

GENERICITY OF CAUSTICS ON A CORNER

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Dedicated to Professor Masahiko Suzuki on his sixtieth birthday

ABSTRACT. We introduce the notions of *caustic-equivalence* and *weak caustic-equivalence relations* of reticular Lagrangian maps in order to give a generic classification of “shapes” of caustics on a corner. We give the figures of all generic caustics on a corner in a smooth manifold of dimension 2 and 3 under these equivalence relations.

1. INTRODUCTION

Lagrangian singularities can be found in many problems of differential geometry, calculus of variations and mathematical physics. One of the most successful applications is the study of singularities of caustics. For example, the particles incident along geodesics from a smooth hypersurface V^{n-1} in a Riemannian manifold M^n to conormal directions define a Lagrangian submanifold at a point in the cotangent bundle. The caustic generated by the hypersurface

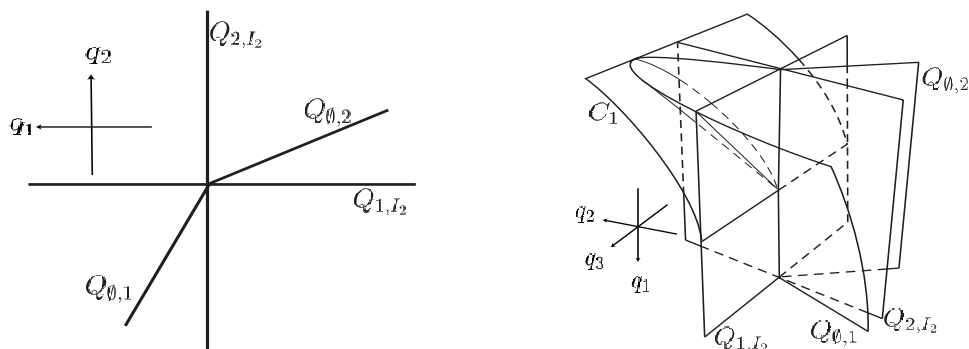


FIGURE 1. Example of caustics on a corner

is regarded as the caustic of the Lagrangian map defined by the restriction of the cotangent bundle projection to the Lagrangian submanifold. In [5] we investigated the theory of *reticular Lagrangian maps*, which is a generalized notion of Lagrangian maps and can be described as stable caustics generated by a hypersurface germ with a boundary ($r = 1$), a corner ($r = 2$), or an r -corner in a smooth manifold. In [6] we gave classification lists of *generic* caustics in the case $r = 0, 1$ respectively. In the case $r = 2$, that is the initial hypersurface has a corner, the method used in [6] does not work well by the modality of generating families since the transversality theorem can not work in this case.

In the case $r = 2$ we consider the following situation: Let V be a 2-dimensional hypersurface germ in a 3-dimensional manifold M . We suppose that V is the light source with a corner and there exist local coordinates (x_1, x_2, x_3) such that $V = \{(x_1, x_2, 0) \in (\mathbb{R}^3, 0) | x_1 \geq 0, x_2 \geq 0\}$. We

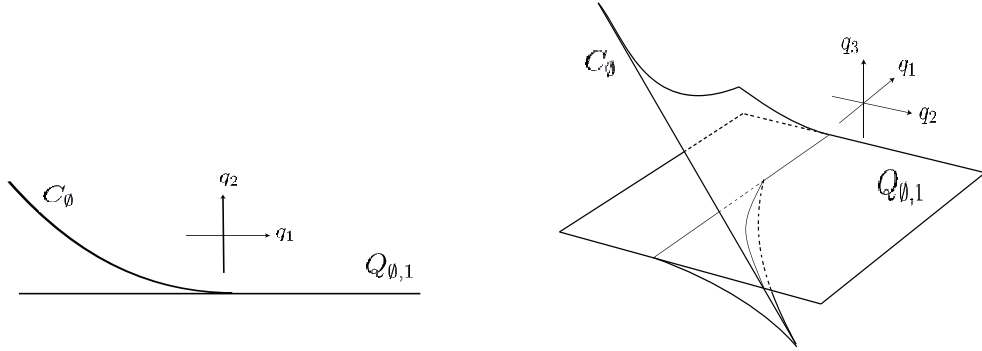


FIGURE 2. Example of caustics on a boundary

consider that the light rays are incident from each of V , $V_1 = V \cap \{x_1 = 0\}$, $V_2 = V \cap \{x_2 = 0\}$, and $V_{I_2} = V \cap \{x_1 = x_2 = 0\}$ to the conormal directions, where $I_2 = \{1, 2\}$. We denote them by $L_\emptyset^0, L_1^0, L_2^0, L_{I_2}^0$ respectively. Then they are reduced to the following normal forms by a suitable symplectic diffeomorphism on $(T^*\mathbb{R}^3, 0)$:

$$L_\sigma^0 = \{(q, p) \in (T^*\mathbb{R}^3, 0) \mid q_\sigma = p_{I_2 \setminus \sigma} = q_3 = 0, q_{I_2 \setminus \sigma} \geq 0\},$$

for $\sigma = \emptyset, 1, 2, I_2$. Then all L_σ^0 are transposed around a point in T^*M by a symplectic diffeomorphism S on T^*M along geodesics. By taking a Lagrangian equivalence around this point, we may assume that S is given by $S : (T^*\mathbb{R}^3, 0) \rightarrow (T^*\mathbb{R}^3, 0)$. Let

$$\mathbb{L} = \{(q, p) \in T^*\mathbb{R}^3 \mid q_1 p_1 = q_2 p_2 = q_3 = 0, q_1 \geq 0, q_2 \geq 0\}$$

be a representative as a germ of the union of all L_σ^0 .

We define $i = S|_{\mathbb{L}}$. Let $\pi : (T^*\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ be the canonical Lagrangian projection. We consider the following map

$$(\mathbb{L}, 0) \xrightarrow{i} (T^*\mathbb{R}^3, 0) \xrightarrow{\pi} (\mathbb{R}^3, 0).$$

We define the caustic of $\pi \circ i$ to be the union of the caustics C_σ of the Lagrangian maps $\pi \circ i|_{L_\sigma^0}$ for all $\sigma \subset I_2$ and the quasi-caustic $Q_{\sigma, \tau} = \pi \circ i(L_\sigma^0 \cap L_\tau^0)$ for all $\sigma, \tau \subset I_r (\sigma \neq \tau)$.

In the case $r = 2$, the caustic of $\pi \circ i$ is

$$C_\emptyset \cup C_1 \cup C_2 \cup C_{\{1,2\}} \cup Q_{0,1} \cup Q_{0,2} \cup Q_{1,\{1,2\}} \cup Q_{2,\{1,2\}}.$$

Then the quasi-caustic $Q_{\sigma, \tau}$ expresses the intersection of light rays incident from V_σ and V_τ .

Our purpose is the investigation of generic caustics under perturbations of S . All functions and maps are smooth, unless stated otherwise.

We here give a review of the theory of reticular Lagrangian maps which is developed in [5].

Reticular Lagrangian maps: Let $I_r = \{1, \dots, r\}$,

$$\mathbb{L} = \{(q, p) \in T^*\mathbb{R}^n \mid q_1 p_1 = \dots = q_r p_r = q_{r+1} = \dots = q_n = 0, q_{I_r} \geq 0\}$$

be a representative of the union of

$$L_\sigma^0 = \{(q, p) \in (T^*\mathbb{R}^n, 0) \mid q_\sigma = p_{I_r \setminus \sigma} = q_{r+1} = \dots = q_n = 0, q_{I_r \setminus \sigma} \geq 0\}$$

for all $\sigma \subset I_r$. We write $q_\sigma = 0$ for the condition $q_i = 0$ for all $i \in \sigma$ and write $q_\sigma \geq 0$ for the condition $q_i \geq 0$ for all $i \in \sigma$ and write other notations analogously. The set L_σ^0 is the normalization of the particles incident from the σ -edge $V_\sigma := V \cap \{q_\sigma = 0\}$ of the light source hypersurface $V = \{(q_1, \dots, q_n) \in (\mathbb{R}^n, 0) \mid q_1 \geq 0, \dots, q_r \geq 0, q_n = 0\}$ with an r -corner for some local coordinate system q of M to conormal directions. Let $\pi : (T^*\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be the canonical Lagrangian projection.

A map germ $\pi \circ i : (\mathbb{L}, 0) \rightarrow (T^*\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ is called a *reticular Lagrangian map* if there exists a symplectic diffeomorphism germ S on $(T^*\mathbb{R}^n, 0)$ such that $i = S|_{\mathbb{L}}$.

We note that the particles incident from all V_σ 's to the conormal directions are mapped along geodesics. This map is extended to *the reticular Lagrangian immersion* i which is the generalized notion of Lagrangian immersion.

