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WEBS AND SINGULARITIES

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Dedicated to Xavier Gómez-Mont on his 60th birthday

ABSTRACT. We investigate the singular web structure of first-order PDEs from the viewpoint of singularity theory. Most of the results given have already appeared in papers by others, as well as the author [28, 29, 30, 31, 32] in various different terminologies. The new results are the construction of mini-versal webs from the deformation of isolated singularities, and their classification. We prove also the existence of the resonance curve for generic 3-webs with cuspidal singular locus. We introduce also Klein-Halphen webs with polyhedral symmetry and Fermat webs, and we investigate their properties.

1. Versal web

Let f be a holomorphic function germ with an isolated singularity at $o \in \mathbb{C}^{n+1}$, i.e., $f(o) = 0$ and $V(\frac{\partial f}{\partial x_0}, \cdots, \frac{\partial f}{\partial x_n}) = \{o\}$, and let us consider the ideal quotients

$$A = \frac{\mathcal{O}_{\mathbb{C}^{n+1},o}}{\langle f,\ \frac{\partial f}{\partial x_1},\ \cdots\ \frac{\partial f}{\partial x_n}\rangle \mathcal{O}_{\mathbb{C}^{n+1},o}}, \qquad B = \frac{\mathcal{O}_{\mathbb{C}^{n+1},o}}{\langle \frac{\partial f}{\partial x_0}\rangle \mathcal{O}_{\mathbb{C},o} + \langle f,\ \frac{\partial f}{\partial x_1},\ \cdots\ \frac{\partial f}{\partial x_n}\rangle \mathcal{O}_{\mathbb{C}^{n+1},o}},$$

where $x_0, \ldots, x_n$ are Cartesian coordinates of $\mathbb{C}^{n+1}$ and $\mathcal{O}_{\mathbb{C},o}$ denotes the local ring of germs of holomorphic functions of x_0 at $0 \in \mathbb{C}$. Clearly, if A is finite over $\mathbb{C}$, then B is also finite over $\mathbb{C}$. On the variety $V' = V(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$, the coordinate x_0 vanishes identically as f has an isolated singularity. If B is finite over $\mathbb{C}$, then A is finite over $\mathcal{O}_{\mathbb{C},o}$; hence V' is isolated and in particular A is finite over $\mathbb{C}$. The dimension of A over $\mathbb{C}$ is called the G_0-*codimension* of f in this paper, and differs slightly from the codimension of the G_0-*equivalence* class of f defined in §4. In this note, a τ-*web structure* (locally a configuration of τ codimension-one foliations) is introduced on A, $\tau = \dim_{\mathbb{C}} A$, and its various properties are investigated.

The quotient A was studied by Teissier [36] from the viewpoint of polar variety, and by Gomez Mont from the viewpoint of Euler obstruction, and also used to compute the dimension of the space of logarithmic vector fields of $f = 0$ (see e.g. [34]). The deformation theory with respect to B was investigated by Goryunov [15], and reviewed in detail in the book [5]. A link of the theory of projection of hypersurface singularities with respect to some intermediate quotient and the classification of first-order ordinary differential equations (ODEs) was investigated by Izumiya, Takahashi et al. (see e.g., [21]). From the viewpoint of the *equivalence problem* for ODEs, the web structure of families of solutions was initiated by Cartan and his followers [11, 14]. This short note is devoted to recollecting various links among these old and new subjects from the viewpoint of web geometry, and also emphasizing the role of versal web structure in the classification problem.

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Let $\{g_1, \ldots, g_\tau\}$ be a $\mathbb{C}$-basis of A, $\tau = \dim_{\mathbb{C}} A$. Consider the linear deformation $V_t : f_t = 0 \subset \mathbb{C}^{n+1}$, $f_t = f + t_1 g_1 + \cdots + t_\tau g_\tau$ and the divergent diagram

$$\mathbb{C} \xleftarrow{\lambda = x_0} V = \cup V_t \xrightarrow{t} \mathbb{C}^\tau .$$

The surface V_t is smooth for a generic t, on which λ has τ critical points of Morse-type. The above diagram may be regarded as a deformation of the projection of hypersurface singularities with parameter $t \in \mathbb{C}^\tau$,

$$\mathbb{C} \xleftarrow{\lambda = x_0} V_t \subset \mathbb{C}^{n+1} .$$

Let D denote the τ-valued function on $\mathbb{C}^\tau$ that assigns these critical values to a t. The *mini-versal τ-web* $W_{\{f_t\}}$ on $\mathbb{C}^\tau$ is defined to be the codimension-one "foliation" by the level hypersurfaces of D. Specifically, $W_{\{f_t\}}$ is locally a configuration of τ foliations, i.e., a τ-web, as D is τ-valued.

It was shown [28] that the web thus constructed is non-singular at a generic point; in other words, the τ-tuple of critical values (*critical-value-map*) $D' = (d_1, \ldots, d_\tau)$ is a local diffeomorphism of $\mathbb{C}^\tau$. This is a consequence of the versality in Theorem 4.1 and the existence of a deformation of f_t with an additional parameter $t' \in \mathbb{C}^{\tau'}$ for which the critical-value-map, defined on the extended parameter space $\mathbb{C}^{\tau+\tau'}$, is submersive at the generic point. By the symmetry quotient, D' induces a map (*classifying map*) $\tilde{D} : \mathbb{C}^\tau \to \mathbb{C}^\tau / S_\tau = \mathbb{C}^\tau$, branched over the discriminant set in the quotient. Thus, by the argument given by Looijenga [25], if the classifying map is proper and finite-to-one, the non-singular locus, i.e., the set of those t for which the critical values are all distinct and D' is a local diffeomorphism, possesses the $K(\pi, 1)$-property (see also [28]). Interestingly, this property was first proved for simple (G-simple) function germs in the weak equivalence relation with respect to B in [15]. The above τ-web is defined also for an arbitrary non-linear deformation of $f = 0$ in the same manner, though a deformation can be linearized if it is *versal*, i.e., $\partial f_t / \partial t$ generates A.

Another feature of the versal web is that all "leaves" are diffeomorphic to a discriminant of a Thom-Mather stable map germ [28]. In other words, a versal web is a complex one-parameter family of discriminant hypersurfaces. A versal deformation with the smallest number (i.e. τ) of parameters is called a *mini-versal* deformation (see also §3).

Brunella [9] investigated the various "real" one-parameter subfamilies of complex one-parameter families of hypersurfaces from the viewpoint of singular Levi-flat surfaces, and called them "*tissus microlocaux*".

2. An example

Let $n = 1$ and $f = x_1^2 + x_0^3$ for simplicity. Then the Jacobian quotients A, B are generated by $1, x_0, x_0^2$ and by $1, x_0$ respectively over $\mathbb{C}$. Our mini-versal linear deformation (with respect to A) with parameter $t = (t_1, t_2, t_3) \in \mathbb{C}^3$ is

$$\mathbb{C}_{x_0} \xleftarrow{x_0} V = \{x_1^2 + x_0^3 + t_1 x_0^2 + t_2 x_0 + t_3 = 0\} \xrightarrow{\pi} \mathbb{C}_t^3.$$

The restriction of x_0 to a fiber over a generic $t \in \mathbb{C}_t^3$

$$x_0 : \mathbb{C}_{x_0} \leftarrow V_t = \{f_t = x_1^2 + x_0^3 + t_1 x_0^2 + t_2 x_0 + t_3 = 0\} \subset \mathbb{C}^5_{x_0 x_1 t_1 t_2 t_3}$$

has three Morse-type singularities. The critical values of the restriction are the solutions in x_0 of $x_0^3 + t_1 x_0^2 + t_2 x_0 + t_3 = 0$. Thus the classifying map $\tilde{D} : \mathbb{C}_t^3 \to \mathbb{C}^3_{XYZ} = \mathbb{C}^3_{xyz}/S_3$ is the diffeomorphism $(X, Y, Z) = (\Sigma_1, \Sigma_2, \Sigma_3) = (-t_1, t_2, -t_3)$, where $\Sigma_1(x, y, z) = x + y + z$, $\Sigma_2(x, y, z) = xy + yz + zx$, and $\Sigma_3(x, y, z) = xyz$. The coordinate foliations by x, y, z on $\mathbb{C}^3_{xyz}$ induce the singular 3-web on the quotient $\mathbb{C}^3_{XYZ}$, which induces the mini-versal web $W_{\{f_t\}}$ on $\mathbb{C}_t^3$ via $\tilde{D}$.

