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EVOLUTES OF FRONTS IN THE EUCLIDEAN PLANE

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Dedicated to Professor Shyuichi Izumiya on the occasion of his 60th birthday

Abstract. The evolute of a regular curve in the Euclidean plane is given by not only the caustics of the regular curve, envelope of normal lines of the regular curve, but also the locus of singular points of parallel curves. In general, the evolute of a regular curve has singularities, since such points correspond to vertices of the regular curve and there are at least four vertices for simple closed curves. If we repeat an evolute, we cannot define the evolute at a singular point. In this paper, we define an evolute of a front and give properties of such an evolute by using a moving frame along a front and the curvature of the Legendre immersion. As applications, repeated evolutes are useful to recognize the shape of curves.

1. Introduction

The evolute of a regular plane curve is a classical object (cf. [5, 8, 9]). It is useful for recognizing the vertex of a regular plane curve as a singularity (generically, a $3/2$ cusp singularity) of the evolute. The caustics (evolutes) are related to general relativity theory, see for instance [6, 10]. The properties of evolutes are discussed by using distance squared functions and the theories of Lagrangian and Legendrian singularities (cf. [1, 2, 3, 13, 14, 17, 20]). Moreover, the singular points of parallel curves of a regular curve sweep out the evolute. By using this property, we define an evolute of a front in §2. In order to consider properties of an evolute of a front, we introduce a moving frame along a front (a Legendre immersion) (cf. [7]). In [7], we give existence and uniqueness for a Legendre curve in the unit tangent bundle like for regular plane curves. It is quite useful to analyze a Legendre curve (or, a frontal) in the unit tangent bundle. In §3, we give another representation for the evolute of a front by using the moving frame and the curvature of the Legendre immersion (Theorem 3.3). By the representation, we give properties of the evolutes of fronts, for example, the evolute of a front is also a front. It follows that we can consider the repeated evolutes, namely, the evolute of an evolute of a front, see Theorem 4.1 in §4. Moreover, we extend the notion of the vertex for a front (or, a Legendre immersion) and give a kind of four vertex theorem for a front, see Proposition 3.11. Furthermore, the evolute of a front is also given by the envelope of normal lines of the front. A singular point of the evolute of the evolute of a regular curve measure to the contact of an involute of a circle. We give the n-th evolute of a front in §5. In §6, we give examples of the evolutes of fronts. In the appendix, we give the condition of contact between regular curves.

All maps and manifolds considered here are differentiable of class C^∞.

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2. Definitions and basic concepts

Let I be an interval or $\mathbb{R}$. Suppose that $\gamma : I \to \mathbb{R}^2$ is a regular plane curve, that is, $\dot{\gamma}(t) \neq 0$ for any $t \in I$. If s is the arc-length parameter of γ, we denote $\boldsymbol{t}(s)$ by the unit tangent vector $\boldsymbol{t}(s) = \gamma'(s) = (d\gamma/ds)(s)$ and $\boldsymbol{n}(s)$ by the unit normal vector $\boldsymbol{n}(s) = J(\boldsymbol{t}(s))$ of $\gamma(s)$, where J is the anticlockwise rotation by $\pi/2$. Then we have the Frenet formula as follows:

$$\begin{pmatrix} \boldsymbol{t}'(s) \\ \boldsymbol{n}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \end{pmatrix},$$

where $\kappa(s) = \boldsymbol{t}'(s) \cdot \boldsymbol{n}(s)$ is the curvature of γ and $\cdot$ is the inner product on $\mathbb{R}^2$.

Even if t is not the arc-length parameter, we have the unit tangent vector $\boldsymbol{t}(t) = \dot{\gamma}(t)/|\dot{\gamma}(t)|$, the unit normal vector $\boldsymbol{n}(t) = J(\boldsymbol{t}(t))$ and the Frenet formula

$$\begin{pmatrix} \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \end{pmatrix} = \begin{pmatrix} 0 & |\dot{\gamma}(t)|\kappa(t) \\ -|\dot{\gamma}(t)|\kappa(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \end{pmatrix},$$

where $\dot{\gamma}(t) = (d\gamma/dt)(t)$, $|\dot{\gamma}(t)| = \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t)}$ and $\kappa(t) = \det(\dot{\gamma}(t), \ddot{\gamma}(t))/|\dot{\gamma}(t)|^3 = \dot{\boldsymbol{t}}(t) \cdot \boldsymbol{n}(t)/|\dot{\gamma}(t)|$. Note that $\kappa(t)$ is independent of the choice of a parametrization.

In this paper, we consider evolutes of plane curves. The *evolute* $Ev(\gamma) : I \to \mathbb{R}^2$ *of a regular plane curve* γ is given by

$$Ev(\gamma)(t) = \gamma(t) + \frac{1}{\kappa(t)}\boldsymbol{n}(t),$$

away from the points where $\kappa(t) = 0$ (cf. [5, 8, 9]).

If γ is not a regular curve, then we cannot define the evolute as above, since the curvature may diverge at a singular point. However, we define an evolute of a front in the Euclidean plane, see Definition 2.10 and Theorem 3.3. It is a generalization of the evolute of regular plane curves.

We say that $\gamma : I \to \mathbb{R}^2$ is a *front* (or, a *wave front*) in the Euclidean plane, if there exists a smooth map $\nu : I \to S^1$ such that the pair $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion, namely, $(\dot{\gamma}(t), \dot{\nu}(t)) \neq (0, 0)$ and $(\gamma(t), \nu(t))^*\theta = 0$ for each $t \in I$. Here θ is the canonical contact structure on $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$, and S^1 is the unit circle. We remark that the second condition is equivalent to $\dot{\gamma}(t) \cdot \nu(t) = 0$ for each $t \in I$ (cf. [1, 2, 3]).

Throughout the paper, we assume that the pair (γ, ν) is co-orientable, the singular points of γ are finite and γ has no inflection points. The first and second conditions can be removed, see Remarks 3.4 and 3.5. However, we add these conditions for the sake of simplicity.

We give examples of fronts. See [1, 4, 11] for other examples.

Example 2.1. One of the typical examples of a front is a regular plane curve. Let $\gamma : I \to \mathbb{R}^2$ be a regular plane curve. In this case, we may take $\nu : I \to S^1$ by $\nu(t) = \boldsymbol{n}(t)$. Then it is easy to check that (γ, ν) is a Legendre immersion.

Example 2.2. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a 3/2 cusp (A_2-singularity) given by $\gamma(t) = (t^2, t^3)$. In this case, 0 is a singular point of γ. If we take $\nu : \mathbb{R} \to S^1$ by $\nu(t) = (1/\sqrt{9t^2 + 4})(-3t, 2)$, then we can show that (γ, ν) is a Legendre immersion. Hence the 3/2 cusp is an example of a front. The 3/2 cusp is the generic singularity of fronts and also evolutes in the Euclidean plane.

Example 2.3. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a 4/3 cusp (E_6-singularity) given by $\gamma(t) = (t^3, t^4)$. In this case, 0 is also a singular point of γ. If we take $\nu : \mathbb{R} \to S^1$ by $\nu(t) = (1/\sqrt{16t^2 + 9})(-4t, 3)$, then we can show that (γ, ν) is a Legendre immersion. Hence the 4/3 cusp is also an example of a front, see Example 6.3.

Example 2.4. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a 5/2 cusp (A_4-singularity) given by $\gamma(t) = (t^2, t^5)$. In this case, 0 is also a singular point of γ. However, the 5/2 cusp is not a front. By the condition

$\dot{\gamma}(t)\cdot\nu(t)=0$, we take $\nu:\mathbb{R}\to S^1$ by $\nu(t)=\pm(1/\sqrt{25t^6+4})(-5t^3,2)$. Then (γ,ν) is not an immersion at $t=0$ and hence γ is not a front (but γ is a frontal, see [7]).

Remark 2.5. By the definition of the Legendre immersion, if (γ,ν) is a Legendre immersion, then $(\gamma,-\nu)$ is also.

We have the following Lemma (cf. [4, 11, 12]).

Lemma 2.6. *Let $\gamma:I\to\mathbb{R}^2$ be a front and $t_0\in I$. If $\gamma^{(i)}(t_0)=0$ for each $1\le i\le k-1$ and $\gamma^{(k)}(t_0)\neq 0$, then γ at t_0 is diffeomorphic to the curve $(t^k,t^{k+1}+o(t^{k+1}))$ at $t=0$. Moreover, if $k=2$ (respectively, $k=3$), the curve at t_0 is diffeomorphic to a $3/2$ (respectively, $4/3$) cusp.*

Let $(\gamma,\nu):I\to\mathbb{R}^2\times S^1$ be a Legendre immersion. We define a *parallel curve* $\gamma_\lambda:I\to\mathbb{R}^2$ of γ by $\gamma_\lambda(t)=\gamma(t)+\lambda\nu(t)$ for each $\lambda\in\mathbb{R}$. Then we have following results.