Equivalence relation: We call a symplectic diffeomorphism germ ϕ on $(T^*\mathbb{R}^n, 0)$ a *reticular diffeomorphism* if $\phi(L_\sigma^0) = L_\sigma^0$ for $\sigma \subset I_r$. We say that reticular Lagrangian maps

$$\pi \circ i_1, \pi \circ i_2 : (\mathbb{L}, 0) \rightarrow (T^*\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$$

are *Lagrangian equivalent* if there exist a reticular diffeomorphism ϕ and a Lagrangian equivalence Θ (a symplectic diffeomorphism which preserves the fiber of π) such that the following diagram is commutative:

$$\begin{array}{ccccc} (\mathbb{L}, 0) & \xrightarrow{i_1} & (T^*\mathbb{R}^n, 0) & \xrightarrow{\pi} & (\mathbb{R}^n, 0) \\ \phi|_{\mathbb{L}} \downarrow & & \Theta \downarrow & & g \downarrow \\ (\mathbb{L}, 0) & \xrightarrow{i_2} & (T^*\mathbb{R}^n, 0) & \xrightarrow{\pi} & (\mathbb{R}^n, 0), \end{array}$$

where g is the diffeomorphism of the base space of π induced by Θ .

It may be thought that a reticular diffeomorphism does not have to be a symplectic diffeomorphism. But a reticular diffeomorphism consists of compositions of two symplectic diffeomorphisms and a Lagrangian equivalence, it follows that a reticular diffeomorphism is automatically a symplectic diffeomorphism. We also remark that if two reticular Lagrangian maps $\pi \circ i_1$ and $\pi \circ i_2$ are Lagrangian equivalent, then the Lagrangian maps $\pi \circ i_1|_{L_\sigma^0}$ and $\pi \circ i_2|_{L_\sigma^0}$ are Lagrangian equivalent for each $\sigma \subset I_r$.

Let U, V be open sets in $\mathbb{R}^{m_1}, \mathbb{R}^{m_2}$ respectively. We denote by $C^\infty(U, V)$ the set which consists of smooth maps from U to V . We define

$$N_f(l, \varepsilon, K) = \{ g \in C^\infty(U, V) \mid |D^\alpha(g - f)_x| < \varepsilon \ \forall x \in K, |\alpha| < l \}$$

for each $f \in C^\infty(U, V), l \in \mathbb{N}, \varepsilon > 0$ and compact set K in U . Then the family of sets $N_f(l, \varepsilon, K)$ forms a basis for the C^∞ -topology on $C^\infty(U, V)$.

Let U be an open set in $T^*\mathbb{R}^n$, and $S(U, T^*\mathbb{R}^n)$ be the set which consists of symplectic embeddings from U to $T^*\mathbb{R}^n$. We equip $S(U, T^*\mathbb{R}^n)$ the induced topology from C^∞ -topology of $C^\infty(U, T^*\mathbb{R}^n)$. We define $S(T^*\mathbb{R}^n, 0)$ to be the set of symplectic diffeomorphism germs on $(T^*\mathbb{R}^n, 0)$.

We say that a reticular Lagrangian map $\pi \circ i : (\mathbb{L}, 0) \rightarrow (T^*\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ is *stable* if the following holds: For any extension $S \in S(T^*\mathbb{R}^n, 0)$ of i and any representative $\tilde{S} \in S(U, T^*\mathbb{R}^n)$ of S , there exists a neighborhood $N_{\tilde{S}}$ of \tilde{S} such that for any $\tilde{T} \in N_{\tilde{S}}$ the reticular Lagrangian maps $\pi \circ (\tilde{T}|_{\mathbb{L}}$ at x_0) and $\pi \circ i$ are Lagrangian equivalent for some $x_0 = (0, \dots, 0, P_{r+1}^0, \dots, P_n^0) \in U$, where the map $(\tilde{T}|_{\mathbb{L}}$ at x_0) is given by $x \in \mathbb{L} \mapsto \tilde{T}(x + x_0) - \tilde{T}(x_0)$.

Generating family: Let $\mathbb{H}^r = \{(x_1, \dots, x_r) \in \mathbb{R}^r \mid x_1 \geq 0, \dots, x_r \geq 0\}$ be an r -corner. We denote by $\mathcal{E}(r; k)$ the set of all germs at 0 in $\mathbb{H}^r \times \mathbb{R}^k$ of smooth maps $\mathbb{H}^r \times \mathbb{R}^k \rightarrow \mathbb{R}$ and set

$\mathfrak{M}(r; k) = \{f \in \mathcal{E}(r; k) \mid f(0) = 0\}$. We write $\mathcal{E}(k)$ for $\mathcal{E}(0; k)$ and $\mathfrak{M}(k)$ for $\mathfrak{M}(0; k)$. Then $\mathcal{E}(r; k)$ is an \mathbb{R} -algebra in the usual way and $\mathfrak{M}(r; k)$ is its unique maximal ideal.

We denote by $J^l(r + k, p)$ the set of l -jets at 0 of germs in $\mathcal{E}(r; k, p)$. There are natural projections:

$$\pi_l : \mathcal{E}(r; k, p) \longrightarrow J^l(r + k, p), \pi_{l_1}^l : J^{l_1}(r + k, p) \longrightarrow J^{l_2}(r + k, p) \quad (l_1 > l_2).$$

We write $j^l f(0)$ for $\pi_l(f)$ for each $f \in \mathcal{E}(r; k, p)$.

A function germ $F(x, y, q) \in \mathfrak{M}(r; k + n)^2$ is called *S-non-degenerate* if

$$x_1, \dots, x_r, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_r}, \frac{\partial F}{\partial y_1}, \dots, \frac{\partial F}{\partial y_k}$$

are independent on $(\mathbb{H}^k \times \mathbb{R}^{k+n}, 0)$, that is

$$\text{rank} \begin{pmatrix} \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x \partial q} \\ \frac{\partial^2 F}{\partial y \partial y} & \frac{\partial^2 F}{\partial y \partial q} \end{pmatrix}_0 = r + k.$$

We say that an *S-non-degenerate* function germ $F(x, y, q) \in \mathfrak{M}(r; k + n)^2$ is a *generating family* of a reticular Lagrangian map $\pi \circ i$ if $F|_{x_\sigma=0}$ is a generating family of the Lagrangian map $\pi \circ i|_{L_\sigma^0}$, that is

$$i(L_\sigma^0) = \left\{ (q, \frac{\partial F}{\partial q}(x, y, q)) \in (T^*\mathbb{R}^n, 0) \mid x_\sigma = \frac{\partial F}{\partial x_{I_r \setminus \sigma}} = \frac{\partial F}{\partial y} = 0, x_{I_r \setminus \sigma} \geq 0 \right\}$$

for $\sigma \subset I_r$.

Generating families of reticular Lagrangian maps with caustics of Figure 1,2 are given as follows:

Figure 1(left): $F(x_1, x_2, q_1, q_2) = x_1^2 - x_1x_2 + x_2^2 + q_1x_1 + q_2x_2$,

Figure 1(right): $F(x_1, x_2, q_1, q_2, q_3) = x_1^2 - x_1x_2 - x_2^3 + q_1x_1 + q_2x_2 + q_3x_2^2$,

Figure 2(left): $F(x, q_1, q_2) = x^3 + q_1x^2 + q_2x$,

Figure 2(right): $F(x, q_1, q_2, q_3) = x^4 + q_1x^3 + q_2x^2 + q_3x$.

We showed that the Lagrangian maps of figure 2 are stable in [5].

We calculate the caustic of the first example: The canonical relation

$$P_i := \{(x, i(x)) \in (\mathbb{L} \times T^*\mathbb{R}, (0, 0))\}$$

of i is given by the generating family $H(Q_1, Q_2, q_1, q_2) = Q_1^2 - Q_1Q_2 + Q_2^2 + Q_1q_1 + Q_2q_2$ such that

$$\begin{aligned} P_i &= \left\{ (Q_1, Q_2, -\frac{\partial H}{\partial Q_1}, -\frac{\partial H}{\partial Q_2}, q_1, q_2, \frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2}) \right\} \\ &= \{Q_1, Q_2, -2Q_1 + Q_2 - q_1, Q_1 - 2Q_2 - q_2, q_1, q_2, Q_1, Q_2\}. \end{aligned}$$

Therefore we have that

$$\begin{aligned} i(L_\emptyset^0) = L_\emptyset &= \{(q_1, q_2, \frac{\partial F}{\partial q_1}, \frac{\partial F}{\partial q_2}) \in (T^*\mathbb{R}^2, 0) \mid \frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = 0\} \\ &= \{(-2x_1 + x_2, x_1 - 2x_2, x_1, x_2)\}, \\ i(L_1^0) = L_1 &= \{(q, \frac{\partial F}{\partial q}) \mid x_1 = \frac{\partial F}{\partial x_2} = 0\} = \{(q_1, -2x_2, 0, x_2)\}, \\ i(L_2^0) = L_2 &= \{(q, \frac{\partial F}{\partial q}) \mid x_2 = \frac{\partial F}{\partial x_1} = 0\} = \{(-2x_1, q_2, x_1, 0)\}, \\ i(L_{\{1,2\}}^0) = L_{\{1,2\}} &= \{(q, \frac{\partial F}{\partial q}) \mid x_1 = x_2 = 0\} = \{(q_1, q_2, 0, 0)\}. \end{aligned}$$

Therefore $C_\emptyset = C_1 = C_2 = C_{\{1,2\}} = \emptyset$.

$$\begin{aligned} Q_{\emptyset,1} &= \{(x_2, -2x_2) \in (\mathbb{R}^2, 0) \mid x_2 \geq 0\}, Q_{\emptyset,2} = \{(-2x_1, x_1) \mid x_1 \geq 0\}, \\ Q_{1,\{1,2\}} &= \{(q_1, 0)\}, Q_{2,\{1,2\}} = \{(0, q_2)\}. \end{aligned}$$