The leaves (*solutions*) satisfy the following implicit first-order PDE

$$\begin{cases} p^3 + (t_1^2 - 2t_2)q^2 + (t_2^2 - 2t_1t_3)p - t_3^2 & = 0 \\ q^3 - t_1q^2 + t_2q - t_3 & = 0 \ , \end{cases}$$

where $p = \frac{\partial t_3}{\partial t_1}$ and $q = \frac{\partial t_3}{\partial t_2}$. The figure on the left in Figure 2 is the cross-section of $W_{\{f_t\}}$ by $T = \mathbb{C}^2_{t_2t_3} : t_1 = 0$. The leaves on T are the solutions of the Clairaut equation

$$q^3 + t_2q - t_3 = 0,$$

in the coordinates t_2, t_3 (c.f. the equation E_1 in §7). This construction of the 3-web by quotient is generalized in §6, 7 and 8. The right figure in Figure 1 is a cross-section of the mini-versal web by a generic $T' : t_1 = s(t_2, t_3)$, where the hexagonal structure is violated so that the closed hexagon with concurrent 3 diagonal curves (Brianchon hexagon, the figure on the left in Figure 1) can not be embedded in a small shape. By this non-embeddability, the *hexagonal web* (for which all hexagons are closed, see §5,) is distinguished from the other apparently similar but non-hexagonal webs. The s is called the *function moduli* following the Russian school. Basically, the same cuspidal web structure was first investigated by Arnold [3, 5], and also independently by Carneiro [10] and Dufour [13] from the viewpoint of web geometry. By a suitable leaf preserving diffeomorphism of the mini-versal web on $\mathbb{C}^3_t$, the hypersurface T' can be transformed so that $s = 0$ on the cuspidal singular locus. The diffeo-type of induced web structure on the plane is then determined by the equivalence class of such an s by the weighted $\mathbb{C}^*$-automorphisms respecting the cusp, which is formally in one-to-one correspondence with an equivalence class of a 2-form on the plane by the same $\mathbb{C}^*$-automorphisms via the web curvature 2-form introduced in §5 [29].

In general, the codimension-one τ-webs with regular first integrals on $\mathbb{C}^s$, $s \leq \tau$, (see the next section for the definition) are described by codimension-($\tau - s$) sections, or pullbacks by maps from the s-space, of a mini-versal τ-web on $\mathbb{C}^\tau$ by the versality theorem (Theorem 4.1).

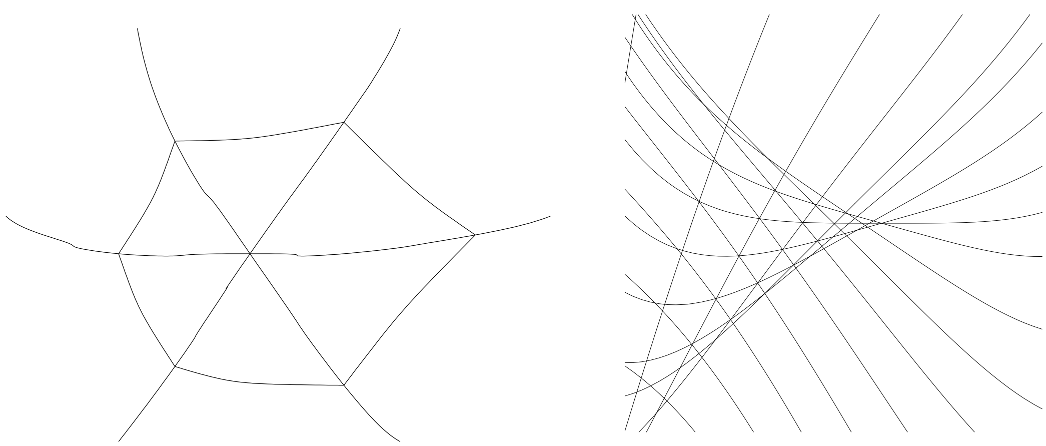

FIGURE 1. Left: a closed hexagon in a 3-web; Right: a non-hexagonal Clairaut 3-web

3. Legendre construction

Let $S^\tau \subset J^1(\mathbb{C}^{\tau-1}_q, \mathbb{C}_{t_\tau}) = \mathbb{C}^{\tau-1}_q \times \mathbb{C}^{\tau-1}_p \times \mathbb{C}_{t_\tau}$, $q = (q_1, \ldots, q_{\tau-1})$, be a germ of a submanifold of dimension τ in the first jet space such that the projection $\pi = \pi_q \times \pi_{t_\tau}$ to the base space (*observation space*) $\mathbb{C}_q^{\tau-1} \times \mathbb{C}_{t_\tau}$ is d-to-one. Assume the canonical contact form $\theta = dt_\tau - \sum p_i dq_i$ of the jet space is integrable on S. Then the push-forward $\pi_*\theta|_S$ defines a d-web on the base space, which we denote by W_S and call S the *skeleton* of W_S. A *first integral* of W_S is a function λ on the skeleton S such that $d\lambda \wedge \theta$ vanishes identically. A *solution* in $\mathbb{C}_q^{\tau-1} \times \mathbb{C}_{t_\tau}$ is an image of the projection of a level surface $\lambda = c$, which we denote by S_c. The family of these solutions $\{S_c\}$ constitutes the web W_S. If λ is non-singular, then $\alpha\ d\lambda = \theta$ with a holomorphic function α on S, thus

$$dt_\tau - \big(\alpha d\lambda + \sum p_i dq_i\big) = 0 \quad \text{ on } S.$$

This equation describes the canonical contact form on the extended jet space $J^1(\mathbb{C}_{x_0} \times \mathbb{C}_q^{\tau-1}, \mathbb{C}_{t_\tau})$ vanishes identically on the image of

$$(\lambda, \pi_q, \alpha, \pi_p, \pi_{t_\tau}) : S^\tau \to (\mathbb{C}_{x_0} \times \mathbb{C}_q^{\tau-1}) \times (\mathbb{C}_{p_0} \times \mathbb{C}_p^{\tau-1}) \times \mathbb{C}_{t_\tau} = J^1(\mathbb{C}_{x_0} \times \mathbb{C}_q^{\tau-1}, \mathbb{C}_{t_\tau}),$$

where π_p denotes the p-coordinates on $S \subset J^1(\mathbb{C}_q^{\tau-1}, \mathbb{C}_{t_\tau})$. Thus the image is a *Legendre sub manifold.*

By a well-known result attributed to Hörmander and Arnold [4], the image of the above inclusion is represented by a Nash blow-up by the tangent hyperplane of a discriminant

$$D(F) \subset \mathbb{C}_{x_0} \times \mathbb{C}_{\tilde{t}}^{\tau-1} \times \mathbb{C}_{t_\tau}$$

of an unfolding of a function on a $\mathbb{C}_x^n$, $F = (t', f_{t'}) : \mathbb{C}_{t'}^\tau \times \mathbb{C}_x^n \to \mathbb{C}_{t'}^\tau \times \mathbb{C}$ by the slopes $p_1, \ldots, p_\tau$ of the critical values of $f_{t'}$ with respect to the parameter $t' = (x_0, \tilde{t}) = (x_0, t_1, \ldots, t_{\tau-1}) \in \mathbb{C}^\tau$. Thus we may suppose

$$(\lambda, \pi_q, \alpha, \pi_p, \pi_{t_\tau}) = (x_0, t_1, \ldots, t_{\tau-1}, \frac{\partial f_{t'}}{\partial x_0}, \frac{\partial f_{t'}}{\partial t_1}, \ldots, \frac{\partial f_{t'}}{\partial t_{\tau-1}}, f_{t'})$$

identifying S and the critical point set $\Sigma(F)$ of F. (The unfolding is possibly nonlinear. The critical locus $\Sigma(F)$ is smooth.) By this identification, the solution S_c is the image of

$$\tilde{F} = (t_1, \ldots, t_{\tau-1}, f_{t'}) : \Sigma(F) \cap \{x_0 = c\} = \Sigma(F|x_0 = c) \to \mathbb{C}_{\tilde{t}}^{\tau-1} \times \mathbb{C}_{t_\tau},$$

which is the discriminant locus of the restriction of $\tilde{F}$ to $x_0 = c$. Let us consider the diagram

$$\mathbb{C} \xleftarrow{\ x_0\ } \mathbb{C}^{\tau+n} \xrightarrow{\tilde{F}=(t_1,\ldots,t_{\tau-1},f_{t'})} \mathbb{C}_{\tilde{t}}^{\tau-1} \times \mathbb{C}_{t_\tau}\ .$$

Here we have the following *Legendre duality*

$$c \in D(x_0|\tilde{F}^{-1}(t_1 \ldots, t_\tau)) \Longleftrightarrow (t_1 \ldots, t_\tau) \in S_c = D(\tilde{F}|x_0 = c)\ .$$

Thus the solution web W_S on the right space $\mathbb{C}_{\tilde{t}}^{\tau-1} \times \mathbb{C}_{t_\tau} = \mathbb{C}_q^{\tau-1} \times \mathbb{C}_{t_\tau}$ in the above divergent diagram is the *Legendre transformation* of the codimension-1 foliation of the left $\mathbb{C}$ by points.