Proposition 2.7. *For a Legendre immersion $(\gamma,\nu):I\to\mathbb{R}^2\times S^1$, the parallel curve $\gamma_\lambda:I\to\mathbb{R}^2$ is a front for each $\lambda\in\mathbb{R}$.*

Proof. We take $\nu_\lambda:I\to S^1$ by $\nu_\lambda(t)=\nu(t)$. Since $\gamma_\lambda(t)=\gamma(t)+\lambda\nu(t)$, it holds that $\dot{\gamma}_\lambda(t)=\dot{\gamma}(t)+\lambda\dot{\nu}(t)$. If $\dot{\gamma}_\lambda(t_0)=0$ at a point $t_0\in I$, then we have $\dot{\gamma}(t_0)+\lambda\dot{\nu}(t_0)=0$. If $\dot{\nu}_\lambda(t_0)=\dot{\nu}(t_0)=0$, then $\dot{\gamma}(t_0)=0$. It contradicts the fact that (γ,ν) is an immersion. Hence $(\gamma_\lambda,\nu_\lambda)$ is an immersion. By $\nu(t)\cdot\nu(t)=1$, we have $\dot{\nu}(t)\cdot\nu(t)=0$. Then

$$\dot{\gamma}_\lambda(t)\cdot\nu_\lambda(t)=(\dot{\gamma}(t)+\lambda\dot{\nu}(t))\cdot\nu(t)=\dot{\gamma}(t)\cdot\nu(t)+\lambda\dot{\nu}(t)\cdot\nu(t)=0$$

holds. It follows that $(\gamma_\lambda,\nu_\lambda)$ is a Legendre immersion and hence γ_λ is a front. □

We denote the curvature of the parallel curve $\gamma_\lambda(t)$ by $\kappa_\lambda(t)$, when γ_λ is a regular curve.

Proposition 2.8. *Let (γ,ν) be a Legendre immersion. If γ is a regular curve and $\lambda\neq 1/\kappa(t)$, then a parallel curve γ_λ is also regular and $Ev(\gamma_\lambda)(t)$ is consistent with $Ev(\gamma)(t)$.*

Proof. Since $\gamma_\lambda(t)=\gamma(t)+\lambda\boldsymbol{n}(t)$, it holds that $\dot{\gamma}_\lambda(t)=|\dot{\gamma}(t)|(1-\lambda\kappa(t))\boldsymbol{t}(t)$. By the assumption $\lambda\neq 1/\kappa(t)$, γ_λ is a regular curve. By a direct calculation, we have

$$\kappa_\lambda(t)=\frac{\kappa(t)}{|1-\lambda\kappa(t)|},\quad \boldsymbol{n}_\lambda(t)=\frac{1-\lambda\kappa(t)}{|1-\lambda\kappa(t)|}\boldsymbol{n}(t).$$

Hence we have

$$\begin{aligned} Ev(\gamma_\lambda)(t) &= \gamma_\lambda(t)+\frac{1}{\kappa_\lambda(t)}\boldsymbol{n}_\lambda(t)=\gamma(t)+\lambda\boldsymbol{n}(t)+\frac{|1-\lambda\kappa(t)|}{\kappa(t)}\frac{1-\lambda\kappa(t)}{|1-\lambda\kappa(t)|}\boldsymbol{n}(t)\\ &= \gamma(t)+\frac{1}{\kappa(t)}\boldsymbol{n}(t)=Ev(\gamma)(t)\end{aligned}$$

□

Remark 2.9. Let (γ,ν) be a Legendre immersion. If t_0 is a singular point of the front γ, then $\lim_{t\to t_0}|\kappa(t)|=\infty$. By the equality $\kappa_\lambda(t)=\kappa(t)/|1-\lambda\kappa(t)|$, we have $\lim_{t\to t_0}\kappa_\lambda(t)\neq 0$, see also Remark 3.2.

We now define an evolute of a front in the Euclidean plane.

Definition 2.10. Let $(\gamma,\nu):I\to\mathbb{R}^2\times S^1$ be a Legendre immersion. We define an *evolute* $\mathcal{E}v(\gamma):I\to\mathbb{R}^2$ *of the front* γ as follows:

$$\mathcal{E}v(\gamma)(t)=\begin{cases}\gamma(t)+\frac{1}{\kappa(t)}\boldsymbol{n}(t) & \text{if}\quad t \text{ is a regular point,}\\ \gamma_\lambda(t)+\frac{1}{\kappa_\lambda(t)}\boldsymbol{n}_\lambda(t) & \text{if}\quad t\in(t_0-\delta,t_0+\delta),\ t_0 \text{ is a singular point of } \gamma,\end{cases}$$

where δ is a sufficiently small positive real number, $\lambda \in \mathbb{R}$ is satisfied the condition $\lambda \neq 1/\kappa(t)$ and $\kappa(t) \neq 0$.

Remark 2.11. By the assumption of the finiteness of singularities of a front, there exists $\lambda \in \mathbb{R}$ with the condition $\lambda \neq 1/\kappa(t)$. Moreover, by Proposition 2.8, we can glue on the regular interval of γ and γ_λ. Then the evolute of a front is well-defined. Furthermore, by definition, the evolute of a front $\mathcal{E}v(\gamma)$ is a C^∞ map.

In order to consider properties of the evolute of a front, we need a moving frame along a front (or, a Legendre immersion) (cf. [7]). Let $(\gamma, \nu): I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion. If γ is a regular curve around a point t_0, then we have the Frenet formula of γ in §2. On the other hand, if γ is singular at a point t_0, then we don't define such a frame. However, ν is always defined even if t is a singular point of γ. Therefore, we have the Frenet formula of a front as follows. We put $\boldsymbol{\mu}(t) = J(\nu(t))$. We call the pair $\{\nu(t), \boldsymbol{\mu}(t)\}$ is a *moving frame along a front* $\gamma(t)$ in $\mathbb{R}^2$ and we have the Frenet formula of a front which is given by

$$\begin{pmatrix} \dot{\nu}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \tag{1}$$

where $\ell(t) = \dot{\nu}(t) \cdot \boldsymbol{\mu}(t)$. Moreover, if $\dot{\gamma}(t) = \alpha(t)\nu(t) + \beta(t)\boldsymbol{\mu}(t)$ for some smooth functions $\alpha(t), \beta(t)$, then $\alpha(t) = 0$ follows from the condition $\dot{\gamma}(t) \cdot \nu(t) = 0$. Hence, there exists a smooth function $\beta(t)$ such that

$$\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t). \tag{2}$$

Since (γ, ν) is an immersion, we have $(\ell(t), \beta(t)) \neq (0, 0)$ for each $t \in I$. The pair (ℓ, β) is an important invariant of Legendre curves (or, frontals) in the unit tangent bundle like as the curvature of a regular plane curve, for more detail, see [7]. We call the pair (ℓ, β) *the curvature of the Legendre curve*. Since we assume that (γ, ν) is a Legendre immersion, so that we call (ℓ, β) *the curvature of the Legendre immersion*. For the related properties, see [15, 16].

3. Properties of the evolutes of fronts

In this section, we consider properties of the evolutes of fronts. Let $(\gamma, \nu): I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature of the Legendre immersion (ℓ, β).

First we give a relationship between the curvature of the Legendre immersion $(\ell(t), \beta(t))$ and the curvature $\kappa(t)$ if γ is a regular curve.

Lemma 3.1. (1) *If γ is a regular curve, then $\ell(t) = |\beta(t)|\kappa(t)$.*
(2) *If γ_λ is a regular curve, then $\ell(t) = |\beta(t) + \lambda\ell(t)|\kappa_\lambda(t)$.*

Proof. (1) By a direct calculation, $\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t)$, $\ddot{\gamma}(t) = \dot{\beta}(t)\boldsymbol{\mu}(t) - \beta(t)\ell(t)\nu(t)$ and

$$\kappa(t) = \frac{\det(\dot{\gamma}(t), \ddot{\gamma}(t))}{|\dot{\gamma}(t)|^3} = \frac{\det\Big(\beta(t)\boldsymbol{\mu}(t), \dot{\beta}(t)\boldsymbol{\mu}(t) - \beta(t)\ell(t)\nu(t)\Big)}{|\beta(t)|^3} = \frac{\beta(t)^2\ell(t)}{|\beta(t)|^3} = \frac{\ell(t)}{|\beta(t)|}.$$

Therefore we have $\ell(t) = |\beta(t)|\kappa(t)$.