Stability of unfoldings: We recall the theory of unfolding which is developed in [5, p.583 Section 4]. Let $(x, y) = (x_1, \dots, x_r, y_1, \dots, y_k)$ be a fixed coordinate system of $(\mathbb{H}^r \times \mathbb{R}^k, 0)$. We denote by $\mathcal{B}(r; k)$ the group of diffeomorphism germs on $(\mathbb{H}^r \times \mathbb{R}^k, 0)$ of the form:

$$\Phi(x, y) = (x_1\phi_1^1(x, y), \dots, x_r\phi_r^1(x, y), \phi_2^1(x, y), \dots, \phi_2^k(x, y)).$$

We also denote by $\mathcal{B}_n(r; k+n)$ the group of diffeomorphism germs on $(\mathbb{H}^r \times \mathbb{R}^{k+n}, 0)$ of the form:

$$\begin{aligned} \Phi(x, y, q) &= (x_1\phi_1^1(x, y, q), \dots, x_r\phi_r^1(x, y, q), \phi_2^1(x, y, q), \dots, \phi_2^k(x, y, q), \\ &\quad \phi_3^1(q), \dots, \phi_3^n(q)). \end{aligned}$$

We write $\Phi(x, y, q) = (x\phi_1(x, y, q), \phi_2(x, y, q), \phi_3(q))$, $x\frac{\partial f}{\partial x} = (x_1\frac{\partial f}{\partial x_1}, \dots, x_r\frac{\partial f}{\partial x_r})$ and write other notations analogously.

We say that $f, g \in \mathcal{E}(r; k)$ are *reticular \mathcal{R} -equivalent* if there exists $\phi \in \mathcal{B}(r; k)$ such that $g = f \circ \phi$.

We say that $F, G \in \mathcal{E}(r; k+n)$ are *reticular \mathcal{P} - \mathcal{R}^+ -equivalent* if there exist $\Phi \in \mathcal{B}_n(r; k+n)$ and $a \in \mathfrak{M}(n)$ such that $G(x, y, q) = F \circ \Phi(x, y, q) + a(q)$ for $(x, y, q) \in (\mathbb{H}^r \times \mathbb{R}^{k+n}, 0)$. We say that $F \in \mathcal{E}(r; k_1+n)$ and $G \in \mathcal{E}(r; k_2+n)$ are *stably reticular \mathcal{P} - \mathcal{R}^+ -equivalent* if F and G are reticular \mathcal{P} - \mathcal{R}^+ -equivalent after the addition of nondegenerate quadratic forms of y .

We say that a function germ $f \in \mathfrak{M}(r; k)$ is *\mathcal{R} -simple* if the following holds: For a sufficiently large integer l , there exists a neighborhood N of $j^l f(0)$ in $J^l(r+k, 1)$ such that N intersects with a finite number of \mathcal{R} -orbits. By [1] we have that:

Theorem 1.1. (see also [2, p.279]) *An \mathcal{R} -simple function germ in $\mathfrak{M}(1; k)^2$ is stably \mathcal{R} -equivalent to one of the following function germs:*

$$B_l^\pm : \pm x^l \ (l \geq 2), \quad C_l^\pm : xy \pm y^l \ (l \geq 3), \quad F_4^\pm : \pm x^2 + y^3.$$

Let U be an open set in \mathbb{R}^n . We equip $C^\infty(U, \mathbb{R}^n)$ with the C^∞ -topology. Let $f \in \mathfrak{M}(r; k)$ and $F \in \mathfrak{M}(r; k+n)$ be an unfolding of f . We say that F is *reticular \mathcal{P} - \mathcal{R}^+ -stable* if the following condition holds: For any neighborhood U of 0 in \mathbb{R}^{r+k+n} and any representative $\tilde{F} \in C^\infty(U, \mathbb{R})$ of F , there exists a neighborhood $N_{\tilde{F}}$ of \tilde{F} such that, for any element $\tilde{G} \in N_{\tilde{F}}$, the germ $\tilde{G}|_{\mathbb{H}^r \times \mathbb{R}^{k+n}}$ at $(0, y_0, q_0)$ is reticular \mathcal{P} - \mathcal{R}^+ -equivalent to F for some $(0, y_0, q_0) \in U$.

We say that F is *reticular $\mathcal{P}\text{-}\mathcal{R}^+$ -infinitesimal versal* if

$$\mathcal{E}(r; k) = \langle x \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle_{\mathcal{E}(r; k)} + \langle 1, \frac{\partial F}{\partial q} |_{q=0} \rangle_{\mathbb{R}}.$$

In [5] we define other stabilities of unfoldings: versatility, infinitesimal stability, homotopical stability.

As a consequence of Damon’s theory of good action, we have the following theorem:

Theorem 1.2. (see [5, Theorem 4.5]) *Let $F \in \mathfrak{M}(r; k + n)$ be an unfolding of $f \in \mathfrak{M}(r; k)$. Then the following are all equivalent.*

- (1) F is *reticular $\mathcal{P}\text{-}\mathcal{R}^+$ -stable*.
- (2) F is *reticular $\mathcal{P}\text{-}\mathcal{R}^+$ -versal*.
- (3) F is *reticular $\mathcal{P}\text{-}\mathcal{R}^+$ -infinitesimally versal*.
- (4) F is *reticular $\mathcal{P}\text{-}\mathcal{R}^+$ -infinitesimally stable*.
- (5) F is *reticular $\mathcal{P}\text{-}\mathcal{R}^+$ -homotopically stable*.

The relation between reticular Lagrangian maps and their generating families are given in the following theorems:

Theorem 1.3. (see [5, Theorem 3.2]) (1) *For any reticular Lagrangian map $\pi \circ i$, there exists a function germ $F \in \mathfrak{M}(r; k + n)^2$ which is a generating family of $\pi \circ i$.*

(2) *For any S -non-degenerate function germ $F \in \mathfrak{M}(r; k + n)^2$, there exists a reticular Lagrangian map of which F is a generating family.*

(3) *Two reticular Lagrangian maps are Lagrangian equivalent if and only if their generating families are stably reticular $\mathcal{P}\text{-}\mathcal{R}^+$ -equivalent.*

We remark that Nguyen Huu Duc, Nguyen Tien Dai and F. Pham proved the same theorem in the complex analytic category (cf., [3]). But their method does not work well for the C^∞ -category because $F_t = (1 - t)F + tF''$ on p.14 may be degenerate at some point in $[0, 1]$. We solved this problem in [5, p.577-582]. Our method is available for all of C^∞ , real analytic, and complex analytic categories.

Theorem 1.4. (see [5, Theorem 5.5] or [6, Theorem 3.1]) *Let $\pi \circ i : (\mathbb{L}, 0) \rightarrow (T^*\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a reticular Lagrangian map with the generating family $F(x, y, q) \in \mathfrak{M}(r; k + n)^2$. Then $\pi \circ i$ is stable if and only if F is a reticular $\mathcal{P}\text{-}\mathcal{R}^+$ -stable unfolding of $F|_{q=0}$.*

We investigated the genericity of caustics on an r -corner and gave the generic classification for the cases $r = 0$ and 1 in [6]. We also showed that the method in [6] does not work well for the case $r = 2$ by the modality of generating families. In this paper we introduce two equivalence relations of reticular Lagrangian maps which are weaker than Lagrangian equivalence in order to give a generic classification of caustics on a corner.

2. CAUSTIC- AND WEAK CAUSTIC-EQUIVALENCES

We introduce equivalence relations on reticular Lagrangian maps and their generating families.

Let $\pi \circ i_j$ be reticular Lagrangian maps for $j = 1, 2$. We say that they are *caustic-equivalent* if there exists a diffeomorphism germ g on $(\mathbb{R}^n, 0)$ such that

$$(1) \quad g(C_\sigma^1) = C_\tau^2, \quad g(Q_{\sigma, \tau}^1) = Q_{\sigma, \tau}^2 \quad \text{for all } \sigma, \tau \subset I_r \ (\sigma \neq \tau).$$

When all C_σ^i and $Q_{\sigma,\tau}^i$ are smooth, we may define weak caustic-equivalence. We say that reticular Lagrangian maps $\pi \circ i_1$ and $\pi \circ i_2$ are *weakly caustic-equivalent* if all C_σ^i and $Q_{\sigma,\tau}^i$ are smooth and there exists a homeomorphism germ g on $(\mathbb{R}^n, 0)$ such that g is smooth on all C_σ^1 , $Q_{\sigma,\tau}^1$, and satisfies (1).

We shall define the stabilities of reticular Lagrangian maps under the above equivalence relations and define the corresponding equivalence relations and stabilities of their generating families.

