minimal contour of f_0.

Note that for a C^∞ map $h_0\colon \mathbb{R}P^2 \to S^2$ of modulo two degree one, a 1-minimal (or 1-essential) contour of h_0 is not 1-essential (resp. 1-minimal), see [2] for details. Thus, we pose the following problem.

***Problem* 6.4.** Study the i-essential contours of C^∞ maps from non-orientable surfaces into S^2. Then, compare an i-minimal contour of h_0 and an i-essential contour of h_0.

References

[1] S. Demoto, *Stable maps between 2-spheres with a connected fold curve.* Hiroshima Math. J. 35 (2005), no. 1, 93–113.
[2] A. Kamenosono and T. Yamamoto, *The minimal numbers of singularities of stable maps between surfaces*, Topology Appl. **156** (2009), pp. 2390-2405. DOI: 10.1016/j.topol.2009.06.010
[3] J. Milnor, *Topology from the differentiable viewpoint. Based on notes by David W. Weaver*, The University Press of Virginia, Charlottesville, Va. 1965

[4] R. Pignoni, *Projections of surfaces with a connected fold curve*, Topology Appl. **49** (1993), no. 1, 55–74.

[5] J. R. Quine, *A global theorem for singularities of maps between oriented* 2*-manifolds*, Trans. Amer. Math. Soc. **236** (1978), 307–314.

[6] R. Thom, *Les singularités des applications différentiables.* (French) Ann. Inst. Fourier, Grenoble **6** (1955–1956), 43–87.

[7] H. Whitney, *On regular families of curves*, Bull. Amer. Math. Soc. **47**, (1941). 145–147. DOI: 10.1090/S0002-9904-1941-07395-7

[8] H. Whitney, *On singularities of mappings of euclidean spaces. I. Mappings of the plane into the plane.* Ann. of Math. (2) **62** (1955), 374–410. DOI: 10.2307/1970070

[9] T. Yamamoto, *Apparent contours with minimal number of singularities*, Kyushu J. Math. **64**(2010), no. 1, 1–16.

[10] T. Yamamoto, *Apparent contours of stable maps between closed surfaces with a connected fold curve*, In preparation.

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