(2) We can also prove by the same calculations of (1). □

Remark 3.2. Since $(\ell(t), \beta(t)) \neq (0, 0)$, if t_0 is a singular point of γ, then γ_λ is a regular curve. By Lemma 3.1 (2), $\ell(t_0) = |\lambda\ell(t_0)|\kappa_\lambda(t_0)$. It follows from $\lambda\ell(t_0) \neq 0$ that $\kappa_\lambda(t_0) \neq 0$.

We give another representation of the evolute of a front by using the moving frame of a front $\{\nu(t), \boldsymbol{\mu}(t)\}$ and the curvature of the Legendre immersion $(\ell(t), \beta(t))$.

Theorem 3.3. *Under the above notations, the evolute of a front* $\mathcal{E}v(\gamma)(t)$ *is represented by*

$$\mathcal{E}v(\gamma)(t) = \gamma(t) - \frac{\beta(t)}{\ell(t)}\nu(t), \tag{3}$$

and $\mathcal{E}v(\gamma)$ *is a front. More precisely,* $(\mathcal{E}v(\gamma)(t), J(\nu(t)))$ *is a Legendre immersion with the curvature*

$$\left(\ell(t),\ \frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right)\right).$$

Proof. First suppose that γ is a regular curve. Since $\dot\gamma(t) = \beta(t)\boldsymbol{\mu}(t)$, we have $|\beta(t)| \neq 0$ and

$$\boldsymbol{t}(t) = \frac{\beta(t)}{|\beta(t)|}\boldsymbol{\mu}(t), \quad \boldsymbol{n}(t) = -\frac{\beta(t)}{|\beta(t)|}\nu(t).$$

By Lemma 3.1 (1), $\kappa(t) = \ell(t)/|\beta(t)|$ and $\ell(t) \neq 0$. Then

$$\mathcal{E}v(\gamma)(t) = \gamma(t) + \frac{1}{\kappa(t)}\boldsymbol{n}(t) = \gamma(t) + \frac{|\beta(t)|}{\ell(t)}\left(-\frac{\beta(t)}{|\beta(t)|}\right)\nu(t) = \gamma(t) - \frac{\beta(t)}{\ell(t)}\nu(t).$$

Second suppose that t_0 is a singular point of γ and γ_λ is a regular curve with $\lambda \neq 1/\kappa(t)$. Since $\dot\gamma_\lambda(t) = (\beta(t) + \lambda\ell(t))\boldsymbol{\mu}(t)$, we have $|\beta(t) + \lambda\ell(t)| \neq 0$ and

$$\boldsymbol{t}_\lambda = \frac{\beta(t) + \lambda\ell(t)}{|\beta(t) + \lambda\ell(t)|}\boldsymbol{\mu}(t), \quad \boldsymbol{n}_\lambda = -\frac{\beta(t) + \lambda\ell(t)}{|\beta(t) + \lambda\ell(t)|}\nu(t).$$

By Lemma 3.1 (2), $\kappa_\lambda(t) = \ell(t)/|\beta(t) + \lambda\ell(t)|$ and $\ell(t) \neq 0$. Then

$$\begin{aligned}\mathcal{E}v(\gamma_\lambda)(t) &= \gamma_\lambda(t) + \frac{1}{\kappa_\lambda(t)}\boldsymbol{n}_\lambda(t) = \gamma(t) + \lambda\nu(t) + \frac{|\beta(t) + \lambda\ell(t)|}{\ell(t)}\left(-\frac{\beta(t) + \lambda\ell(t)}{|\beta(t) + \lambda\ell(t)|}\right)\nu(t) \\ &= \gamma(t) - \frac{\beta(t)}{\ell(t)}\nu(t).\end{aligned}$$

If we take $\widetilde{\nu}(t) = J(\nu(t)) = \boldsymbol{\mu}(t)$, then $(\mathcal{E}v(\gamma)(t), \widetilde{\nu}(t))$ is a Legendre immersion. In fact, $\dot{\widetilde{\nu}}(t) = \ell(t)J(\boldsymbol{\mu}(t)) \neq 0$ and by the form of

$$\dot{\mathcal{E}v}(\gamma)(t) = -\frac{\dot\beta(t)\ell(t) - \beta(t)\dot\ell(t)}{\ell(t)^2}\nu(t) = \frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right)J(\boldsymbol{\mu}(t)), \tag{4}$$

we have $\dot{\mathcal{E}v}(\gamma)(t) \cdot \widetilde{\nu}(t) = 0$. It follows that $(\mathcal{E}v(\gamma)(t), J(\nu(t)))$ is a Legendre immersion with the curvature $(\ell(t), (d/dt)(\beta(t)/\ell(t)))$ and hence $\mathcal{E}v(\gamma)$ is a front. This completes the proof of the Theorem. □

Remark 3.4. By the representation (3), we may define the evolute of a front even if γ have non-isolated singularities, under the condition $\ell(t) \neq 0$.

By Lemma 3.1 and Remark 3.4, for a Legendre immersion (γ, ν) with the curvature of the Legendre immersion (ℓ, β), we say that t_0 is *an inflection point of the front* γ (or, *the Legendre immersion* (γ, ν)) if $\ell(t_0) = 0$. Since $\beta(t_0) \neq 0$ and Proposition 3.1, $\ell(t_0) = 0$ is equivalent to the condition $\kappa(t_0) = 0$.

Remark 3.5. Let (γ, ν) be a Legendre immersion, then $(\gamma, -\nu)$ is also (Remark 2.5). However, $\mathcal{E}v(t)$ does not change. It follows that we can define an evolute of a non co-orientable front, by taking double covering of γ.

Remark 3.6. By Definition 2.10, the evolute of a front is independent on the parametrization of (γ,ν). The curvature of the Legendre immersion (ℓ,β) is depended on the parametrization of (γ,ν), see [7]. If $s=s(t)$ is a parameter changing on I to $\overline{I}$, then $\ell(t)=\ell(s(t))\dot{s}(t)$ and $\beta(t)=\beta(s(t))\dot{s}(t)$. It also follows from the representation (3) that the evolute of a front is independent on the parametrization of (γ,ν).

If t_0 is a singular point of γ, then $\beta(t_0)=0$. As a corollary of Theorem 3.3, we have the following.

Corollary 3.7. *If t_0 is a singular point of γ, then $\mathcal{E}v(\gamma)(t_0)=\gamma(t_0)$.*

Proposition 3.8. *Let $(\gamma,\nu): I\to\mathbb{R}^2\times S^1$ be a Legendre immersion without inflection points. Suppose that t_0 is a singular point of γ. Then t_0 is a regular point of $\mathcal{E}v(\gamma)(t)$ if and only if $\ddot{\gamma}(t_0)\neq 0$.*

Proof. By the assumption, $\beta(t_0)=0$. Let t_0 be a regular point of $\mathcal{E}v(\gamma)(t)$. Since (4) and $\ell(t_0)\neq 0$, we have $\dot{\beta}(t_0)\neq 0$. By the differentiate of $\dot{\gamma}(t)=\beta(t)\boldsymbol{\mu}(t)$, we have

$$\ddot{\gamma}(t)=\dot{\beta}(t)\boldsymbol{\mu}(t)-\beta(t)\ell(t)\nu(t)$$

It follows that $\dot{\gamma}(t_0)=0$ and $\ddot{\gamma}(t_0)=\dot{\beta}(t_0)\boldsymbol{\mu}(t_0)\neq 0$. The converse is also holded by reversing the arguments. □

Note that by Lemma 2.6 and Proposition 3.8, the conditions follows that γ is diffeomorphic to the 3/2 cusp at t_0. Hence, we can recognize the 3/2 cusp of original curve by the regularity of the evolute of a front, see Examples 6.2 and 6.3.