The purpose of this paper is to show the following theorem:

Theorem 2.1. *Let $n = 2, 3$, U a neighborhood of 0 in $T^*\mathbb{R}^n$, $S(T^*\mathbb{R}^n, 0)$ be the set of symplectic diffeomorphism germs on $(T^*\mathbb{R}^n, 0)$, and $S(U, T^*\mathbb{R}^n)$ be the space of symplectic embeddings from U to $T^*\mathbb{R}^n$ with the C^∞ -topology. Then there exists a residual set $O \subset S(U, T^*\mathbb{R}^n)$ such that for any $\tilde{S} \in O$ and $x \in U$, the reticular Lagrangian map $\pi \circ \tilde{S}_x|_{\mathbb{L}}$ is weakly caustic-stable or caustic-stable, where $\tilde{S}_x \in S(T^*\mathbb{R}^n, 0)$ is defined by the map $x_0 \mapsto \tilde{S}(x_0 + x) - \tilde{S}(x)$.*

A reticular Lagrangian map $\pi \circ \tilde{S}_x|_{\mathbb{L}}$ for any $\tilde{S} \in O$ and $x \in U$ is weakly caustic-equivalent to one which has a weak reticular \mathcal{P} - \mathcal{C} -stable generating family $B_{2,2}^{\pm,+1}$, $B_{2,2}^{\pm,+2}$, $B_{2,2}^{\pm,-}$, or is caustic equivalent to one which has a reticular \mathcal{P} - \mathcal{C} -stable generating family $B_{2,2}^{\pm,0}$, $B_{2,2,3}^{\pm,\pm}$, $B_{2,3}^{\pm,\pm}$, $B_{3,2}^{\pm,\pm}$, $C_{3,2}^{\pm,\pm}$:

$$\begin{aligned} B_{2,2}^{\pm,+1}: F(x_1, x_2, q_1, q_2) &= x_1^2 \pm x_1x_2 + \frac{1}{5}x_2^2 + q_1x_1 + q_2x_2, \\ B_{2,2}^{\pm,+2}: F(x_1, x_2, q_1, q_2) &= x_1^2 \pm x_1x_2 + x_2^2 + q_1x_1 + q_2x_2, \\ B_{2,2}^{\pm,-}: F(x_1, x_2, q_1, q_2) &= x_1^2 \pm x_1x_2 - x_2^2 + q_1x_1 + q_2x_2, \\ B_{2,2}^{\pm,0}: F(x_1, x_2, q_1, q_2, q_3) &= x_1^2 \pm x_2^2 + q_1x_1 + q_2x_2 + q_3x_1x_2, \\ B_{2,2,3}^{\pm,\pm}: F(x_1, x_2, q_1, q_2, q_3) &= (x_1 \pm x_2)^2 \pm x_2^3 + q_1x_1 + q_2x_2 + q_3x_2^2, \\ B_{2,3}^{\pm,\pm}: F(x_1, x_2, q_1, q_2, q_3) &= x_1^2 \pm x_1x_2 \pm x_2^3 + q_1x_1 + q_2x_2 + q_3x_2^2, \\ B_{3,2}^{\pm,\pm}: F(x_1, x_2, q_1, q_2, q_3) &= x_1^3 \pm x_1x_2 \pm x_2^2 + q_1x_1 + q_2x_2 + q_3x_1^2, \\ C_{3,2}^{\pm,\pm}: F(y, x_1, x_2, q_1, q_2, q_3) &= \pm y^3 + x_1y \pm x_2y + x_2^2 + q_1y + q_2x_1 + q_3x_2. \end{aligned}$$

In order to describe the caustic-equivalence of reticular Lagrangian maps by their generating families, we introduce the following equivalence relation of function germs. We say that function germs $f, g \in \mathcal{E}(r; k)$ are *reticular \mathcal{C} -equivalent* if there exist $\phi \in \mathcal{B}(r; k)$ and a non-zero number $a \in \mathbb{R}$ such that $g = a \cdot f \circ \phi$. We construct the theory of unfoldings with respect to the corresponding equivalence relation. Then the relation of unfoldings is given as follows: Two function germs $F(x, y, q), G(x, y, q) \in \mathcal{E}(r; k + n)$ are *reticular \mathcal{P} - \mathcal{C} -equivalent* if there exist $\Phi \in \mathcal{B}_n(r; k + n)$ and a unit $a \in \mathcal{E}(n)$ and $b \in \mathcal{M}(n)$ such that $G = a \cdot F \circ \Phi + b$. We define the *stable reticular (\mathcal{P} -) \mathcal{C} -equivalence* in the usual way. We remark that a reticular \mathcal{P} - \mathcal{C} -equivalence class includes the reticular \mathcal{P} - \mathcal{R}^+ -equivalence classes.

We review the results of the theory of function germs under this equivalence relation. Let $F(x, y, q) \in \mathcal{M}(r; k + n)$ be an unfolding of $f(x, y) \in \mathcal{M}(r; k)$.

We say that F is *reticular \mathcal{P} - \mathcal{C} -stable* if the following condition holds: For any neighborhood U of 0 in \mathbb{R}^{r+k+n} and any representative $\tilde{F} \in C^\infty(U, \mathbb{R})$ of F , there exists a neighborhood $N_{\tilde{F}}$ of \tilde{F} in the C^∞ -topology such that for any element $\tilde{G} \in N_{\tilde{F}}$ the germ $\tilde{G}|_{\mathbb{H}^r \times \mathbb{R}^{k+n}}$ at $(0, y_0, q_0)$ is reticular \mathcal{P} - \mathcal{C} -equivalent to F for some $(0, y_0, q_0) \in U$.

We say that F is *reticular \mathcal{P} - \mathcal{C} -versal* if all unfoldings of f are reticular \mathcal{P} - \mathcal{C} - f -induced from F . That is, for any unfolding $G \in \mathfrak{M}(r; k + n')$ of f , there exist a map germ

$$\Phi : (\mathbb{H}^r \times \mathbb{R}^{k+n'}, 0) \rightarrow (\mathbb{H}^r \times \mathbb{R}^{k+n}, 0)$$

and a unit $a \in \mathcal{E}(n')$ and $b \in \mathfrak{M}(n')$ satisfying the following conditions:

- (1) $\Phi(x, y, 0) = (x, y, 0)$ for all $(x, y) \in (\mathbb{H}^r \times \mathbb{R}^k, 0)$ and $a(0) = 1, b(0) = 0$,
- (2) Φ can be written in the form: $\Phi(x, y, q) = (x\phi_1(x, y, q), \phi_2(x, y, q), \phi_3(q))$,
- (3) $G(x, y, q) = a(q) \cdot F \circ \Phi(x, y, q) + b(q)$ for all $(x, y, q) \in (\mathbb{H}^r \times \mathbb{R}^{k+n'}, 0)$.

We say that F is *reticular \mathcal{P} - \mathcal{C} -infinitesimally versal* if

$$(2) \quad \mathcal{E}(r; k) = \langle x \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle_{\mathcal{E}(r; k)} + \langle 1, f, \frac{\partial F}{\partial q} |_{q=0} \rangle_{\mathbb{R}}.$$

We say that F is *reticular \mathcal{P} - \mathcal{C} -infinitesimally stable* if

$$\mathcal{E}(r; k + n) = \langle x \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \rangle_{\mathcal{E}(r; k+n)} + \langle 1, F, \frac{\partial F}{\partial q} \rangle_{\mathcal{E}(n)}.$$

We say that F is *reticular \mathcal{P} - \mathcal{C} -homotopically stable* if for any smooth path-germ

$$(\mathbb{R}, 0) \rightarrow \mathcal{E}(r; k + n), t \mapsto F_t$$

with $F_0 = F$, there exists a smooth path-germ

$$(\mathbb{R}, 0) \rightarrow \mathcal{B}_n(r; k + n) \times \mathcal{E}(n) \times \mathcal{E}(n), t \mapsto (\Phi_t, a_t, b_t)$$

with $(\Phi_0, a_0, b_0) = (id, 1, 0)$ such that each (Φ_t, a_t, b_t) is a reticular \mathcal{P} - \mathcal{C} -isomorphism from F to F_t , that is $F_t = a_t \cdot F \circ \Phi_t + b_t$ for t around 0.

The following theorem is used to prove that the stabilities of reticular Lagrangian maps and their generating families are equivalent.

Theorem 2.2. (cf., [5, Theorem 4.5]) *Let $F \in \mathfrak{M}(r; k + n)$ be an unfolding of $f \in \mathfrak{M}(r; k)$. Then the following are all equivalent.*

- (1) F is reticular \mathcal{P} - \mathcal{C} -stable.
- (2) F is reticular \mathcal{P} - \mathcal{C} -versal.
- (3) F is reticular \mathcal{P} - \mathcal{C} -infinitesimally versal.
- (4) F is reticular \mathcal{P} - \mathcal{C} -infinitesimally stable.
- (5) F is reticular \mathcal{P} - \mathcal{C} -homotopically stable.

For a non-quasihomogeneous function germ $f(x, y) \in \mathfrak{M}(r; k)$, if $1, f, a_1, \dots, a_n \in \mathcal{E}(r; k)$ is a representative of a basis of the vector space

$$\mathcal{E}(r; k) / \langle x \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle_{\mathcal{E}(r; k)},$$

then the function germ $f + a_1q_1 + \dots + a_nq_n \in \mathfrak{M}(r; k + n)$ is a reticular \mathcal{P} - \mathcal{C} -stable unfolding of f by (2). We call n the reticular \mathcal{C} -codimension of f . We remark that the dimension of the vector space is $(n + 2)$, but the reticular \mathcal{C} -codimension is n .