Let

$$f_t = f_{(*,t_1,\ldots,t_{\tau-1})}(*)\ -\ t_\tau \in \mathcal{O}_{\mathbb{C}_{x_0} \times \mathbb{C}_x^n, o}\ ,$$

where $t = (t_1, \ldots, t_{\tau-1}, t_\tau) \in \mathbb{C}_{\tilde{t}}^{\tau-1} \times \mathbb{C}_{t_\tau}$. Then $W_S = W_{\{f_t\}}$. The following theorem is stated in [28] and also found in the paper by Hayakawa, et al [17] in an apparently different form.

Theorem 3.1. *Let $S \subset J^1(\mathbb{C}^{\tau-1}, \mathbb{C})$ be a germ of a submanifold at a (o, p, o) such that the projection to $\mathbb{C}^{\tau-1} \times \mathbb{C}$ is d-to-1 and the fiber over the origin is (o, p, o). Assume there exists a regular (nonsingular) first integral on S. Then there exists a family of functions $f_t \in \mathcal{O}_{\mathbb{C}^{n+1}, o}$, $n > 0$, with parameter $t \in \mathbb{C}^\tau$ such that $W_S = W_{\{f_t\}}$.*

The deformation $\{f_t\}$ above constructed is unique up to *stable equivalence* [5], which is called the *generating function* of the web with a first integral W_S. If $\{f_t\}$ is versal, as will be described in the next section, the restriction $F_c = \tilde{F} : \{x_0 = c\} \to \mathbb{C}^\tau$ is stable in the sense of Thom-Mather theory, and the solutions $S_c, c \in \mathbb{C}$, are all diffeomorphic—trivial by an ambient isotopy. The family $\{F_c\}$ is also *stable as a family* [28].

4. Versality

Let us consider the group G of triples (ϕ, ψ, h), where ϕ and ψ are respectively germs of diffeomorphisms of $\mathbb{C}^{n+1}, o$ and the x_0-line $\mathbb{C}, o$ compatible via the projection of $\mathbb{C}^{n+1}$ onto the first x_0-factor $\mathbb{C}$, and h is a function germ on $\mathbb{C}^{n+1}, o$ with $h(o) \neq 0$. The *product* $\circ$ on G is defined by

$$(\phi, \psi, h) \circ (\phi', \psi', h') = (\phi \circ \phi', \psi \circ \psi', \phi'^* h \times h').$$

Let $G_0 \subset G$ be the normal subgroup of triples (ϕ, id, h). These groups act on the local ring $\mathcal{O}_{\mathbb{C}^{n+1},o}$ of function germs defined at $o \in \mathbb{C}^{n+1}$ from the right by $f \cdot (\phi, \psi, h) = h \cdot \phi^* f$.

Two function germs f, g at $o \in \mathbb{C}^{n+1}$ are *G-equivalent* (respectively *G_0-equivalent*) if they lay on a common G-(resp. G_0-)orbit. Note that function germs non-vanishing at o are all G_0-equivalent, and hence G-equivalent. These notions are variants of the contact equivalence stated in terms of fibered diffeomorphisms of $\mathbb{C}^{n+1}$ over $\mathbb{C}$ and found in various articles (see cf. [15, 35]). For instance, G-equivalence is called *parameterized contact equivalence* in some papers. However the deformation theory was developed mostly involving an intermediate equivalence relation, for instance, the $\mathbb{R}^+$-equivalence, which has resulted in some confusion in terminology. Some incorrect conclusions were drawn, for instance, the "classification" of webs with two functional moduli in [35]. This matter will be explained in Example 1.

The present author investigated the theory in the latter equivalence relation and arrived at a natural versality notion suitable for classifying singular τ-webs with first integrals [28]. It is worth recalling the theory developed and restating it in a common language in terms of deformation of singularities. Theorem 4.3 was stated in [28] in an alternative form; its proof is for the first given in the present paper.

G_0-orbits are contained in G-orbits by definition. Because $G_0 \subset G$ is a normal subgroup, all G-orbits are foliated by G_0-orbits (possibly of codimension 0) in a finite jet level, and a triple (ϕ, ψ, h) in G sends a G_0-orbit to a G_0-orbit, thus it leaves each G-orbit invariant respecting the foliation by G_0-orbits. Here we can build two deformation theories with respect to G_0- and G-equivalence relations. If an f is G_0-simple, i.e., there exist only finitely many G_0-orbits on a sufficiently small neighborhood of f, then it is also G-simple and the G_0-orbit locally coincides with its G-orbit, hence the two theories coincide.

The above actions induce those of the Lie groups of their k-jets, denoted G_0^k, G^k, on the k-jet space of function germs $J^k(\mathbb{C}^{n+1}, o) = \mathcal{O}_{\mathbb{C}^{n+1},o}/m^{k+1}_{\mathbb{C}^{n+1},o}$, where $m_{\mathbb{C}^{n+1},o} \subset \mathcal{O}_{\mathbb{C}^{n+1},o}$ denotes the maximal ideal consisting of function germs vanishing at the origin. Of course, the tangent space of the G_0^k-orbit $\mathcal{O}_{G_0^k}(J^k f(o))$ as well as the G^k-orbit $\mathcal{O}_{G^k}(J^k f(o))$ at a k-jet $J^k f(0)$ is spanned by the tangent lines of the action of one-parameter subgroups generated by vector fields in a suitable form. The respective normal spaces of these orbits are presented as ideal quotients

$$N_{J^k f(0)} \mathcal{O}_{G_0^k}(J^k f(o)) = \frac{\mathcal{O}_{\mathbb{C}^{n+1},o}}{\langle f \rangle_{\mathcal{O}_{\mathbb{C}^{n+1},o}} + \langle \frac{\partial f}{\partial x_1}, \cdots \frac{\partial f}{\partial x_n} \rangle_{m_{\mathbb{C}^{n+1},o}} + m^{k+1}_{\mathbb{C}^{n+1},o}},$$

$$N_{J^k f(o)} \mathcal{O}_{G^k}(J^k f(o)) = \frac{\mathcal{O}_{\mathbb{C}^{n+1},o}}{\langle f \rangle_{\mathcal{O}_{\mathbb{C}^{n+1},o}} + \langle \frac{\partial f}{\partial x_0} \rangle_{m_{\mathbb{C},o}} + \langle \frac{\partial f}{\partial x_1}, \cdots \frac{\partial f}{\partial x_n} \rangle_{m_{\mathbb{C}^{n+1},o}} + m^{k+1}_{\mathbb{C}^{n+1},o}}.$$

Assume $\mathbb{C}$-dimensions of the Jacobian quotients A, B are finite. Then the ideals in the denominators are decreasing and stabilized as $k \to \infty$, hence the $\mathbb{C}$-dimensions of the quotients converge to certain limits, which are invariant under the G-equivalence relation. We say a deformation $f_t \in \mathcal{O}_{\mathbb{C}^{n+1},o}$ with parameter $t \in \mathbb{C}^r$ (possibly nonlinear in t) is *versal* (more precisely G_0-*versal*) if the differential $\delta : \mathbb{C}^r \to A$, $\delta(t) = \sum t_i \frac{\partial f_t}{\partial t_i}|_{t=o} \in A$ is surjective, and *mini-versal* if δ is an isomorphism.