The most degenerate case of the evolute of a front, we have classified as follows:

Proposition 3.9. *If $\dot{\mathcal{E}v}(\gamma)(t)\equiv 0$, then γ is a part of a circle or a point.*

Proof. By the condition $\dot{\mathcal{E}v}(\gamma)(t)\equiv 0$, there exists a constant $c\in\mathbb{R}$ such that $\beta(t)/\ell(t)\equiv c$, if and only if $\beta(t)=c\ell(t)$. If $c=0$, then $\dot{\gamma}(t)=\beta(t)\boldsymbol{\mu}(t)=0$. It follows that γ is a point. Suppose that $c\neq 0$. By the existence and the uniqueness of a front in [7], we take

$$\nu(t)=\left(\cos\left(\int\ell(t)dt\right),\ \sin\left(\int\ell(t)dt\right)\right),\ \boldsymbol{\mu}(t)=\left(-\sin\left(\int\ell(t)dt\right),\ \cos\left(\int\ell(t)dt\right)\right).$$

By $\dot{\gamma}(t)=\beta(t)\boldsymbol{\mu}(t)$, we have

$$\begin{aligned}\gamma(t) &= \left(-c\int\ell(t)\sin\left(\int\ell(t)dt\right)dt+a,\ c\int\ell(t)\cos\left(\int\ell(t)dt\right)dt+b\right)\\ &= \left(c\cos\left(\int\ell(t)dt\right)+a,\ c\sin\left(\int\ell(t)dt\right)+b\right)\end{aligned}$$

for some constants $a,b\in\mathbb{R}$. Therefore, γ is a part of a circle. □

As a well-known result, a singular point of $Ev(\gamma)$ of a regular plane curve γ is corresponding to a vertex of γ, namely, $\dot{\kappa}(t)=0$ (cf. [5, 8, 18, 19]).

We extend the notion of vertex. For a Legendre immersion (γ,ν) with the curvature of the Legendre immersion (ℓ,β), t_0 is *a vertex of the front* γ (or *a Legendre immersion* (γ,ν)) if $(d/dt)(\beta/\ell)(t_0)=0$, namely, $(d/dt)\mathcal{E}v(t_0)=0$. Note that if t_0 is a regular point of γ, the definition of the vertex coincides with usual vertex for regular curves. Therefore, this is a generalization of the notion of the vertex of regular plane curves.

Remark 3.10. Let (γ,ν) be a Legendre immersion. If t_0 is a singular point of γ which degenerate more than $3/2$ cusp, then t_0 is a vertex of a front γ. In fact,

$$\frac{d}{dt}\left(\frac{\beta}{\ell}\right)(t_0)=\frac{\dot\beta(t_0)\ell(t_0)-\beta(t_0)\dot\ell(t_0)}{\ell(t_0)^2}=0,$$

since $\beta(t_0)=\dot\beta(t_0)=0$ by Proposition 3.8.

In this paper, a Legendre immersion $(\gamma,\nu):[a,b]\to\mathbb{R}^2\times S^1$ is a *closed* Legendre immersion if $(\gamma^{(n)}(a),\nu^{(n)}(a))=(\gamma^{(n)}(b),\nu^{(n)}(b))$ for all $n\in\mathbb{N}\cup\{0\}$ where $\gamma^{(n)}(a)$, $\nu^{(n)}(a)$, $\gamma^{(n)}(b)$ and $\nu^{(n)}(b)$ means one-sided differential. If $(\gamma,\nu):[a,b]\to\mathbb{R}^2\times S^1$ is a closed Legendre immersion, then both a and b are regular points or both a and b are singular points of γ. When a and b are singular points of γ, we treat these singular points as one singular point.

Proposition 3.11. *Let $(\gamma,\nu):[a,b]\to\mathbb{R}^2\times S^1$ be a closed Legendre immersion without inflection points.*

(1) *If γ has at least two singular points which degenerate more than $3/2$ cusp, then γ has at least four vertices.*

(2) *If γ has at least four singular points, then γ has at least four vertices.*

Proof. (1) Suppose that γ has at least two singular points which degenerate more than $3/2$ cusp. By Remark 3.10, these singularities are vertices of γ, therefore it is sufficient to show that there is at least one vertex between two adjacent singular points. Since γ has no inflection points, the sign of the curvature of γ on regular points is constant. Therefore, either $\lim_{t\to t_0}\kappa(t)=\infty$ for all $t_0\in\Sigma(\gamma)$ or $\lim_{t\to t_0}\kappa(t)=-\infty$ for all $t_0\in\Sigma(\gamma)$, where $\Sigma(\gamma)$ is the set of singular points of γ. This concludes there exist $t\in(t_1,t_2)$ such that $\dot\kappa(t)=0$ for singular points t_1 and t_2 of γ.

Suppose that a and b are singular points which degenerate more than $3/2$ cusp. Since we treat a and b as the one singular point, there exists at least one singular point $t_1\in(a,b)$ which degenerate more than $3/2$ cusp by the assumption. In this case, there exist at least two vertices $v_1\in(a,t_1)$ and $v_2\in(t_1,b)$. Moreover, a and t_1 are also vertices. Therefore, there exist at least four vertices.

Next, suppose that a and b are regular points or $3/2$ cusps. Then there exist at least two singular points t_1 and t_2 (we assume $t_1<t_2$) in (a,b) which degenerate more than $3/2$ cusp. In this case, there exists at least one vertex $v_1\in(t_1,t_2)$. Moreover, since (γ,ν) is closed, there exists a point $v_2\in[a,t_1)\cup(t_2,b]$ such that $\dot\kappa(v_2)=0$. Therefore, γ has at least four vertices.

(2) Suppose that γ has at least four singular points. Since γ has no inflection points, the sign of the curvature of γ on regular points is constant. Therefore, either $\lim_{t\to t_0}\kappa(t)=\infty$ for all $t_0\in\Sigma(\gamma)$ or $\lim_{t\to t_0}\kappa(t)=-\infty$ for all $t_0\in\Sigma(\gamma)$. This concludes there exist $t\in(t_1,t_2)$ such that $\dot\kappa(t)=0$, that is, there is at least one vertex between two adjacent singular points.

Suppose that a and b are singular points of γ. Since we treat a and b as the one singular point, there exist at least three singular points t_1, t_2 and t_3 of γ in (a,b), which we assume to be ordered so that $a<t_1<t_2<t_3<b$. Since there is at least one vertex between two adjacent singular points, there exist at least four vertices $v_1\in(a,t_1)$, $v_2\in(t_1,t_2)$, $v_3\in(t_2,t_3)$ and $v_4\in(t_3,b)$.

Next, suppose that a and b are regular points of γ. Let t_1, t_2, t_3 and t_4 be singular points of γ (we assume $a<t_1<t_2<t_3<t_4<b$). Since there is at least one vertex between two adjacent singular points, there exist at least three vertices $v_1\in(t_1,t_2)$, $v_2\in(t_2,t_3)$, $v_3\in(t_3,t_4)$. Moreover, since (γ,ν) is closed, there exists a point $v_4\in[a,t_1)\cup(t_4,b]$ such that $\dot\kappa(v_4)=0$. Therefore, γ has at least four vertices. □

Finally, in this section, we consider the evolute of a front as a (wave) front of a Legendre immersion by using a family of functions.

We define a family of functions

$$F : I \times \mathbb{R}^2 \to \mathbb{R}$$

by $F(t, x, y) = (\gamma(t) - (x, y)) \cdot \boldsymbol{\mu}(t)$.

Proposition 3.12. *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature of the Legendre immersion (ℓ, β).*

(1) *$F(t, x, y) = 0$ if and only if there exists a real number λ such that $(x, y) = \gamma(t) - \lambda\nu(t)$.*
(2) *$F(t, x, y) = (\partial F/\partial t)(t, x, y) = 0$ if and only if $\ell(t) \neq 0$ and $(x, y) = \gamma(t) - (\beta(t)/\ell(t))\nu(t)$.*

Proof. (1) $(\gamma(t)-(x, y))\cdot\boldsymbol{\mu}(t) = 0$ if and only if there exists $\lambda \in \mathbb{R}$ such that $\gamma(t)-(x, y) = \lambda\nu(t)$.

(2) $(\partial F/\partial t)(t, x, y) = \dot{\gamma}(t)\cdot\boldsymbol{\mu}(t)+(\gamma(t)-(x, y))\cdot\dot{\boldsymbol{\mu}}(t) = \beta(t)-\lambda\ell(t)$. If $\ell(t) = 0$, then $\beta(t) = 0$. This is a contradiction for $(\ell(t), \beta(t)) \neq (0, 0)$. It follows that $\lambda = \beta(t)/\ell(t)$. The converse is also holded. □

One can show that F is a *Morse family*, in the sense of Legendrian singularity theory (cf. [1, 14, 20]), namely, $(F, \partial F/\partial t) : I \times \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}$ is a submersion at $(t, x, y) \in D(F)$, where

$$D(F) = \{(t, x, y) \mid F(t, x, y) = (\partial F/\partial t)(t, x, y) = 0\}.$$

It follows that the evolute of a front $\mathcal{E}v(\gamma)$ is a (wave) front of a Legendre immersion and is given by the envelope of normal lines of the front.