If f is quasihomogeneous then f is included in $\langle x \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle_{\mathcal{E}(r; k)}$. This means that the reticular \mathcal{C} -codimension of a quasihomogeneous function germ is equal to its reticular \mathcal{R}^+ -codimension. Therefore if $1, a_1, \dots, a_n \in \mathcal{E}(r; k)$ is a representative of a basis of the vector space, the function germ $f + a_1q_1 + \dots + a_nq_n \in \mathfrak{M}(r; k + n)$ is a reticular \mathcal{P} - \mathcal{C} -stable unfolding of f . In this case the dimension of the vector space is $(n + 1)$, but the reticular \mathcal{C} -codimension is n .

We define the *simplicity* of function germs under the reticular \mathcal{C} -equivalence in the usual way (cf., the definition before Theorem 1.1).

Theorem 2.3. (cf., Theorem 1.1) *A reticular \mathcal{C} -simple function germ in $\mathfrak{M}(1; k)^2$ is stably reticular \mathcal{C} -equivalent to one of the following function germs:*

$$B_l : x^l \ (l \geq 2), \quad C_l^\varepsilon : xy + \varepsilon y^l \ (\varepsilon^{l-1} = 1, l \geq 3), \quad F_4 : x^2 + y^3.$$

The relation between reticular Lagrangian maps and their generating families under the caustic-equivalence are given as follows:

Proposition 2.4. *Let $\pi \circ i_j$ be reticular Lagrangian maps with generating families F_j for $j = 1, 2$. If F_1 and F_2 are stably reticular \mathcal{P} - \mathcal{C} -equivalent then $\pi \circ i_1$ and $\pi \circ i_2$ are caustic-equivalent.*

Proof. The function germ F_2 may be written as $F_2(x, y, q) = a(q)F_3(x, y, q)$, where a is a unit and F_1 and F_3 are stably reticular \mathcal{P} - \mathcal{R}^+ -equivalent. Then the reticular Lagrangian map $\pi \circ i_3$ given by F_3 and $\pi \circ i_1$ are Lagrangian equivalent and the caustic of $\pi \circ i_2$ and $\pi \circ i_3$ coincide with each other. ■

This proposition shows that it is enough to classify function germs under stable reticular \mathcal{P} - \mathcal{C} -equivalence in order to classify reticular Lagrangian maps under caustic-equivalence. We give here the following classification list:

Theorem 2.5. (see [5, p.592]) *Let $f \in \mathfrak{M}(2; k)^2$ have reticular \mathcal{C} -codimension ≤ 3 . Then f is stably reticular \mathcal{C} -equivalent to one in the following list.*

k	Normal form	codim	Conditions	Notation
0	$x_1^2 \pm x_1x_2 + ax_2^2$	3	$0 < a < \frac{1}{4}$	$B_{2,2,a}^{\pm,+}$
	$x_1^2 \pm x_1x_2 + ax_2^2$	3	$a > \frac{1}{4}$	$B_{2,2,a}^{\pm,+;2}$
	$x_1^2 \pm x_1x_2 + ax_2^2$	3	$a < 0$	$B_{2,2,a}^{\pm,-}$
	$x_1^2 \pm x_2^2$	3		$B_{2,2}^{\pm,0}$
	$(x_1 \pm x_2)^2 \pm x_2^3$	3		$B_{2,2,3}^{\pm,\pm}$
	$x_1^2 \pm x_1x_2 \pm x_2^3$	3		$B_{2,3}^{\pm,\pm}$
	$x_1^3 \pm x_1x_2 \pm x_2^2$	3		$B_{3,2}^{\pm,\pm}$
1	$\pm y^3 + x_1y \pm x_2y + x_2^2$	3		$C_{3,2}^{\pm,\pm}$

We remark that the stable reticular \mathcal{C} -equivalence class $B_{2,3}^{+,+}$ of $x_1^2 + x_1x_2 + x_2^3$ consists of the union of the stable reticular \mathcal{R} -equivalence classes of $x_1^2 + x_1x_2 + ax_2^3$ and $-x_1^2 - x_1x_2 - ax_2^3$ for $a > 0$. Since $x_1 \frac{\partial f}{\partial x_1}, x_2 \frac{\partial f}{\partial x_2}, f$ are linear independent, this means that the \mathcal{C} -equivalence class of f is simple. The same things hold for $B_{2,2,3}^{\pm,\pm}, B_{2,3}^{\pm,\pm}, B_{3,2}^{\pm,\pm}, C_{3,2}^{\pm,\pm}$.

We also remark that the classification list looks like that of D.Siersma [4, p.126]. But our equivalence relation differs from his.

3. CAUSTIC-STABILITY

We define the *caustic-stability* of reticular Lagrangian maps and reduce our investigation to finite-dimensional jet spaces of symplectic diffeomorphism germs.

We say that a reticular Lagrangian map $\pi \circ i$ is *caustic-stable* if the following condition holds: For any extension $S \in S(T^*\mathbb{R}^n, 0)$ of i and any representative $\tilde{S} \in S(U, T^*\mathbb{R}^n)$ of S , there exists a neighborhood $N_{\tilde{S}}$ of \tilde{S} such that for any $\tilde{S}' \in N_{\tilde{S}}$ the reticular Lagrangian map $\pi \circ \tilde{S}'|_{\mathbb{L}}$ at x_0 and $\pi \circ i$ are caustic-equivalent for some $x_0 = (0, \dots, 0, p_{r+1}^0, \dots, p_n^0) \in U$.

Definition 3.1. Let $\pi \circ i$ be a reticular Lagrangian map and l be a non-negative number. We say that $\pi \circ i$ is *caustic l -determined* if the following condition holds: For any extension S of

i , the reticular Lagrangian map $\pi \circ S'|_{\mathbb{L}}$ and $\pi \circ i$ are caustic-equivalent for any symplectic diffeomorphism germ S' on $(T^*\mathbb{R}^n, 0)$ satisfying $j^l S(0) = j^l S'(0)$.

Lemma 3.2. *Let $\pi \circ i : (\mathbb{L}, 0) \rightarrow (T^*\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be a reticular Lagrangian map. If a generating family of $\pi \circ i$ is reticular \mathcal{P} - \mathcal{C} -stable then $\pi \circ i$ is caustic $(n + 2)$ -determined.*

Proof. This is proved in a manner analogous to that of [6, Theorem 5.3]. We give a sketch of the proof. Let S be an extension of i . Then we may assume that there exists a function germ $H(Q, p)$ such that the canonical relation P_S has the form:

$$P_S = \left\{ (Q, -\frac{\partial H}{\partial Q}(Q, p), -\frac{\partial H}{\partial p}(Q, p), p) \in (T^*\mathbb{R}^n \times T^*\mathbb{R}^n, (0, 0)) \right\}.$$

Then the function germ $F(x, y, q) = H_0(x, y) + \langle y, q \rangle$ is a reticular \mathcal{P} - \mathcal{C} -stable generating family of $\pi \circ i$, and H_0 is reticular \mathcal{R} - $(n + 3)$ -determined, where $H_0(x, y) = H(x, 0, y)$. Let a symplectic diffeomorphism germ S' on $(T^*\mathbb{R}^n, 0)$ satisfying $j^{n+2} S(0) = j^{n+2} S'(0)$ be given. Then there exists a function germ $H'(Q, p)$ such that the canonical relation $P_{S'}$ has the same form for H' and the function germ $G(x, y, q) = H'_0(x, y) + \langle y, q \rangle$ is a generating family of $\pi \circ S'|_{\mathbb{L}}$. Then it holds that $j^{n+3} H_0(0) = j^{n+3} H'_0(0)$. There exists a function germ G' such that G and G' are reticular \mathcal{P} - \mathcal{R} -equivalent and F and G' are reticular \mathcal{P} - \mathcal{C} -infinitesimal versal unfoldings of $H_0(x, y)$. It follows that F and G are reticular \mathcal{P} - \mathcal{C} -equivalent by Theorem 1.2. Therefore $\pi \circ i$ and $\pi \circ S'|_{\mathbb{L}}$ are caustic-equivalent. ■

Let $S^l(n)$ be the Lie group which consists of l -jets of symplectic diffeomorphisms on $(T^n\mathbb{R}, 0)$.

Orbits of the caustic-equivalence classes $B_{2,2,3}^{\pm,\pm}$, $B_{2,3}^{\pm,\pm}$, $B_{3,2}^{\pm,\pm}$, $C_{3,2}^{\pm,\pm}$: Let $[S_X]_c$ be the caustic-equivalence class of $S_X \in S(T^*\mathbb{R}^3, 0)$ for each $X = B_{2,2,3}^{\pm,\pm}$, $B_{2,3}^{\pm,\pm}$, $B_{3,2}^{\pm,\pm}$, $C_{3,2}^{\pm,\pm}$ in Theorem 2.5 such that $\pi \circ S_X|_{\mathbb{L}}$ has a generating family which is a reticular \mathcal{P} - \mathcal{C} -stable unfolding of X . Since the above singularities are reticular \mathcal{C} -simple, this means that $[j^l S_X(0)]_c (= j^l [S_X(0)]_c)$ are immersed manifolds in $S^l(3)$ for $l \geq 2$.

4. WEAK CAUSTIC-EQUIVALENCE

There exist modalities in the classification list of Theorem 2.5. This means that caustic-equivalence is still too strong for a generic classification of caustics on a corner. In order to obtain a generic classification, we need to admit weak caustic-equivalence and the corresponding equivalence relation on generating families.