It is not difficult to see $f = \{f_t\}$ is versal if and only if

$$\frac{\partial f_0}{\partial x_i}, \quad i = 1, \dots, n, \qquad \frac{\partial f_t}{\partial t_j}|_{t=0}, \quad j = 1, \dots, r$$

generate the normal space of the $\mathcal{O}_{G_0^k}$-orbit over $\mathbb{C}$ for any sufficiently large k; in other words, the k-jet section $J^k f : \mathbb{C}^{n+r} \to J^k(\mathbb{C}^{n+1}, o)$ defined by

$$J^k f(a,t) = \text{“}k\text{-jet of } f_t(x + (0,a)) \text{ at } x = o \in \mathbb{C}^{n+1}\text{”}$$

is transverse to the G_0^k-orbit at $o \in \mathbb{C}^{n+r}$.

We say a d-web W_S with a smooth skeleton S and a non-singular first integral is *versal* (respectively *mini-versal*) if its generating function $\{f_t\}$, constructed in the previous section, is versal (resp. mini-versal) in the above sense.

Given a generic map μ of $\mathbb{C}^\sigma$ to a parameter space $\mathbb{C}^\tau$ of a deformation $\{f_t\}, t \in \mathbb{C}^r$, the pullback deformation $\{f_{\mu(s)}\}$, $s \in \mathbb{C}^\sigma$, defines a singular d-web on $\mathbb{C}^\sigma$. This does not require the transversality of μ to the leaves of the web $W_{\{f_t\}}$. We denote the pullback web by $\mu^* W_{\{f_t\}}$ or $W_{\mu^*\{f_t\}} = W_{\{f_{\mu(s)}\}}$ and call μ the *classifying map*. The pullback $\mu^* W_{\{f_t\}}$ has the natural first integral induced from $W_{\{f_t\}}$, which is tautologically defined assigning the critical values of $f_{\mu(s)}$ to $s \in \mathbb{C}^\sigma$ and lifting it to the skeleton in a natural manner.

Theorem 4.1. *A germ of a smooth codimension-one τ-web on $\mathbb{C}^\sigma$ with a smooth skeleton, a regular first integral, and a finite G_0-codimension is a pullback of a mini-versal τ-web on $\mathbb{C}^\tau$ via a map germ $\mu : \mathbb{C}^\sigma, o \to \mathbb{C}^\tau, o$.*

This theorem follows immediately from the next theorem and its corollary in the deformation theory.

Theorem 4.2. *Let $f \in \mathcal{O}_{\mathbb{C}^{n+1},o}$ be an isolated singularity and assume f has a finite G_0-codimension, or equivalently, $\dim_{\mathbb{C}} A$ is finite. Let $f_t, g_t, t \in \mathbb{C}^\tau$, be versal deformations of an equal $f = f_o = g_o$. Then there exist a germ of diffeomorphism χ of $\mathbb{C}^\tau, o$ and a family of diffeomorphisms ϕ_t of $\mathbb{C}^{n+1}, o$ leaving the first coordinate x_0 invariant such that*

$$\phi_t(\{g_t = 0\}) = \{f_{\chi(t)} = 0\}, \quad t \in \mathbb{C}^\tau.$$

In particular, $\chi^* W_{\{f_t\}} = W_{\{g_t\}}$.

The theorem states in particular the mini-versal web $W_{\{f_t\}}$ is determined by f_o up to diffeomorphism. The proof is routine in Thom-Mather theory, being based on first constructing the one-parameter family of versal deformations joining $\{f_t\}$ to $\{g_t\}$, and second the trivialization of the family. A detailed proof is found in [28] in a different terminology.

Corollary 4.1. *Let $f \in \mathcal{O}_{\mathbb{C}^{n+1},o}$ be an isolated singularity with a finite G_0-codimension, and let $\dim_{\mathbb{C}} A = \tau$. Let $f_t, t \in \mathbb{C}^\tau$, $g_s, s \in \mathbb{C}^\sigma$ be deformations of $f_o = g_o = f$. Assume $\{f_t\}$ is versal. Then there exists a map germ $\mu : \mathbb{C}^\sigma, o \to \mathbb{C}^\tau, o$ such that*

$$W_{\{g_s\}} = \mu^* W_{\{f_t\}}.$$

Proof. Let $\{g_{rs}\}$ be a versal deformation with $g_{os} = g_s$ with an additional parameter $r \in \mathbb{C}^\rho$, and put $f_{ut} = f_t$ for $u \in \mathbb{C}^{\rho+\sigma-\tau}$. Here we suppose $\rho + \sigma - \tau \geq 0$ choosing a large ρ. By Theorem 4.2, these versal deformation with an equal dimension of parameters $\{f_{ut}\}$, $\{g_{rs}\}$ are equivalent, thus there exists a diffeomorphism χ of $\mathbb{C}^{\rho+\sigma}, 0$ such that $W_{\{g_{rs}\}} = \chi^* W_{\{f_{ut}\}}$. Let $i : \mathbb{C}^\sigma \to \mathbb{C}^{\rho+\sigma}$ be the natural embedding and $\pi : \mathbb{C}^{\rho+\tau} \to \mathbb{C}^\tau$ be the natural projection. Then $W_{\{g_s\}} = i^* W_{\{g_{rs}\}}$ and $W_{\{f_{ut}\}} = \pi^* W_{\{f_t\}}$. Therefore $W_{\{g_s\}} = (\pi \circ \chi \circ i)^* W_{\{f_t\}}$. □

Example 1. *Let us consider the versal web $W_{\{f_t\}}$ constructed in §2. The graph of the critical values of x_0 on $f_{t_1t_2t_3} = x_1^2 + x_0^3 + t_1x_0^2 + t_2x_0 + t_3 = 0 \subset \mathbb{C}^2_{x_0x_1}$ is the set*

$$\{x_0^3 + t_1x_0^2 + t_2x_0 + t_3 = 0\} \subset \mathbb{C}^4_{x_0t_1t_2t_3}$$

and the versal web on $\mathbb{C}^3_{t_1t_2t_3}$ is given by the family of x_0-level sets. A 3-web $W_{\{g_{uv}\}}$ on the plane $\mathbb{C}^2_{uv}$ with $g_o = f_o = x_1^2 + x_0^3$ is of the form $\mu^* W_{\{f_t\}}$ induced from the versal web by Theorem 4.1. A generic embedding $\mu : \mathbb{C}^2_{uv} \to \mathbb{C}^3_{t_1t_2t_3}$ can be presented as $\mu(u, v) = (\alpha(u, v), u, v)$. Thus the critical-value-graph of the induced web $W_{\{g_{uv}\}} = \mu^* W_{\{f_t\}}$ is presented as

$$\mathbb{C} \xleftarrow{x_0} \{x_0^3 + \alpha(u, v)x_0^2 + ux_0 + v = 0\} \subset \mathbb{C}^3_{x_0uv} \xrightarrow{\pi} \mathbb{C}^2_{uv}.$$

Here the graph in the middle is smooth with coordinates (u, x_0); the first integral is fixed to x_0, while the second projection π varies as the embedding μ varies. If we normalize the projection π to the Whitney cusp map, we obtain the normal form of 3-webs with the cuspidal singular locus due to Carneiro [10] *and Dufour* [13]*. Similarly, a generic 4-web on $\mathbb{C}^2_{uv}$ with $g_o = x_1^2 + x_0^4$ is given by*

$$\mathbb{C} \xleftarrow{x_0} \{x_0^4 + \alpha(u, v)x_0^3 + \beta(u, v)x_0^2 + ux_0 + v = 0\} \subset \mathbb{C}^3_{x_0uv} \xrightarrow{\pi} \mathbb{C}^2_{uv}.$$

These α and β are function moduli of the web structures. This normal form conflicts with a classification result in [35]*, and waits for a better explanation.*

By Theorem 4.2, a deformation of a versal web is trivial, i.e., equivalent to a trivial family of the versal web. In particular, a versal web is a cylinder of a mini-versal web. Thus the complement of the discriminant locus of a versal web possesses the $K(\pi, 1)$-property, where the fundamental group π is the braid subgroup of τ strings given by the τ critical values of the first integral x_0 on $V_t : f_t = 0$.