4. Evolutes of the evolutes of fronts

By Theorem 3.3, the evolute of a front is also a front without inflection points. We consider a repeated evolute of an evolute of a front and give properties of a singular point of it. Let (γ, ν) be a Legendre immersion with the curvature of the Legendre immersion (ℓ, β) and without inflection points.

Theorem 4.1. *The evolute of an evolute of a front is given by*

$$\mathcal{E}v(\mathcal{E}v(\gamma))(t) = \mathcal{E}v(\gamma)(t) - \frac{\dot{\beta}(t)\ell(t) - \beta(t)\dot{\ell}(t)}{\ell(t)^3}\boldsymbol{\mu}(t). \tag{5}$$

Proof. At this proof, we denote $\widetilde{\gamma}(t) = \mathcal{E}v(\gamma)(t)$. By the proof of Theorem 3.3,

$$(\widetilde{\gamma}(t), \widetilde{\nu}(t)) = (\mathcal{E}v(\gamma)(t), \boldsymbol{\mu}(t))$$

is a Legendre immersion. Since $\widetilde{\boldsymbol{\mu}}(t) = J(\widetilde{\nu}(t)) = -\nu(t)$ and the derivative of the evolute of the front (4), we have

$$\widetilde{\beta}(t) = \frac{\dot{\beta}(t)\ell(t) - \beta(t)\dot{\ell}(t)}{\ell(t)^2},$$

where $\dot{\widetilde{\gamma}}(t) = \widetilde{\beta}(t)\widetilde{\boldsymbol{\mu}}(t)$. Moreover $\widetilde{\ell}(t) = \ell(t)$ by the Frenet formula of a front (1). It follows that

$$\mathcal{E}v(\mathcal{E}v(\gamma))(t) = \mathcal{E}v(\widetilde{\gamma})(t) = \widetilde{\gamma}(t) - \frac{\widetilde{\beta}(t)}{\widetilde{\ell}(t)}\widetilde{\nu}(t) = \mathcal{E}v(\gamma)(t) - \frac{\dot{\beta}(t)\ell(t) - \beta(t)\dot{\ell}(t)}{\ell(t)^3}\boldsymbol{\mu}(t).$$

□

We can also prove Theorem 4.1 by a direct calculation of the definition of the evolute of a front (Definition 2.10). We need to divide into four cases, that is, γ is a regular or a singular, and $\mathcal{E}v(\gamma)$ is a regular or a singular. All cases coincide with (5). We also call $\mathcal{E}v(\mathcal{E}v(\gamma))$ the *second evolute of a front*.

Now we consider a geometric meaning of a singular point of $\mathcal{E}v(\mathcal{E}v(\gamma))(t)$.

Lemma 4.2. *Suppose that γ and $\mathcal{E}v(\gamma)$ are both regular curves. If $\dot{\mathcal{E}v}(\mathcal{E}v(\gamma))(t) \equiv 0$, then γ is an involute of a circle.*

Proof. We may assume that t is the arc-length parameter of γ. It follows that $|\beta(t)| = 1$ and hence $\ell(t) = \kappa(t)$ by Lemma 3.1. Moreover, we have $\beta(t)^2 = 1$ and $\dot{\beta}(t) = 0$. Since $\boldsymbol{t}(t) = \beta(t)\boldsymbol{\mu}(t)$ and $\boldsymbol{n}(t) = -\beta(t)\nu(t)$, we have $\boldsymbol{\mu}(t) = \beta(t)\boldsymbol{t}(t)$ and $\nu(t) = -\beta(t)\boldsymbol{n}(t)$. Then

$$\mathcal{E}v(\gamma)(t) = \gamma(t) - \frac{\beta(t)}{\ell(t)}\nu(t) = \gamma(t) - \frac{\beta(t)}{\kappa(t)}(-\beta(t)\boldsymbol{n}(t)) = \gamma(t) + \frac{1}{\kappa(t)}\boldsymbol{n}(t)$$

and

$$\mathcal{E}v(\mathcal{E}v(\gamma))(t) = \mathcal{E}v(\gamma)(t) + \frac{\beta(t)\dot{\kappa}(t)}{\kappa(t)^3}\beta(t)\boldsymbol{t}(t) = \mathcal{E}v(\gamma)(t) + \frac{\dot{\kappa}(t)}{\kappa(t)^3}\boldsymbol{t}(t)$$

hold. It follows that

$$\dot{\mathcal{E}v}(\gamma)(t) = -\frac{\dot{\kappa}(t)}{\kappa(t)^2}\boldsymbol{n}(t),\ \dot{\mathcal{E}v}(\mathcal{E}v(\gamma))(t) = \frac{\ddot{\kappa}(t)\kappa(t) - 3\dot{\kappa}(t)^2}{\kappa(t)^4}\boldsymbol{t}(t).$$

By the assumptions, $\kappa(t) \neq 0, \dot{\kappa}(t) \neq 0$ and $\ddot{\kappa}(t)\kappa(t) - 3\dot{\kappa}(t)^2 \equiv 0$, it follows that

$$\frac{d}{dt}\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right) = 2\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right)^2.$$

Solving this differential equation, there exist constants $C_1, C_2 \in \mathbb{R}$ with $C_2 \neq 0$ such that

$$\kappa(t) = C_2\frac{1}{\sqrt{2t + C_1}}.$$

A curve having the curvature $1/\sqrt{2ct}$ for a constant $c \in \mathbb{R} \setminus \{0\}$ is an involute of a circle with radius c. By the existence and the uniqueness theorems of regular plane curves, see for example [8, 9], γ is an involute of a circle (cf. [9, P.138]). □

Let $\gamma : I \to \mathbb{R}^2$ be a regular curve and $t_0 \in I$. The *involute of a regular curve* is defined by $Inv(\gamma, t_0) : I \to \mathbb{R}^2$;

$$Inv(\gamma, t_0)(t) = \gamma(t) - \left(\int_{t_0}^{t} |\dot{\gamma}(u)|\, du\right)\boldsymbol{t}(t).$$

Note that $\mathcal{E}v(Inv(\gamma, t_0))(t) = \gamma(t)$, for more detail see [5, 8, 9].

Theorem 4.3. *Suppose that γ and $\mathcal{E}v(\gamma)$ are regular curves. If t_0 is a singular point of $\mathcal{E}v(\mathcal{E}v(\gamma))$, then γ is at least 4-th order contact to an involute of a circle at the point $t = t_0$ up to congruent.*

Proof. We may assume that t is the arc-length parameter of γ. By the same arguments in the proof of Lemma 4.2, we have $\kappa(t_0) \neq 0, \dot{\kappa}(t_0) \neq 0$ and $\ddot{\kappa}(t_0)\kappa(t_0) - 3\dot{\kappa}(t_0)^2 = 0$. We set $\kappa(t_0) = a$ and $\dot{\kappa}(t_0) = b$. Then we define a curve $\widetilde{\gamma}(t)$ whose curvature is given by

$$\widetilde{\kappa}(t) = a\sqrt{\frac{a}{b}}\frac{1}{\sqrt{-2t + 2t_0 + \frac{a}{b}}}, \quad \left(\text{respectively, } \widetilde{\kappa}(t) = a\sqrt{-\frac{a}{b}}\frac{1}{\sqrt{2t - 2t_0 - \frac{a}{b}}}\right)$$

if $ab > 0$ (respectively, $ab < 0$). Then $\kappa(t_0) = \widetilde{\kappa}(t_0) = a$ and $\dot{\kappa}(t_0) = \dot{\widetilde{\kappa}}(t_0) = b$. Since $\ddot{\kappa}(t_0)\kappa(t_0) - 3\dot{\kappa}(t_0)^2 = 0$ and $\ddot{\widetilde{\kappa}}(t)\widetilde{\kappa}(t) - 3\dot{\widetilde{\kappa}}(t)^2 \equiv 0$, we have $\ddot{\kappa}(t_0) = \ddot{\widetilde{\kappa}}(t_0)$. By the Theorem A.1 in the appendix, γ and $\widetilde{\gamma}$ are at least 4-th order contact at the point $t = t_0$ up to congruent. It follows that γ and an involute of a circle are at least 4-th order contact at the point $t = t_0$ up to congruent. This completes the proof of Theorem. □

Remark 4.4. Suppose that γ is a regular curve. If t_0 is a singular point of $\mathcal{E}v(\gamma)(t)$ and $\mathcal{E}v(\mathcal{E}v(\gamma))(t)$, then $\dot{\kappa}(t_0) = \ddot{\kappa}(t_0) = 0$ by the same calculations of the proof of Lemma 4.2. It follows that γ and the osculating circle are at least 4-th order contact at the point $t = t_0$.