We say that a reticular Lagrangian map $\pi \circ i$ is *weakly caustic-stable* if the following condition holds: For any extension $S \in S(T^*\mathbb{R}^n, 0)$ of i and any representative $\tilde{S} \in S(U, T^*\mathbb{R}^n)$ of S , there exists a neighborhood $N_{\tilde{S}}$ of \tilde{S} such that for any $\tilde{S}' \in N_{\tilde{S}}$ the reticular Lagrangian map $\pi \circ \tilde{S}'|_{\mathbb{L}}$ at x_0 and $\pi \circ i$ are weakly caustic-equivalent for some $x_0 = (0, \dots, 0, p_{r+1}^0, \dots, p_n^0) \in U$.

We say that two function germs in $\mathfrak{M}(r; k + n)^2$ are *weakly reticular \mathcal{P} - \mathcal{C} -equivalent* if they are generating families of weakly caustic-equivalent reticular Lagrangian maps. We say that two function germs in $\mathfrak{M}(r; k)^2$ are *weakly reticular \mathcal{C} -equivalent* if they have unfoldings which are weakly reticular \mathcal{P} - \mathcal{C} -equivalent. We define the *stable* weakly reticular (\mathcal{P} -) \mathcal{C} -equivalence in the usual way.

We say that a function germ $F(x, y, q) \in \mathfrak{M}(r; k + n)$ is *weakly reticular \mathcal{P} - \mathcal{C} -stable* if the following condition holds: For any neighborhood U of 0 in \mathbb{R}^{r+k+n} and any representative $\tilde{F} \in C^\infty(U, \mathbb{R})$ of F , there exists a neighborhood $N_{\tilde{F}}$ of \tilde{F} in the C^∞ -topology such that for any

element $\tilde{G} \in N_{\tilde{F}}$ the germ $\tilde{G}|_{\mathbb{H}^r \times \mathbb{R}^{k+n}}$ at $(0, y_0, q_0)$ is weakly reticular $\mathcal{P}\text{-}\mathcal{C}$ -equivalent to F for some $(0, y_0, q_0) \in U$.

Orbits of weak caustic-equivalence classes $B_{2,2,a}^{\pm,+1}, B_{2,2,a}^{\pm,+2}, B_{2,2,a}^{\pm,-}$: We investigate here the weak reticular \mathcal{C} -equivalence classes $B_{2,2,a}^{+,+2}$ of function germs. The same methods are valid for the classes $B_{2,2,a}^{\pm,+1}, B_{2,2,a}^{\pm,+2}, B_{2,2,a}^{\pm,-}$. So we discuss only the classes $B_{2,2,a}^{+,+2}$.

We consider the reticular Lagrangian maps $\pi \circ i_a : (\mathbb{L}, 0) \rightarrow (T^*\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ with the generating families $F_a(x_1, x_2, q_1, q_2) = x_1^2 + x_1x_2 + ax_2^2 + q_1x_1 + q_2x_2$ ($a > \frac{1}{4}$). We give the caustics of $\pi \circ i_a$ and $\pi \circ i_b$ for $\frac{1}{4} < a < b$. In these figures $Q_{1,I_2}, Q_{2,I_2}, Q_{\emptyset,2}$ are in the same positions.

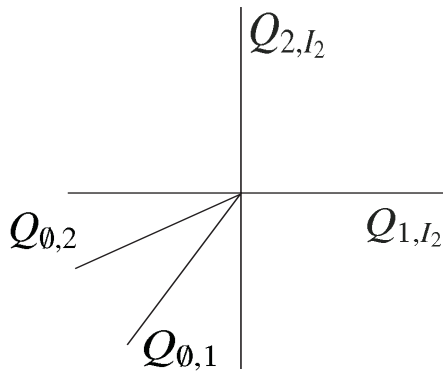


FIGURE 3. the caustics of $\pi \circ i_a$

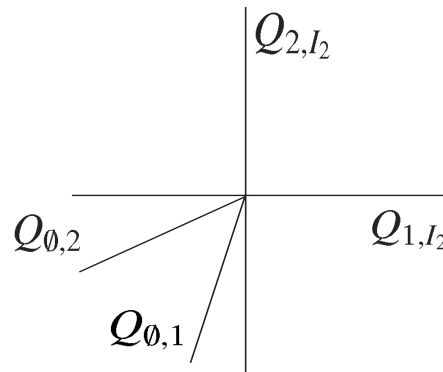


FIGURE 4. the caustics of $\pi \circ i_b$

Suppose that there exists a diffeomorphism germ g on $(\mathbb{R}^2, 0)$ such that $Q_{1,I_2}, Q_{2,I_2}, Q_{\emptyset,2}$ are invariant under g . Then g can not map $Q_{\emptyset,1}$ from one to the other. This implies that caustic-equivalence is too strong for generic classifications. But these caustics are equivalent under weak caustic-equivalence. The homeomorphism germ Φ_a^b on $(\mathbb{R}^2, 0)$ is given as follows: We consider the unit circle U with center 0 and let U_a, U_b be the intersection of U and the caustics of $\pi \circ i_a, \pi \circ i_b$ respectively. Then there exists a diffeomorphism ϕ_a^b on U such that $\phi_a^b(U_a) = U_b$ and ϕ_a^b depends smoothly on a, b . We extend naturally the source space of ϕ_a^b to $\mathbb{R}^2 - \{0\}$ and define

$$\Phi_a^b(x) = \begin{cases} \phi_a^b(x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Then the map germ Φ_a^b at 0 gives a weak caustic-equivalence of $\pi \circ i_a$ and $\pi \circ i_b$. We remark that Φ_a^b is smooth and depends smoothly on a, b except at the origin. This means that the weak caustic-equivalence Φ_a^1 is naturally extended for weak caustic equivalence from the (caustic) stable reticular Lagrangian map with the generating families

$$F'_a(x_1, x_2, q_1, q_2, q_3) = x_1^2 + x_1x_2 + ax_2^2 + q_1x_1 + q_2x_2 + q_3x_2^2$$

to $F'(x_1, x_2, q_1, q_2, q_3) = x_1^2 + x_1x_2 + x_2^2 + q_1x_1 + q_2x_2$. The figure of the corresponding caustic is given in Figure 8. We also remark that the functions $x_1^2 + x_1x_2 + x_2^2 + q_1x_1 + q_2x_2$ and $x_1^2 + x_1x_2 + \frac{1}{5}x_2^2 + q_1x_1 + q_2x_2$ are not weakly reticular $\mathcal{P}\text{-}\mathcal{C}$ -equivalent because $Q_{\emptyset,1}$ and $Q_{\emptyset,1}$ of their caustics are in the opposite positions to each other.

Then we have that the function germs $f_a(x) = x_1^2 + x_1x_2 + ax_2^2$ ($a > \frac{1}{4}$) are all weakly reticular \mathcal{C} -equivalent to each other.

We define the weak reticular \mathcal{C} -equivalence class $[f_a]_w$ of f_a by $\bigcup_{a>\frac{1}{4}} [f_a]_c$. We also define the weak reticular \mathcal{C} -equivalence class $[j^l f_a(0)]_w$ of $j^l f_a(0)$ by $\bigcup_{a>\frac{1}{4}} j^l([f_a]_c)$.

Since $x_1 \frac{\partial f_a}{\partial x_1}, x_2 \frac{\partial f_a}{\partial x_2}, x_2^2$ are linear independent and span the tangent space of the weak reticular \mathcal{C} -equivalence class $[f_a]_w$, we have that $[f_a]_w$ is an immersed manifold in $J^3(2, 1)$.

We classify function germs in $\mathfrak{M}(2; k)^2$ with respect to the reticular \mathcal{C} -equivalence and weak reticular \mathcal{C} -equivalence with the codimension ≤ 3 . Then we have the following list:

k	Normal form	codim	Notation
0	$x_1^2 \pm x_1 x_2 + \frac{1}{5} x_2^2$	2	$B_{2,2}^{\pm,+1}$
	$x_1^2 \pm x_1 x_2 + x_2^2$	2	$B_{2,2}^{\pm,+2}$
	$x_1^2 \pm x_1 x_2 - x_2^2$	2	$B_{2,2}^{\pm,-}$
	$x_1^2 \pm x_2^2$	3	$B_{2,2}^{\pm,0}$
	$(x_1 \pm x_2)^2 \pm x_2^3$	3	$B_{2,2,3}^{\pm,\pm}$
	$x_1^2 \pm x_1 x_2 \pm x_2^3$	3	$B_{2,3}^{\pm,\pm}$
1	$\pm y^3 + x_1 y \pm x_2 y + x_2^2$	3	$C_{3,2}^{\pm,\pm}$

We need to show the following proposition:

Proposition 4.1. *Let $\pi \circ i_a : (\mathbb{L}, 0) \rightarrow (T^*\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be the reticular Lagrangian map with the generating family $x_1^2 + x_1 x_2 + a x_2^2 + q_1 x_1 + q_2 x_2 (B_{2,2}^{+,+2})$. Let $S_a \in S(T^*\mathbb{R}^2, 0)$ be extensions of i_a . Then the weak caustic-equivalence class*

$$[j^l S_1(0)]_w := \bigcup_{a>\frac{1}{4}} [j^l S_a(0)]_c$$

is an immersed manifold in $S^l(2)$ for $l \geq 1$.