The following theorem is remarkable as it reduces the classification of versal webs to that of functions on varieties by G-equivalence, which is weaker than G_0-equivalence.

Theorem 4.3. *Let $f, g \in \mathcal{O}_{\mathbb{C}^{n+1},o}$ be G-equivalent. Then f, g have an equal G_0-codimension and their mini-versal webs $W_{\{f_t\}}$ and $W_{\{g_t\}}$ are diffeomorphic.*

Proof. Assume f, g are equivalent by a triple (ϕ, ψ, h): $h\phi^* f = g$. By a straight forward calculation,

$$\frac{\partial g}{\partial x_i} = \frac{\partial h\phi^* f}{\partial x_i} = \frac{\partial h}{\partial x_i}\phi^* f + h \sum_{j=1,\ldots,n} \frac{\partial \phi_j}{\partial x_i} \phi^* \frac{\partial f}{\partial x_j}.$$

This states that ϕ^* sends the $\mathcal{O}_{\mathbb{C}^{n+1},o}$-submodule $< f > + J_0 f$ to $< g > + J_0 g$, where $< g > + J_0 g$ stands for the denominator of the quotient A for g. If $\{s_1, \ldots, s_\tau\}$ is a $\mathbb{C}$-basis of the Jacobian quotient A of f, its pullback $\{\phi^* s_1, \ldots, \phi^* s_\tau\}$ is a $\mathbb{C}$-basis of the quotient for g. Thus

$$g_t = g + \sum t_i \phi^* s_i = h\phi^* f + \sum t_i \phi^* s_i = \phi^*(h' f + \sum t_i s_i)$$

is a mini-versal deformation of g, where h' is a unit. If we write $f'_t = h' f + \sum t_i s_i$, then the solution $S_c \in W_{\{g_t\}}$ coincides with the solution $S_{\psi(c)} \in W_{\{f'_t\}}$. Thus $W_{\{g_t\}} = W_{\{f'_t\}}$. By a similar calculation, we have also $< h' f > + J_0 h' f = < f > + J_0 f$ as h' is a unit. Thus $\{f'_t\}$ is versal, and $\{f''_t = h' f + \sum t_i h' s_i\}$ is also versal as h' is a unit. Thus $W_{\{f'_t\}}$ and $W_{\{f''_t\}}$ are diffeomorphic by Theorem 4.2. As $f''_t = h' f_t$, we obtain $W_{\{f''_t\}} = W_{\{f_t\}}$. This completes the proof. $\square$

A result of Matsuoka [26] asserts that functions on a variety are classified by associated homomorphisms of $\mathbb{C}$-algebras.

5. Affine Connection of 3-webs on the plane

A *complex first-order ODE of one valuable*, local in $p = dy/dx$, is

$$f(x, y, p) = 0 \tag{$*$}$$

where f is a germ of complex analytic function at a $(0, 0, p_0)$. In suitable coordinates one may assume $p_0 = 0$. The *skeleton* $S \in J^1(\mathbb{C}, \mathbb{C}) = \mathbb{C}_x \times \mathbb{C}_y \times \mathbb{C}_p$ is defined by $f = 0$. From now on we do not assume S to be smooth.

To obtain the solutions of $(*)$, we assume the equation is locally solved in p as

$$p = f_i(x, y)\ ,\ \ i = 1, \dots, d$$

on a domain nearby the origin with implicit functions f_i. The solutions of each explicit differential equation as above form a germ of curvilinear foliations, hence the entire family of solutions of $(*)$ form a configuration of d foliations, i.e., a d-web. We recall some classic results obtained by Cartan and Blaschke (for details, see e.g. [11, 7, 8, 12]).

One of the basic ideas to extract geometric invariants from a web is to extend the Bott connection (parallel translation of normal vectors along leaves) of its constituent foliations (if possible) to an equal affine connection ∇ of the xy-plane. For $d = 3$, such a connection exists and called the *Chern connection*. This connection is defined on the complement of the discriminant of the equation (in p), and extends meromorphically to the discriminant [4]. The singularity of the connection depends subtly on that of the equation in general. Hence one may expect to classify the equations in terms of connection. Indeed, the transverse sections of the mini-versal 3-webs $W_{\{f_t\}}$, $f_o = x_1^2 + x_0^3$, in §2 are classified by their curvature 2-forms [29].

To introduce such a common affine connection, let us consider

$$\omega_i = U_i\ (dy - f_i\ dx), \quad i = 1, \dots, d$$

with units $U_i \neq 0$. In the simplest non-trivial case $d = 3$, one may impose the *normalization condition*

$$\omega_1 + \omega_2 + \omega_3 = 0\ .$$

Then it is seen that there exists a unique θ such that

$$d \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta \end{bmatrix} \wedge \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + (T = 0) \tag{$**$}$$

The i-th row of the equation

$$d\omega_i = \theta \wedge \omega_i$$

is just the integrability condition for ω_i. Omitting the i-th row for any i, the equation $(**)$ is regarded as the structure equation for an affine connection without torsion. By the above normalization condition, the resulting connection is independent of the choice of i and U_1, U_2, and U_3. The affine connection thus defined is called the *Chern connection* of the 3-web of ω_1, ω_2, and ω_3; it has connection form $\Theta = \theta I$ and curvature form

$$\Omega = d\Theta + \Theta \wedge \Theta = d\theta\ I$$

where I stands for the 2×2 identity matrix. The curvature form is independent of the choice of co-frame because it is a similarity matrix. The $d\theta$ is called the *web curvature* by Blaschke [8]. It is not difficult to see that if $\omega_1 = -f_x dx, \omega_2 = -f_y dy$ and $\omega_3 = df$, then

$$d\theta = \frac{\partial^2}{\partial x \partial y} \log \frac{f_x}{f_y} dx \wedge dy.$$

The skeleton S is locally identified with the xy-plane via the natural projection. The above method is generalized to define an affine connection on (the smooth part of) S, which is an extension of the Bott connection of the foliation given by the contact form for any $d \geq 3$.

The following proposition was proved by Lins Neto and the author [24].

Proposition 5.1. *Assume the natural projection of a germ of skeleton $f(x, y, p) = 0$ to xy-plane has multiplicity d. Then f is equivalent to the polynomial equation of degree d in p,*

$$p^d + B_2\ p^{d-2} + B_3 p^{d-3} + \cdots + B_d = 0$$

where B_i are germs of analytic functions of x, y.

This is seen simply by changing the coordinate y with a first integral of the mean slope equation $y' = -B_1/d$ for a polynomial $f(x, y, p) = p^d + B_1 p^{d-1} + \cdots + B_d$. The general case is reduced to the polynomial case by the Weierstrass preparation theorem.

For $d = 3$, our normal form is

$$p^3 + B\ p + C = 0. \qquad (***)$$

Mignard[7] calculated the curvature form of 3-webs given by ODEs without this normalization and using computer produced a large formula. Henaut [6] provided insight into the web curvature form from the D-module theory. The following curvature form for the above normal form was presented by Lins Neto and the author in [24, 32].

Theorem 5.1. The web curvature form $d\theta$ of the normal form $(***)$ is

$$\frac{1}{6}\ (\log \Delta)_{xy}\ dx \wedge dy + d \left\{ \frac{(6BB_yC - 4B^2C_y)\ dx + (6BC_x - 9B_xC)\ dy}{\Delta} \right\},$$

where $\Delta = 4B^3 + 27C^2$ is the discriminant of the cubic polynomial in p.

From the theorem we immediately obtain

Theorem 5.2. (Resonance Curve Theorem (Lins Neto, Nakai) [24]**)** Assume $d = 3$, $B = C = 0$ at the origin and the germ of discriminant Δ at the origin is diffeomorphic to the $(2, 3)$ cusp; assume also the skeleton is smooth and transverse to the canonical contact element $dy - pdx = 0$ in the first jet space. Then the curvature form vanishes on a union of two transverse non-singular curves passing through the origin; one is tangent to the discriminant at the origin and the other is transverse. The statement remains valid also in the real smooth case.