Proposition 4.5. *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion without inflection points. Suppose that t_0 is a singular point of both γ and $\mathcal{E}v(\gamma)$. Then t_0 is a regular point of $\mathcal{E}v(\mathcal{E}v(\gamma))$ if and only if $\dddot{\gamma}(t_0) \neq 0$.*

Proof. Let t_0 be a regular point of $\mathcal{E}v(\mathcal{E}v(\gamma))$. By Proposition 3.8, $\beta(t_0) = \dot{\beta}(t_0) = 0$ and $\ell(t_0) \neq 0$. Since

$$\frac{d}{dt}\mathcal{E}v(\mathcal{E}v(\gamma))(t) = -\frac{\ddot{\beta}(t)\ell(t)^2 - \beta(t)\ell(t)\ddot{\ell}(t) - 3\dot{\beta}(t)\ell(t)\dot{\ell}(t) + 3\beta(t)\dot{\ell}(t)^2}{\ell(t)^4}\boldsymbol{\mu}(t),$$

it holds that $(d/dt)\mathcal{E}v(\mathcal{E}v(\gamma))(t_0) = -\ddot{\beta}(t_0)\ell(t_0)^{-2}\boldsymbol{\mu}(t_0) \neq 0$ if and only if $\ddot{\beta}(t_0) \neq 0$. By the differentiate of $\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t)$, we have

$$\dddot{\gamma}(t) = (\ddot{\beta}(t) - \beta(t)\ell(t)^2)\boldsymbol{\mu}(t) - (2\dot{\beta}(t)\ell(t) + \beta(t)\dot{\ell}(t))\nu(t)$$

It follows that $\dddot{\gamma}(t_0) = \ddot{\beta}(t_0)\boldsymbol{\mu}(t_0) \neq 0$. The converse is also shown by reversing the arguments. □

Note that by Lemma 2.6 and Proposition 4.5, the conditions follows that γ is diffeomorphic to the 4/3 cusp at t_0.

5. The n-th evolutes of fronts

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature (ℓ, β) and without inflection points. We give the form of the n-th evolute of a front, where n is a natural number. We denote $\mathcal{E}v^0(\gamma)(t) = \gamma(t)$ and $\mathcal{E}v^1(\gamma)(t) = \mathcal{E}v(\gamma)(t)$ for convenience. We define

$$\mathcal{E}v^n(\gamma)(t) = \mathcal{E}v(\mathcal{E}v^{n-1}(\gamma))(t), \quad \beta_0(t) = \beta(t), \quad \text{and} \quad \beta_n(t) = \frac{d}{dt}\left(\frac{\beta_{n-1}(t)}{\ell(t)}\right)$$

inductively.

Theorem 5.1. *$(\mathcal{E}v^n(\gamma), J^n(\nu)) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion with the curvature (ℓ, β_n), where the n-th evolute of the front is given by*

$$\mathcal{E}v^n(\gamma)(t) = \mathcal{E}v^{n-1}(\gamma)(t) - \frac{\beta_{n-1}(t)}{\ell(t)}J^{n-1}(\nu(t)),$$

where J^n is n-times operations of J.

Proof. Let $n = 1$ and $n = 2$, then

$$\mathcal{E}v^1(\gamma)(t) = \mathcal{E}v^0(\gamma)(t) - \frac{\beta_0(t)}{\ell(t)}J^0(\nu(t)) = \gamma(t) - \frac{\beta(t)}{\ell(t)}\nu(t)$$

and

$$\begin{aligned}\mathcal{E}v^2(\gamma)(t) &= \mathcal{E}v^1(\gamma)(t) - \frac{\beta_1(t)}{\ell(t)}J^1(\nu(t)) = \mathcal{E}v(\gamma)(t) - \frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right)\frac{1}{\ell(t)}J(\nu(t)) \\ &= \mathcal{E}v(\gamma)(t) - \frac{\dot{\beta}(t)\ell(t) - \beta(t)\dot{\ell}(t)}{\ell(t)^3}\boldsymbol{\mu}(t).\end{aligned}$$

These are nothing but the evolute of a front (3) and the second evolute of a front (5).

Next suppose that $1 \leq j \leq k$ is holded, namely,

$$\mathcal{E}v^j(\gamma)(t) = \mathcal{E}v^{j-1}(\gamma)(t) - \frac{\beta_{j-1}(t)}{\ell(t)} J^{j-1}(\nu(t))$$

for $1 \leq j \leq k$. We consider $\mathcal{E}v(\mathcal{E}v^k(\gamma))(t)$. Suppose that $(\mathcal{E}v^k(\gamma)(t), J^k(\nu(t)))$ is a Legendre immersion with the curvature $(\ell(t), \beta_k(t))$. By Theorem 3.3, we have $(k+1)$-th evolute of the front

$$\mathcal{E}v^{k+1}(\gamma)(t) = \mathcal{E}v^k(\gamma)(t) - \frac{\beta_k(t)}{\ell(t)} J^k(\nu(t)).$$

Since

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}v^{k+1}(\gamma)(t) &= \frac{d}{dt}\mathcal{E}v^k(\gamma)(t) - \frac{d}{dt}\left(\frac{\beta_k(t)}{\ell(t)}\right) J^k(\nu(t)) - \frac{\beta_k(t)}{\ell(t)} J^k(\dot{\nu}(t)) \\
&= \beta_k(t) J^{k+1}(t) + \beta_{k+1}(t) J^{k+2}(\nu(t)) - \beta_k(t) J^{k+1}(\nu(t)) \\
&= \beta_{k+1}(t) J^{k+2}(\nu(t)), \\
\frac{d}{dt} J^{k+1}(\nu(t)) &= J^{k+1}(\dot{\nu}(t)) = J^{k+1}(\ell(t)\boldsymbol{\mu}(t)) = \ell(t) J^{k+1}(J(\nu(t))) \\
&= \ell(t) J^{k+2}(\nu(t)),
\end{aligned}$$

it holds that $(\mathcal{E}v^{k+1}(\gamma), J^{k+1}(\nu))$ is a Legendre immersion with the curvature $(\ell(t), \beta_{k+1}(t))$. By the induction, this completes the proof of Theorem. □

As a generalization of Propositions 3.8 and 4.5, we have the following result:

Proposition 5.2. *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature of the Legendre immersion (ℓ, β) and without inflection points. Suppose that t_0 is a singular point of γ. Then the following are equivalent:*
(1) *t_0 is a singular point of $\mathcal{E}v^i(\gamma)(t)$ for $i = 1, \dots, n$.*
(2) *$(d^i\beta/dt^i)(t_0) = 0$ for $i = 1, \dots, n$.*
(3) *$(d^i\gamma/dt^i)(t_0) = 0$ for $i = 2, \dots, n+1$.*

Proof. First, we show that $\beta_i(t)$ is given by the form $\beta^{(i)}(t)$ and lower terms of $\beta^{(i)}(t)$, namely,

$$\beta_i(t) = \frac{\beta^{(i)}(t)}{\ell(t)^i} + L(\beta(t), \dots, \beta^{(i-1)}(t)) \tag{6}$$

for some smooth function L which contain $\ell(t)$ and derivatives of $\ell(t)$.

Since

$$\beta_1(t) = \frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right) = \frac{\dot{\beta}(t)}{\ell(t)} + \beta(t)\frac{d}{dt}\left(\frac{1}{\ell(t)}\right),$$

the case of $i = 1$ is holded. Suppose that $i = k$ is holded, namely, there exists a smooth function L such that

$$\beta_k(t) = \frac{\beta^{(k)}(t)}{\ell(t)^k} + L(\beta(t), \dots, \beta^{(k-1)}(t)).$$

Then

$$\beta_{k+1}(t) = \frac{d}{dt}\left(\frac{\beta_k(t)}{\ell(t)}\right) = \frac{\beta^{(k+1)}(t)}{\ell(t)^{k+1}} + \widetilde{L}(\beta(t), \dots, \beta^{(k)}(t)),$$

for some smooth function $\widetilde{L}$. By the induction, we conclude the assertion.

Second, assume that t_0 is a singular point of $\mathcal{E}v^i(\gamma)(t)$ for $i = 1, \dots, n$. By Theorem 5.1, $(d/dt)\mathcal{E}v^i(\gamma)(t_0) = 0$ if and only if $\beta_i(t_0) = 0$. Since (6) and $\beta(t_0) = 0$, it holds that $\beta_i(t_0) = 0$ for $i = 1, \dots, n$ if and only if $\beta^{(i)}(t_0) = 0$ for $i = 1, \dots, n$. It follows that (1) implies (2). By the reversing arguments, the converse (1) follows from (2).