Proof. Let $f(x_1, x_2) = x_1^2 + x_1 x_2 + a x_2^2$. The tangent space of $[j^l f(0)]_w$ is spanned by $j^l(x_1 \frac{\partial f}{\partial x_1})(0), j^l(x_2 \frac{\partial f}{\partial x_2})(0), j^l(x_2^2)(0)$ and these are linearly independent for $l \geq 2$. This means that $[j^l f(0)]_w$ is an immersed manifold of $J^l(2, 1)$ for $l \geq 2$. This means that $[j^l S_1(0)]_w$ is an immersed manifold of $S^l(2)$ for $l \geq 1$. ■

We consider the (caustic) stable reticular Lagrangian map

$$\pi \circ i_a : (\mathbb{L}, 0) \rightarrow (T^*\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$$

with the generating family

$$x_1^2 + x_1 x_2 + a x_2^2 + q_1 x_1 + q_2 x_2 + q_3 x_2^2$$

and take an extension $S'_a \in S(T^*\mathbb{R}^2, 0)$ of i_a , then we have by the analogous method that:

Corollary 4.2. *Let S'_a be as above. Then the weak caustic-equivalence class*

$$[j^l S'_1(0)]_w := \bigcup_{a>\frac{1}{4}} [j^l S'_a(0)]_c$$

is an immersed manifold in $S^l(3)$ for $l \geq 1$.

Theorem 4.3. *The function germ $F(x_1, x_2, q_1, q_2) = x_1^2 + x_1 x_2 + x_2^2 + q_1 x_1 + q_2 x_2$ is a weakly reticular \mathcal{P} - \mathcal{C} -stable unfolding of $f(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$*

Proof. We define $F' \in \mathfrak{M}(2; 3)^2$ by $F'(x_1, x_2, q_1, q_2, q_3) = F(x_1, x_2, q_1, q_2) + q_3 x_2^2$. Then F' is a reticular \mathcal{P} - \mathcal{R}^+ -stable unfolding of f . It follows that for any neighborhood U' of 0 in \mathbb{R}^5 and any representative $\hat{F}' \in C^\infty(U, \mathbb{R})$, there exists a neighborhood $N_{\hat{F}'}$ such that for

any $\tilde{G}' \in N_{\tilde{F}'}$, the function germ $\tilde{G}'|_{\mathbb{H}^2 \times \mathbb{R}^3}$ at p'_0 is reticular $\mathcal{P}\text{-}\mathcal{R}^+$ -equivalent to F' for some $p'_0 = (0, 0, q_1^0, q_2^0, q_3^0) \in U'$.

Let a neighborhood U of 0 in \mathbb{R}^4 and a representative $\tilde{F} \in C^\infty(U, \mathbb{R})$ be given. We set the open interval $I = (-0.5, 0.5)$ and set $U' = U \times I$. Then there exists $N_{\tilde{F}'}$, for which the above condition holds. We can choose a neighborhood $N_{\tilde{F}}$ of \tilde{F} such that for any $\tilde{G} \in N_{\tilde{F}}$ the function $\tilde{G} + q_3 x_2^2 \in N_{\tilde{F}'}$. Let a function $\tilde{G} \in N_{\tilde{F}}$ be given. Then the function germ $G' = (\tilde{G} + q_3 x_2^2)|_{\mathbb{H}^2 \times \mathbb{R}^3}$ at p'_0 is reticular $\mathcal{P}\text{-}\mathcal{R}^+$ -equivalent to F' for some

$$p'_0 = (0, 0, q_1^0, q_2^0, q_3^0) \in U'.$$

We define $G \in \mathfrak{M}(2; 2)^2$ by $\tilde{G}|_{\mathbb{H}^2 \times \mathbb{R}^2}$ at $p_0 = (0, 0, q_1^0, q_2^0) \in U$. Then it holds that

$$G'(x, q) = G(x, q_1, q_2) + (q_3 + q_3^0)x_2^2,$$

and $G'|_{q=0} = G(x, 0) + q_3^0 x_2^2$ is reticular \mathcal{R} -equivalent to f . Let (Φ, a) be the reticular $\mathcal{P}\text{-}\mathcal{R}^+$ -equivalence from G' to F' . We write $\Phi(x, q) = (x\phi_1(x, q), \phi_1^2(q), \phi_2^2(q), \phi_3^2(q))$. By shrinking U if necessary, we may assume that the map germ

$$(q_1, q_2) \mapsto (\phi_1^2(q_1, q_2, 0), \phi_2^2(q_1, q_2, 0)) \text{ on } (\mathbb{R}^2, 0)$$

is a diffeomorphism germ. Then F is reticular $\mathcal{P}\text{-}\mathcal{R}^+$ -equivalent to $G_1 \in \mathfrak{M}(2; 2)^2$ given by $G_1(x, q) = G(x_1, x_2, q_1, q_2) + (\phi_3^2(q_1, q_2, 0) + q_3^0)x_2^2$. It follows that the reticular Lagrangian maps defined by F and G_1 are Lagrangian equivalent. We have that

$$j^2(G + q_3^0 x_2^2)(0) = j^2 G_1(0), \quad q_3^0 > -0.5.$$

This means that the caustic of G_1 is weakly caustic-equivalent to the caustic of G because the reticular Lagrangian maps of G_1 and F are the same weak caustic-equivalence class that is 1-determined under weak caustic-equivalence. This means that F and G are weakly reticular $\mathcal{P}\text{-}\mathcal{C}$ -equivalent. Therefore F is weakly reticular $\mathcal{P}\text{-}\mathcal{C}$ -stable. ■

By the above consideration, we have that: For each singularity $B_{2,2}^{\pm,+1}, B_{2,2}^{\pm,+2}, B_{2,2}^{\pm,-}$, if we take the symplectic diffeomorphism germ $S_a(S'_a)$ as the above method, then the weak caustic-equivalence class $[j^l S_a(0)]_w([j^l S'_a(0)]_w)$ is one class and immersed manifold in $S^l(2)(S^l(3))$ for $l \geq 1$ respectively. Since the caustics of $\pi \circ S_a|_{\mathbb{L}}$ and $\pi \circ S'_a|_{\mathbb{L}}$ are determined by their linear parts, this means that the reticular Lagrangian maps are weakly caustic 1-determined.

We now start to prove the main theorem: We choose the weakly caustic-stable reticular Lagrangian maps $\pi \circ i_X : (\mathbb{L}, 0) \rightarrow (T^*\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ for

$$(3) \quad X = B_{2,2}^{\pm,+1}, B_{2,2}^{\pm,+2}, B_{2,2}^{\pm,-}.$$

We also choose the caustic-stable reticular Lagrangian maps $\pi \circ i_X : (\mathbb{L}, 0) \rightarrow (T^*\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ for

$$(4) \quad X = B_{2,2}^{\pm,0}, B_{2,2,3}^{\pm,\pm}, B_{2,3}^{\pm,\pm}, B_{3,2}^{\pm,\pm}, C_{3,2}^{\pm,\pm}.$$

Then other reticular Lagrangian maps are not caustic-stable since other singularities have reticular \mathcal{C} -codimension > 3 . We choose extensions $S_X \in S(T^*\mathbb{R}^n, 0)$ of i_X for all X . We define

$$O'_1 = \{\tilde{S} \in S(U, T^*\mathbb{R}^n) \mid j_0^1 \tilde{S} \text{ is transversal to } [j^1 S_X(0)]_w \text{ for all } X \text{ in (3)}\},$$

$$O'_2 = \{\tilde{S} \in S(U, T^*\mathbb{R}^n) \mid j_0^{n+2} \tilde{S} \text{ is transversal to } [j^{n+2} S_X(0)]_c \text{ for all } X \text{ in (4)}\},$$

where $j_0^l \tilde{S}(x) = j^l \tilde{S}_x(0)$. Then O'_1 and O'_2 are residual sets. We set

$$Y = \{j^{n+2}S(0) \in S^{n+2}(n) \mid \text{the codimension of } [j^{n+2}S(0)]_L > 8\}.$$

Then Y is an algebraic set in $S^{n+2}(n)$ by [6, Theorem 6.6 (a')]. Therefore we can define

$$O'' = \{\tilde{S} \in S(U, T^*\mathbb{R}^n) \mid j_0^{n+2}\tilde{S} \text{ is transversal to } Y\}.$$

For any $S \in S(T^*\mathbb{R}^n, 0)$ with $j^{n+2}S(0) \in Y$ and any generating family F of $\pi \circ S|_L$, the function germ $F|_{q=0}$ has reticular \mathcal{R}^+ -codimension > 4 . This means that $F|_{q=0}$ has reticular C -codimension > 3 . It follows that $j^{n+2}S(0)$ does not belong to the above equivalence classes. Then Y has codimension > 6 because all elements in Y are adjacent to one of the list (4) which are caustic-simple. Then we have that

$$O'' = \{\tilde{S} \in S(U, T^*\mathbb{R}^n) \mid j_0^{n+2}\tilde{S}(U) \cap Y = \emptyset\}.$$

We define $O = O'_1 \cap O'_2 \cap O''$. Since all $\pi \circ i_X$ for X in (3) are weak caustic 1-determined, and all $\pi \circ i_X$ in (4) are caustic 5-determined by Lemma 3.2. Then O has the required condition. ■