Proof. The first assumption implies that (B, C) is a local diffeomorphism of $\mathbb{C}^2$ and the second assumption tells $C_x \neq 0$ at the origin. By straightforward calculation one sees that the curvature multiplied by Δ^2 has a trivial linear part, the second-degree part is non-degenerate, and its zero splits into the tangent line of the cusp and another transverse line. □

In the real case, the resonance curve is also real; moreover, the theorem provides a law of positivity and negativity of the curvature in the sectorial areas between the cusp and the component of the resonance curve passing through inside. In the 3-web on the left in Fig 4, the curvature is negative and positive on the respective upper and lower sectorial domains inside the cusp separated by the resonance curve (see also [29] for the web curvature nearby singular locus).

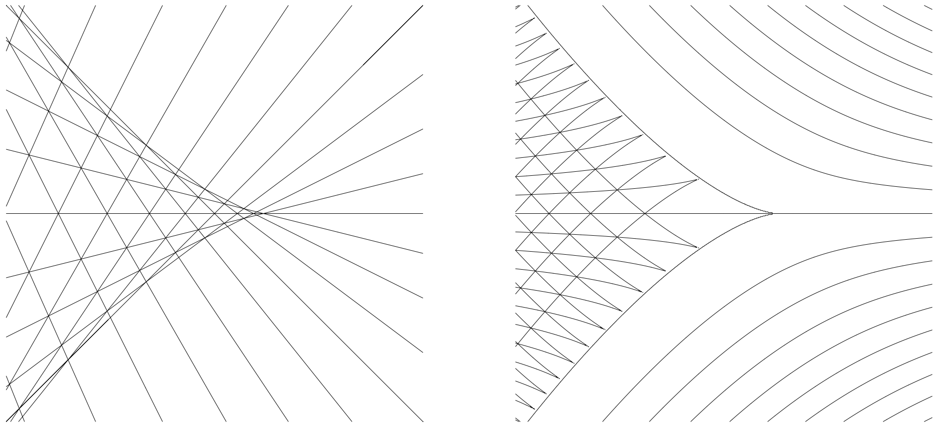

FIGURE 2. Left:Cusp-Clairaut 3-web: Exponent 1; Right:Rectangular 3-web: Exponent $\frac{1}{2}$

For $d \geq 4$, there is no canonical affine connection associated to a d-web, unless the cross ratio of the tangent lines of d leaves is constant. Henaut [18, 19, 20] defines a curvature from the viewpoint of Abelian relations. Recently Henaut proved his curvature form is the sum of web curvatures of all subtracted 3-subwebs in [20].

6. Flat Differential Equations

A 3-web is *hexagonal* if it is locally flat, in other words, the web curvature form vanishes identically. The next fact is classically known.

Theorem 6.1. (Linearization Theorem [1,2,4]) *A non-singular flat 3-web is locally diffeomorphic to the linear 3-web defined by*

$$dx\ ,\ dy\ ,\ -(dx+dy).$$

From the intuitive geometric point of view, it is interesting to classify singular hexagonal 3-webs on the plane. The following theorem was announced by Lins Neto and the author in [31, 32] without proof, which has now been given by Agafonov [1].

Theorem 6.2. *Assume the solution web of the local first-order ODE* $(*)$ *in* §5 *is a hexagonal 3-web and the discriminant locus in* p *is diffeomorphic to the* $(2,3)$*-cusp. Then the equation* $(*)$ *is equivalent to one of the following two equations by transformation of the* xy*-plane.*

(Cusp-Clairaut) $$p^3 + xp - y = 0,$$

(Rectangular) $$p^3 + \frac{1}{4}xp + \frac{1}{8}y = 0.$$

These equations have smooth skeletons and their projections onto the xy-plane are the Whitney cusp map. The portraits of the solution hexagonal 3-webs of these equations are drawn in Figure 2. The reader may appreciate the affine (linear) structure on the complement of the discriminant. Curiously these hexagonal webs appear in geometric optics: The figure on the left is apparently the most symmetric 3-web by ray lines tangent to the cuspidal caustics, and the figure on the right is give by the contour lines of the differences in phases (critical values) $d_1 - d_2$, $d_2 - d_3$, and $d_3 - d_1$ in the Pearcey web in §10, where d_1, d_2, and d_3 are the critical values of the potential function $\frac{1}{4}p^4 + \frac{1}{2}xp^2 + yp$.

Agafonov [2] explains these singular hexagonal 3-webs from the viewpoint of Frobenius manifolds.

7. Klein-Halphen webs and Fermat webs

In this section we introduce some other singular hexagonal 3-webs on the plane. The *coordinate 3-web* on an analytic surface $V \subset \mathbb{C}^3$ is defined by the coordinate 1-forms dx, dy, dz. The coordinate web is hexagonal on the generalized Brieskorn variety

$$V_{\alpha,\beta,\gamma} : X^\alpha + Y^\beta + Z^\gamma = 0 \ , \quad \alpha, \beta, \gamma \in \mathbb{Q}^*$$

as $d\theta = \frac{\partial^2}{\partial x \partial y} \log \frac{z_x}{z_y} dx \wedge dy$ vanishes identically on the variety. Clearly a pullback of this coordinate 3-web by any non-degenerate map germ $\phi : \mathbb{C}^2, o \to V_{\alpha,\beta,\gamma}, o$ is hexagonal, but it is highly singular at the preimage of the origin in general.

For positive integers α, β, γ, Halphen [16], Klein [22] and Lins Neto [23] proved that the germ $V_{\alpha,\beta,\gamma}, o$ admits a finite-to-one dominating map germ ϕ from $\mathbb{C}^2, o$ if and only if

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} > 1 \ .$$

Moreover, for such a triple (α, β, γ) with $\alpha, \beta, \gamma \geq 2$, i.e., one of $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$, the variety is a quotient of $\mathbb{C}^2$ by a certain subgroup of the binary polyhedral group in $SU(2)$ corresponding to the triple, i.e., a cyclic group of order n, binary Klein quartic group, binary alternating group of order 24 or the whole binary octahedral group, respectively. Thus the hexagonal coordinate 3-web on the variety lifts to a singular hexagonal 3-web on $\mathbb{C}^2$ with symmetry of the subgroup (see c.f. [22] for the explicit form of the quotient map). It is seen that the web thus constructed possesses also the natural $\mathbb{C}^*$-symmetry and moreover the symmetry of the whole binary polyhedral group corresponding to the triple. The complete quotients of $\mathbb{C}^2$ by these binary polyhedral groups are known as *duVal singularities* of type D_n, E_6, E_7, and E_8. Therefore every duVal singularity of type ADE, including type A_n, admits a hexagonal singular web structure, which carries affine structure off the singular locus. (The lift of the coordinate 3-web on $V_{2,3,3}$ possesses also the symmetry of binary octahedral group if we allow transposition of the constituent coordinate foliations of the web. Therefore the E_7 singularity admits two distinct singular affine structures induced from $V_{2,3,3}$ and $V_{2,3,4}$.) We call these webs on the source $\mathbb{C}^2$ lifted from $V_{\alpha,\beta,\gamma}$, *Klein-Halphen* webs. This construction may be generalized to certain other values of α, β, γ.

Let us consider the generalized homogeneous *Fermat surface*

$$V_\alpha : X^\alpha + Y^\alpha + Z^\alpha = 0 \ ,$$

where $\alpha \neq 0$ is a rational number. This is the only symmetric germ of an analytic subspace, up to diffeomorphisms of Cartesian form $\phi(X) \times \phi(Y) \times \phi(Z)$, such that the coordinate 3-web is hexagonal. The coordinate web on V_α induces a hexagonal web W_α on its S_3-quotient $\tilde{V}_\alpha$. The coordinate functions X, Y, Z induce a 3-valued function on the quotient, which is the "defining level function" of the quotient 3-web. The natural $\mathbb{C}^*$-action on V_α induces that on $\tilde{V}_\alpha$, and then $\tilde{V}_{p/q}$ is the $\mathbb{Z}_q$-quotient of $\tilde{V}_p$, and the $\mathbb{Z}_p$-quotient of $\tilde{V}_p$ is $\tilde{V}_1$.