Finally, since $\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t)$, we can also show that (2) is equivalent to (3) by the induction.
$\square$

6. Examples

We give examples to understand the phenomena for evolutes of fronts.

Example 6.1. Let $\gamma(t) = (a\cos t, b\sin t)$ be an ellipse with $a, b > 0$ and $a \neq b$. Since

$$\nu(t) = \frac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}(-b\cos t, a\sin t),\ \boldsymbol{\mu}(t) = \frac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}(-a\sin t, -b\cos t),$$

we have

$$\ell(t) = \frac{ab}{a^2\sin^2 t + b^2\cos^2 t},\ \beta(t) = -\sqrt{a^2\sin^2 t + b^2\cos^2 t}.$$

The evolute, the second evolute and the third evolute of the ellipse γ are given by

$$\begin{aligned}
\mathcal{E}v(\gamma)(t) &= \left(\frac{a^2-b^2}{a}\cos^3 t, -\frac{a^2-b^2}{b}\sin^3 t\right), \\
\mathcal{E}v(\mathcal{E}v(\gamma))(t) &= \Big(\frac{a^2-b^2}{ab^2}\cos t\left(b^2\cos^4 t + 3a^2\sin^4 t + b^2\sin^2 2t\right), \\
&\quad -\frac{a^2-b^2}{a^2 b}\sin t\left(a^2\sin^4 t + 3b^2\cos^4 t + a^2\sin^2 2t\right)\Big),
\end{aligned}$$

and $\mathcal{E}v^3(\gamma)(t) =$

$$\begin{aligned}
&\Big(\frac{a^2-b^2}{8a^3b^2}\cos^3 t\left(45a^4 - 10a^2b^2 - 3b^4 + 12(-5a^4 + 4a^2b^2 + b^4)\cos 2t + 15(a^2-b^2)^2\cos 4t\right), \\
&\quad \frac{a^2-b^2}{8a^2b^3}\sin^3 t\left(3a^4 + 10a^2b^2 - 45b^4 + 12(a^4 + 4a^2b^2 - 5b^4)\cos 2t - 15(a^2-b^2)^2\cos 4t\right)\Big).
\end{aligned}$$

The ellipse γ and its evolute (red curve) are showed in Figures 1 left and 2 center. Moreover, the second evolute (yellow curve), see Figure 1 center, and the third evolute (green curve), see Figures 1 right and 2 right.

The evolute is useful to recognize the difference of the sharp of curves. In Figure 2, the left is a circle and the center is an ellipse and its evolute. We can observe the evolute of the ellipse, however, it is very small (red curve). If we consider the repeated evolute, we can easy to observe it. The right in Figure 2 is the second and the third evolute of the ellipse.

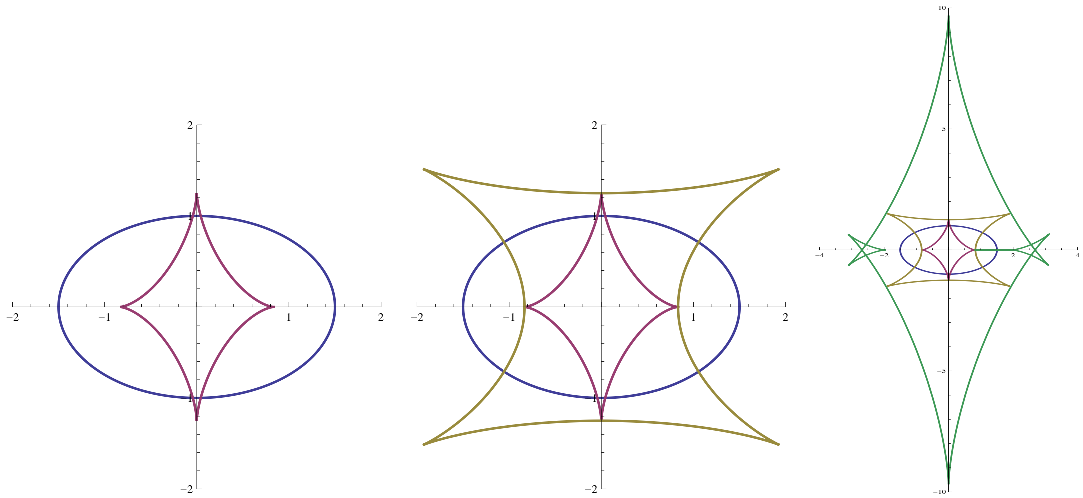

Figure 1. The ellipse and evolutes.

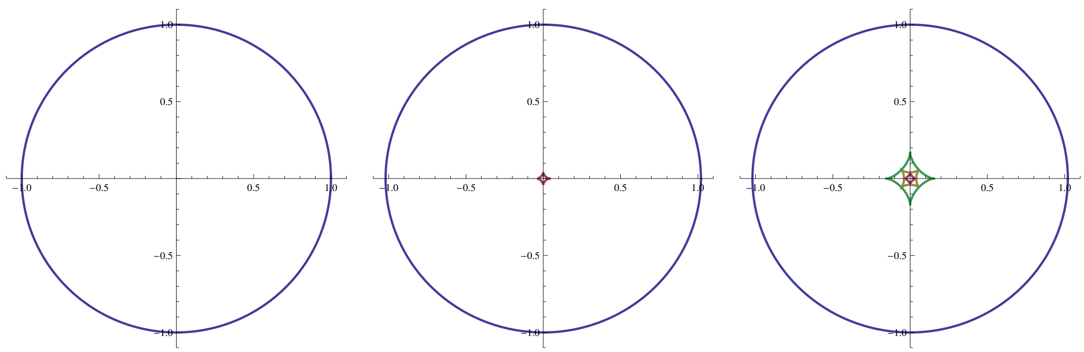

Figure 2.

Example 6.2. Let $\gamma(t) = (3\cos t - \cos 3t, 3\sin t - \sin 3t) = (6\cos t - 4\cos^3 t, 4\sin^3 t)$ be the nephroid, see Figure 3 left. Since $\nu(t) = (-\sin 2t, \cos 2t)$ and $\boldsymbol{\mu}(t) = (-\cos 2t, \sin 2t)$, we have $\ell(t) = 2, \beta(t) = -6\sin t$. The evolute and the second evolute of the nephroid are as follows, see Figure 3 center and right:

$$
\begin{aligned}
\mathcal{E}v(\gamma)(t) &= \left(2\cos^3 t, 3\sin t - 2\sin^2 t\right), \\
\mathcal{E}v(\mathcal{E}v(\gamma))(t) &= \left(\frac{3}{2}\cos t - \cos^3 t, \sin^3 t\right).
\end{aligned}
$$

We can observe that $\gamma(t)/4 = \mathcal{E}v(\mathcal{E}v(\gamma))(t)$.

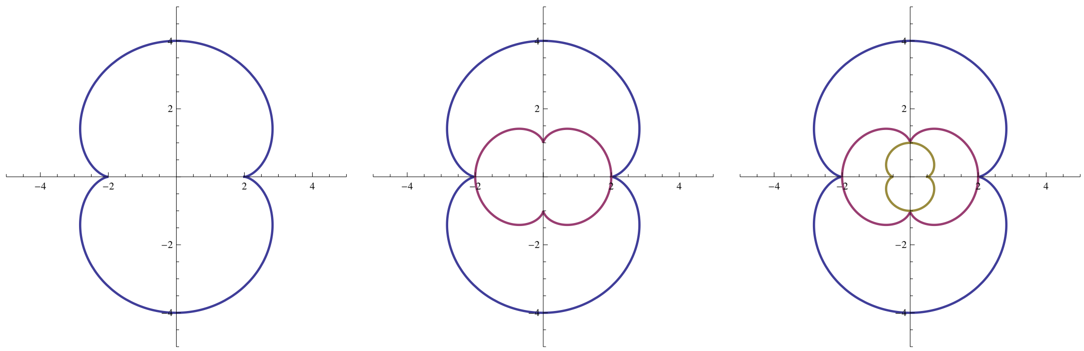

Figure 3. The nephroid and evolutes.