$B_{2,2}^{*,*}$ in 2D

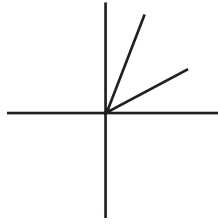


FIGURE 5. $B_{2,2}^{+,+,1}, B_{2,2}^{+,+,2}$

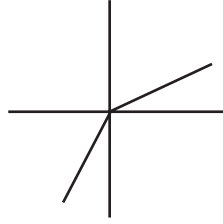


FIGURE 6. $B_{2,2}^{-,+,1}, B_{2,2}^{-,+,2}$

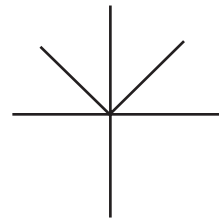


FIGURE 7. $B_{2,2}^{+,-}, B_{2,2}^{-,-}$

$B_{2,2}^{*,*}$ in 3D

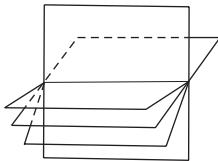


FIGURE 8. $B_{2,2}^{+,+,1}, B_{2,2}^{+,+,2}$

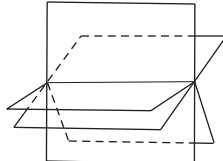


FIGURE 9. $B_{2,2}^{-,+,1}, B_{2,2}^{-,+,2}$

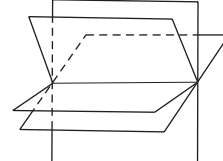


FIGURE 10. $B_{2,2}^{+,-}, B_{2,2}^{-,-}$

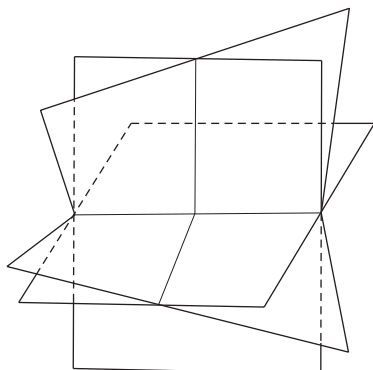


FIGURE 11. $B_{2,2}^{+,0}$

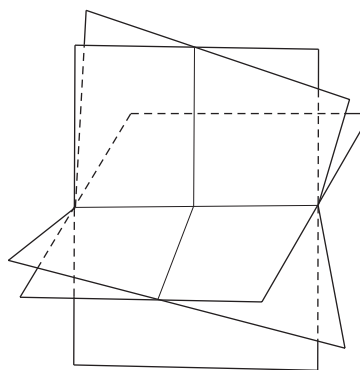


FIGURE 12. $B_{2,2}^{-,0}$

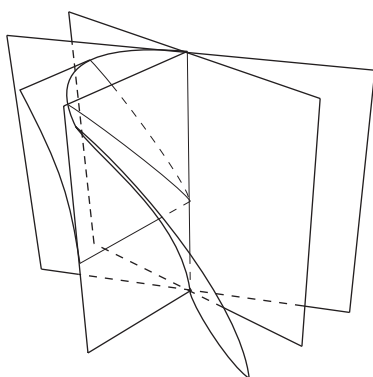


FIGURE 13. $B_{2,2,3}^{+,+}$

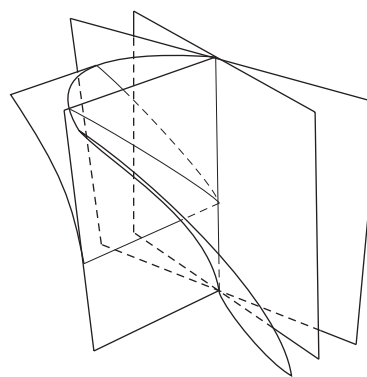


FIGURE 14. $B_{2,2,3}^{+,-}$

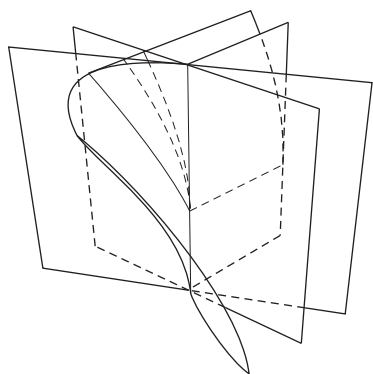


FIGURE 15. $B_{2,2,3}^{-,+}$

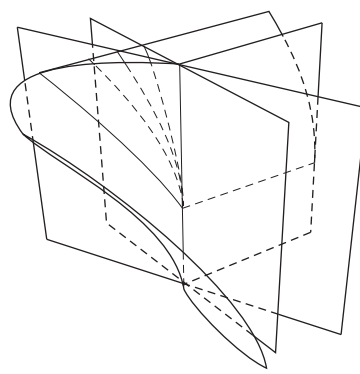
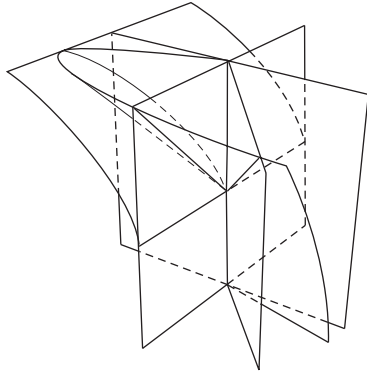
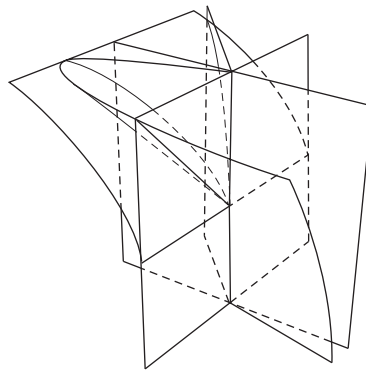
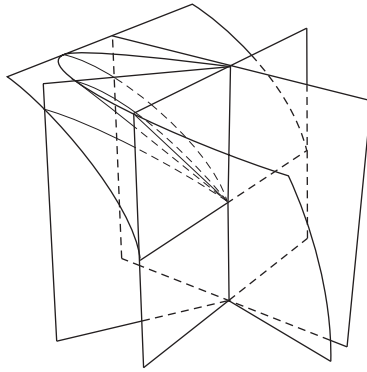
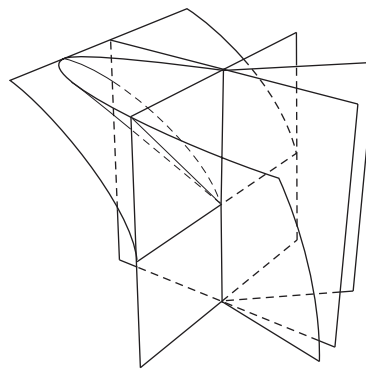


FIGURE 16. $B_{2,2,3}^{-,-}$

FIGURE 17. $B_{2,3}^{+,+}$ FIGURE 18. $B_{2,3}^{+,-}$ FIGURE 19. $B_{2,3}^{-,+}$ FIGURE 20. $B_{2,3}^{-,-}$

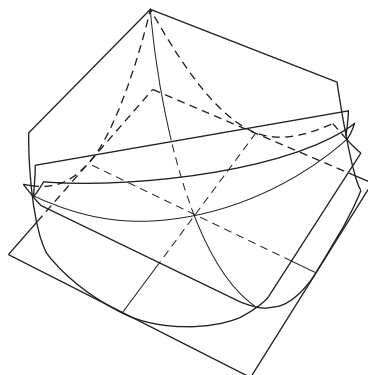


FIGURE 21. $C_{3,2}^{+,+}$

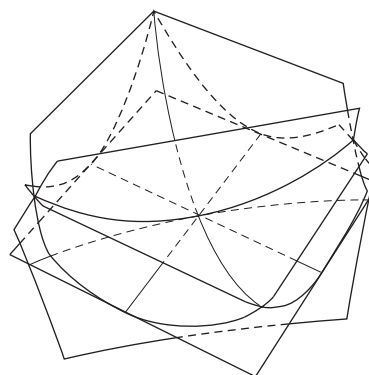


FIGURE 22. $C_{3,2}^{+,-}$

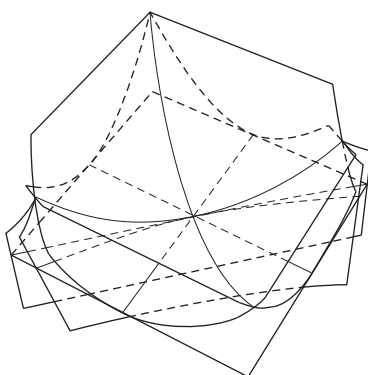


FIGURE 23. $C_{3,2}^{-,+}$

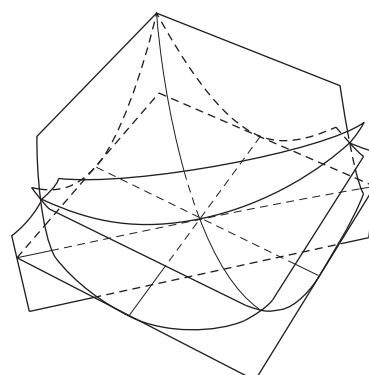


FIGURE 24. $C_{3,2}^{-,-}$

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