For an exponent α in a certain class, there exists a finite covering map $P : \mathbb{C}^2, o \to \tilde{V}_\alpha, o$ branched at o. Among the positive integers, such a covering map exists only for $\alpha = 1, 2, 3, 4$ and 5: $\tilde{V}_4$, $\tilde{V}_5$ are diffeomorphic to the A_1 singularity and A_4 singularity, respectively, which are the quotients of $\mathbb{C}^2$ by cyclic groups of orders 2 and 5. For such an α (not necessarily a positive integer in general) the quotient coordinate 3-web on $\tilde{V}_\alpha, o$ lifts to a hexagonal 3-web on the source $\mathbb{C}^2$, and its first integral is induced from the coordinate functions X, Y, Z. We call the web thus constructed *Fermat web* if it exists, and denote by W_α.

Here we present the first-order differential equations defining the Fermat webs for certain exponents α:

$$
\begin{array}{lll}
E_1 : & p^3 + xp - y = 0 & \text{(Cusp-Clairaut)}, \\
E_{\frac{1}{2}} : & p^3 + 4xyp - 8y^2 = 0, & \\
E_{\frac{1}{3}} : & 27x^2p^3 + 3xp - y = 0, & \\
E_{\frac{1}{6}} : & 27x^2p^3 + 12xyp - 8y^2 = 0, &
\end{array}
$$

where $p = dy/dx$. The portrait of $W_{\frac{1}{2}}$ is the figure on the left in Figure 4. The equations thus obtained are not local in p in general, as is seen in the small list above, and do not fall within the classification scheme given by Theorem 3.1.

The equation E_1 has the natural $\mathbb{C}^*$-symmetry induced from that on the plane

$$V_1 : \Sigma_1 = X + Y + Z = 0$$

in XYZ-space. For instance, the action by -1 induces the involution $(x, y) \to (x, -y)$ on the xy-plane, $x = \Sigma_2 = XY + YZ + ZX, y = \Sigma_3 = XYZ$, and the action of the cubic root of unity ω gives the symmetry of order 3, $(x, y) \to (\omega^{-1}x, y)$. The quotients of E_1 by these symmetries of order 2, 3, and their generating group of order 6 are respectively $E_{\frac{1}{2}}, E_{\frac{1}{3}}$ and $E_{\frac{1}{6}}$.

For $\alpha = -1$, we obtain the following Clairaut equation

$$E_{-1} : \quad p^3 + x^2p^2 - 2xyp + y^2 = 0,$$

where $x = \Sigma_1, y = \Sigma_3$ and $p = dy/dx$. The explicit form of the differential equations for the other α can be calculated by computer but they are too big and unsuitable to present here. The other cases will be published elsewhere.

8. Abelian relation of first integral and exponent

Assume the Fermat web W_α exists for a rational α. By construction, the coordinates X, Y, Z on the Fermat surface V_α induce a single valued first integral λ on the skeleton of E_α, and it enjoys the relation

$$\text{Trace } \lambda^\alpha = \sum \lambda^\alpha = 0$$

on the xy-plane, where the sum in the middle is taken over the fiber of projection of the skeleton onto the xy-plane choosing suitable branches of λ^α. For instance, $\lambda = p$ for the equation E_1 in the previous section, and the relation Trace $p = 0$ is apparent by the presentation of the equation. We call the relation an *Abelian relation* of the first integral λ.

Consider a germ of 3-web at the origin in the xy-plane defined by a first-order ODE $f(x, y, p) = 0$ as in §5, which is not necessarily local in p. The Trace λ^α is well defined in a similar manner to the above on a punctured neighborhood of the origin for a first integral λ on the skeleton $S : f = 0$. Then the Abelian relation Trace $\lambda^\alpha = 0$ implies the web is hexagonal.

Assume another Abelian relation Trace $\mu^\beta = 0$ holds. As the space of Abelian relations for triples of defining 1-forms of a germ of non-singular plane 3-web is of dimension at most 1 (see [7, 8, 20] for the detail), it follows $d\lambda^\alpha = c\, d\mu^\beta$ on S with a constant $c \neq 0$, from which $\lambda^\alpha = c\mu^\beta$. Assume S is smooth at a lift $\tilde{C}$ of a solution $C \subset \mathbb{C}^2$ containing the origin in its closure, and λ, μ vanishe on $\tilde{C}$ in order 1. Then comparing the orders of vanishing of both sides of $\lambda^\alpha = c\mu^\beta$ at $\tilde{C}$, we obtain $\alpha = \beta$. Moreover a first integral with an Abelian relation of exponent α is unique up to multiplication by a constant. The exponent α is uniquely determined by the 3-web if there exists only one solution C with the above property. We call α the *exponent* of the hexagonal 3-web with first integerl. (Remark that the first integral λ^n fulfills the Abelian relation of exponent

α/n for any positive integer n, but it vanishes at $\tilde{C}$ in order n. We define the exponent to be ∞ if the Abelian relation does not exist.)

Theorem 8.1. (Universality of Fermat web) *Assume a germ of a hexagonal* 3*-web on the plane admits an irreducible holomorphic first integral λ with an Abelian relation of exponent α as above, and assume the Fermat web W_α exists. Assume also $\lambda \neq 0$ at a point on each fiber of the projection of skeleton over $(x, y) \neq o$ near the origin o. Then the web and the first integral λ are induced from the Fermat web W_α via a holomorphic map of the plane.*

Proof. Let $\Delta \subset \mathbb{C}^2$ denote the subset of those points where the projection of $\pi : S \to \mathbb{C}^2$ are not regular at the fiber of π or λ is not regular at the fiber. Define $\mu : \mathbb{C}^2 \setminus \Delta \to \mathbb{C}^3$ by $\mu = (\Sigma_1, \Sigma_2, \Sigma_3)$ with the symmetric polynomials Σ_i of degree i of the values of λ at the fiber of π. It is bounded on a neighborhood of the origin, so extends holomorphically to a map of $\mathbb{C}^2$ to the quotient Fermat surface $\tilde{V}_\alpha$, and the web W_S is the pullback of the quotient web on $\tilde{V}_\alpha$. By assumption, the extension μ has the fiber $\mu^{-1}(o) = o \in \mathbb{C}^2$. Therefore the pullback of the branched covering $P : \mathbb{C}^2 \to V_\alpha$ by μ decomposes into a union of non-singular surfaces meeting at the origin. Let s be a section of the pullback, $\tilde{\mu}'$ the natural bundle map covering μ, and set $\tilde{\mu} = \tilde{\mu}' \circ s$. Then $P \circ \tilde{\mu} = \mu$ and $W_S = \tilde{\mu}^* W_\alpha$. By the definition of μ, the lift $\tilde{\mu}$ respects the first integrals of the leaves passing through those points in the source and target. Therefore λ is induced from the first integral of W_α. $\square$

For instance, Rectangular web in Figure 2 has exponent $\frac{1}{2}$ as is seen by the argument for Dual-Cusp-Clairaut web in the end of the next §9, thus it is a pullback of the Fermat web $W_{\frac{1}{2}}$. We remark that Klein-Halphen webs may possess a similar universal property.

In general, a singular d-web admits at most $(d-1)(d-2)/2$ linearly independent Abelian relations of first integrals by the virtue of the theory of Abelian relation of integrable forms (see c.f. [20]). The following Clairaut equation defines a 4-web, and the first integral $p = dy/dx$ enjoys the maximal number (i.e. 3) of Abelian relations with the various exponents

$$p^4 + xp - y = 0, \quad \text{Trace } p = 0, \ \text{Trace } p^2 = 0, \ \text{Trace } p^5 = 0.$$

This suggests generalizing our construction of Fermat webs to Brieskorn varieties of higher codimensions.

9. Dual 3-web

The hexagonal 3-web structure can be also produced by rotation of leaves as follows. The *dual* L^* of a configuration $L = L_1 \cup L_2 \cup L_3$ of lines in the plane passing through the origin is the unique 3-line configuration (different from L) invariant under the linear symmetry group of L. The *dual 3-web* W^* of a 3-web W is defined by integrating the dual 3-line configuration of the tangent 3-line fields of W.

Theorem 9.1. *The bi-duality holds: $W^{**} = W$, and W, W^* share the same Chern connection.*

Corollary 9.1. *A 3-web W is flat if and only if its dual W^* is flat.*

The dual equations of (Cusp-Clairaut), (Rectangular) in Theorem 6.2 are respectively

(Dual-Cusp-Clairaut) $$y'^3 + \frac{2x^2}{3y}\, y'^2 - x\, y' + \frac{2x^3 + 27y^2}{27y} = 0,$$

(Dual-Rectangular) $$y'^3 - \frac{x^2}{3y}\, y'^2 - \frac{x}{4}\, y' - \frac{2x^3 + 27y^2}{216y} = 0.$$

These equations have the exponents $\frac{1}{2}$ and 1 respectively. The portraits of the solution webs of these equations are drawn in Figure 3. Dual-Cusp-Clairaut web is the S_3-quotient of the plane

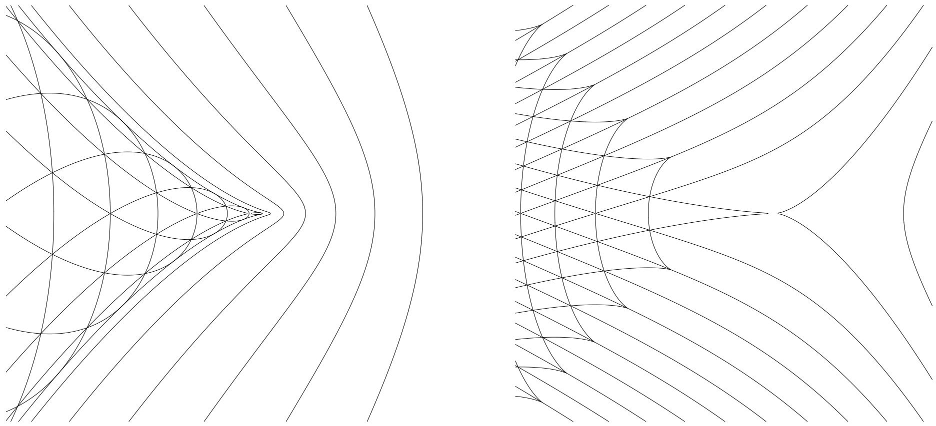

FIGURE 3. Left: Dual-Cusp-Clairaut web: Exponent $\frac{1}{2}$; Right: Dual-Rectangular web: Exponent 1

$\alpha+\beta+\gamma=0$ in the $\alpha\beta\gamma$-space foliated by the level lines of $\alpha-\beta$, and its skeleton is diffeomorphic to the quotient of the plane by the involution transposing α and β. Thus $\lambda=(\alpha-\beta)^2$, well defined on the skeleton, is a first integral of the equation. The Abelian relation of λ of exponent $\frac{1}{2}$ follows from the obvious relation $(\alpha-\beta)+(\beta-\gamma)+(\gamma-\alpha)=0$. Fermat web $W_{\frac{1}{2}}$ is also interesting because it is self-dual, i.e., identical to its dual 3-web. Generalization of the duality to flat webs in higher dimensions would be interesting.

10. Pearcey web and the stationary phase method

Let us consider the Pearcey integral

$$\int_{-\infty}^{\infty} \exp\ \sqrt{-1}\{\frac{1}{4}x^4+\frac{1}{2}t_1x^2+t_2x+h(t_1,t_2)\}\ dx.$$

In the theory of geometric optics, the potential function is supposed to be the distance from a point in the observation plane $(t_1,t_2)\in\mathbb{C}^2$ to a wave front [37]. Huygens principle suggests the intensity of light at the (t_1,t_2) near caustics is well approximated by the absolute value of the integral[6, 37, 33]. This model is associated with the generating function

$$f_t(x_0,x_1)=\frac{1}{4}x_1^4+\frac{1}{2}t_1x_1^2+t_2x_1+h(t_1,t_2)-x_0, \quad t=(t_1,t_2)\in\mathbb{C}^2,$$

which is non-versal in the manner in §1, 4 and embedded into the versal family with an additional parameter

$$F_T(x_0,x_1)=\frac{1}{4}x_1^4+\frac{1}{2}t_1x_1^2+t_2x_1+t_3-x_0, \quad T=(t_1,t_2,t_3)\in\mathbb{C}^3.$$

The stationary phase method reveals that the integral is approximated by the functional value of the integrand at the critical points of the potential function in the exponent. This leads us to the geometry of the family of wavefronts: Phase = constant, which is just our 3-web $W_{\{f_t\}}$ introduced in §1 and illustrated on the right in Figure 4. We call this web of wave fronts, *Pearcey 3-web*.

Clearly the intensity is independent of the constant term h. The contour map of the intensity is given on the left in Figure 5. In the paper of Pearcey [33], the contour map of the phase is also given for the h identically 0. On the right in Figure 5, we present it with a h suitable for the natural wave propagation. The skeleton is smooth if $h_{t_1}(0,0) \neq 0$. It is interesting to compare the figures on the right in Figures 2, 3, and the Pearcey 3-web in Figure 4 with the contour maps of the respective intensity and phase of the Pearcey integral in Figure 5.

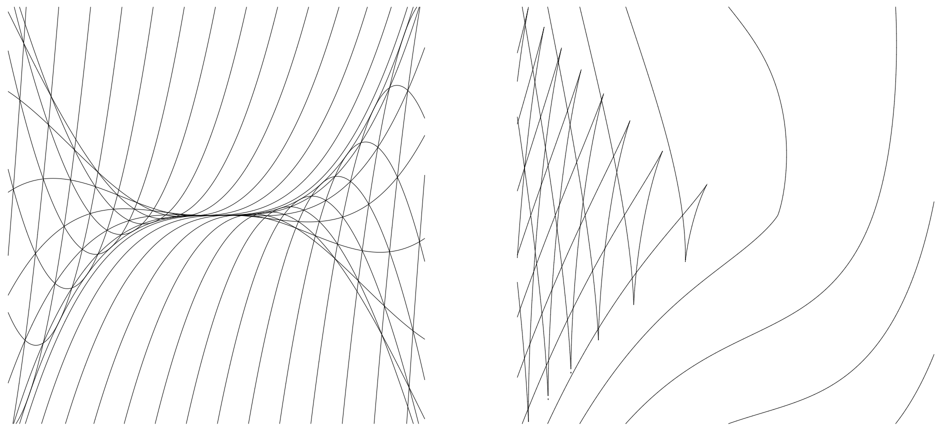

FIGURE 4. Left: Fermat web $W_{\frac{1}{2}}$; Right: non-hexagonal Pearcey 3-web

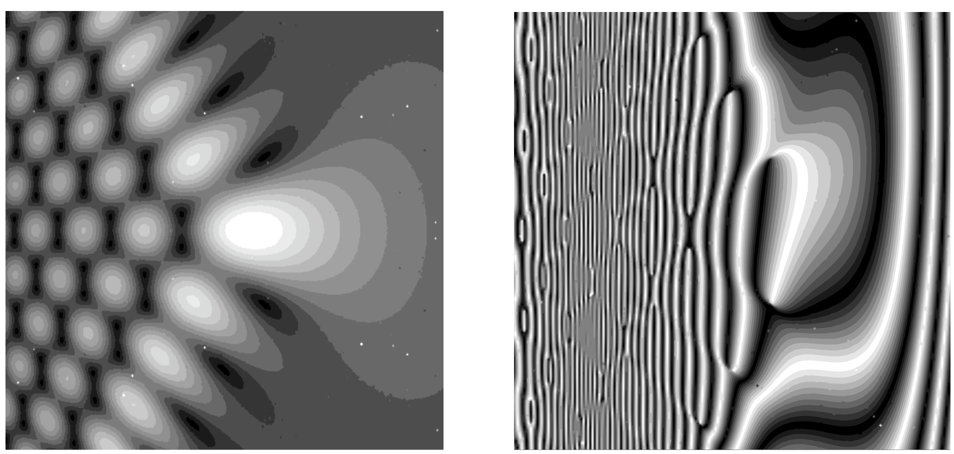

FIGURE 5. Contour maps of the intensity (above left) and phase (above right). In the figure on the left, the black dots represent zeros of the integral (small white dots are computational bugs), whereas in the right figure the black dots represent the centers of whirlpools.

According to Theorem 5.2 (Resonance Curve Theorem), there exist two smooth curves passing through the cusp point, on an infinitesimally small neighborhood of which the 3-web structure is flat. Moreover 3-phases of the wave resonate at some discrete points on those curves. If we suppose the wave in the last figure is propagating from right to left, some "trajectories" seem to be trapped in whirlpools inside the cusp.

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