Example 6.3. Let $\gamma(t) = (t^3, t^4)$ be the 4/3 cusp, Figure 4 left. Since $\nu(t) = (1/\sqrt{16t^2+9})(-4t, 3)$ and $\boldsymbol{\mu}(t) = (1/\sqrt{16t^2+9})(-3, -4t)$, we have $\ell(t) = 12/(16t^2+9), \beta(t) = -t^2\sqrt{16t^2+9}$. The evolute and the second evolute of the 4/3 cusp are as follows, see Figure 4 center and right:

$$\begin{aligned} \mathcal{E}v(\gamma)(t) &= \left(-2t^3 - \frac{16}{3}t^5, \frac{9}{4}t^2 + 5t^4\right), \\ \mathcal{E}v(\mathcal{E}v(\gamma))(t) &= \left(-\frac{27}{8}t - 23t^3 - 32t^5, -\frac{9}{4}t^2 - 23t^4 - \frac{320}{9}t^6\right). \end{aligned}$$

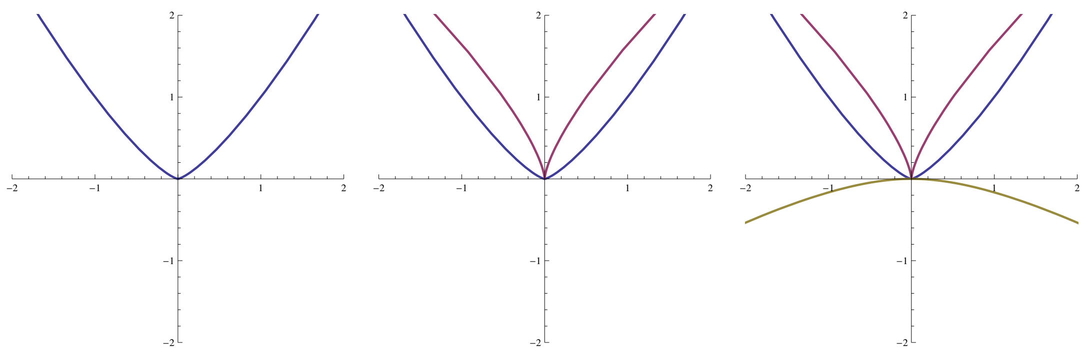

Figure 4. The 4/3 cusp and evolutes.

Appendix A. Contact between regular curves

In this appendix, we discuss contact between regular curves. Let $\gamma : I \to \mathbb{R}^2; t \mapsto \gamma(t)$ and $\widetilde{\gamma} : \widetilde{I} \to \mathbb{R}^2; u \mapsto \widetilde{\gamma}(u)$ be regular plane curves, respectively. We say that γ and $\widetilde{\gamma}$ have *k-th order contact at $t = t_0, u = u_0$* if

$$\gamma(t_0) = \widetilde{\gamma}(u_0),\ \frac{d\gamma}{dt}(t_0) = \frac{d\widetilde{\gamma}}{du}(u_0),\ \cdots,\ \frac{d^k\gamma}{dt^k}(t_0) = \frac{d^k\widetilde{\gamma}}{du^k}(u_0),\ \frac{d^{k+1}\gamma}{dt^{k+1}}(t_0) \neq \frac{d^{k+1}\widetilde{\gamma}}{du^{k+1}}(u_0).$$

Moreover, we say that γ and $\widetilde{\gamma}$ have *at least k-th order contact at $t = t_0, u = u_0$* if

$$\gamma(t_0) = \widetilde{\gamma}(u_0),\ \frac{d\gamma}{dt}(t_0) = \frac{d\widetilde{\gamma}}{du}(u_0),\ \cdots,\ \frac{d^k\gamma}{dt^k}(t_0) = \frac{d^k\widetilde{\gamma}}{du^k}(u_0).$$

Let $\gamma_1, \gamma_2 : I \to \mathbb{R}^2$ be regular plane curves. We say that γ_1 and γ_2 are *congruent* if there exists a congruence C such that $\gamma_2(t) = C(\gamma_1(t)) = A(\gamma_1(t)) + \boldsymbol{b}$ for all $t \in I$, where the congruence is given by a rotation A and a translation $\boldsymbol{b}$ on $\mathbb{R}^2$.

Let $\gamma : I \to \mathbb{R}^2; t \mapsto \gamma(t)$ and $\widetilde{\gamma} : \widetilde{I} \to \mathbb{R}^2; u \mapsto \widetilde{\gamma}(u)$ be regular plane curves. We take the arc-length parameter for $\gamma(t)$ and $\widetilde{\gamma}(u)$, respectively. In general, we may assume that $\gamma(t)$ and

$\widetilde{\gamma}(u)$ have at least first order contact at any point $t = t_0, u = u_0$ up to congruent. We denote the curvatures $\kappa(t)$ of $\gamma(t)$ and $\widetilde{\kappa}(u)$ of $\widetilde{\gamma}(u)$, respectively.

Theorem A.1. *Let $\gamma : I \to \mathbb{R}^2$ and $\widetilde{\gamma} : \widetilde{I} \to \mathbb{R}^2$ be regular plane curves. If $\gamma(t)$ and $\widetilde{\gamma}(u)$ have at least $(k+2)$-th order contact at $t = t_0, u = u_0$ then*

$$\kappa(t_0) = \widetilde{\kappa}(u_0),\ \frac{d\kappa}{dt}(t_0) = \frac{d\widetilde{\kappa}}{du}(u_0),\ \cdots,\ \frac{d^k\kappa}{dt^k}(t_0) = \frac{d^k\widetilde{\kappa}}{du^k}(u_0). \tag{7}$$

Conversely, if t and u are the arc-length parameter of γ and $\widetilde{\gamma}$ respectively, and the condition (7) holds, then γ and $\widetilde{\gamma}$ have at least $(k+2)$-th order contact at $t = t_0, u = u_0$ up to congruent.

Proof. We may assume that t and u are the arc-length parameter of γ and $\widetilde{\gamma}$ respectively. Suppose that γ and $\widetilde{\gamma}$ have at least third order contact. Since the Frenet formula, we have $(d\gamma/dt)(t) = \boldsymbol{t}(t), (d^2\gamma/dt^2)(t) = \kappa(t)\boldsymbol{n}(t)$ and $(d\widetilde{\gamma}/du)(u) = \widetilde{\boldsymbol{t}}(u), (d^2\widetilde{\gamma}/du^2)(u) = \widetilde{\kappa}(u)\widetilde{\boldsymbol{n}}(u)$. It follows that $\boldsymbol{t}(t_0) = \widetilde{\boldsymbol{t}}(u_0), \boldsymbol{n}(t_0) = \widetilde{\boldsymbol{n}}(u_0)$ and $\kappa(t_0) = \widetilde{\kappa}(u_0)$. Hence, the case of $k = 1$ holds.

Suppose that γ and $\widetilde{\gamma}$ have at least $(k+2)$-th order contact and

$$\kappa(t_0) = \widetilde{\kappa}(u_0),\ \frac{d\kappa}{dt}(t_0) = \frac{d\widetilde{\kappa}}{du}(u_0), \cdots,\ \frac{d^{k-1}\kappa}{dt^{k-1}}(t_0) = \frac{d^{k-1}\widetilde{\kappa}}{du^{k-1}}(u_0)$$

hold. Since $(d^3\gamma/dt^3)(t) = (d\kappa/dt)(t)\boldsymbol{n}(t) - \kappa(t)^2\boldsymbol{t}(t)$, the form of $(d^{k+1}\gamma/dt^{k+1})(t)$ is given by

$$\frac{d^{k-1}\kappa}{dt^{k-1}}(t)\boldsymbol{n}(t) + f\left(\kappa(t), \cdots, \frac{d^{k-2}\kappa}{dt^{k-2}}(t)\right)\boldsymbol{t}(t) + g\left(\kappa(t), \cdots, \frac{d^{k-2}\kappa}{dt^{k-2}}(t)\right)\boldsymbol{n}(t),$$

for some smooth functions f and g. Then

$$\frac{d^{k+2}\gamma}{dt^{k+2}}(t) = \frac{d^k\kappa}{dt^k}(t)\boldsymbol{n}(t) + F\left(\kappa(t), \cdots, \frac{d^{k-1}\kappa}{dt^{k-1}}(t)\right)\boldsymbol{t}(t) + G\left(\kappa(t), \cdots, \frac{d^{k-1}\kappa}{dt^{k-1}}(t)\right)\boldsymbol{n}(t)$$

for some smooth functions F and G. By the same calculations, we have

$$\frac{d^{k+2}\widetilde{\gamma}}{du^{k+2}}(u) = \frac{d^k\widetilde{\kappa}}{du^k}(u)\widetilde{\boldsymbol{n}}(u) + F\left(\widetilde{\kappa}(u), \cdots, \frac{d^{k-1}\widetilde{\kappa}}{du^{k-1}}(u)\right)\widetilde{\boldsymbol{t}}(u) + G\left(\widetilde{\kappa}(u), \cdots, \frac{d^{k-1}\widetilde{\kappa}}{du^{k-1}}(u)\right)\widetilde{\boldsymbol{n}}(u).$$

It follows that $(d^k\kappa/dt^k)(t_0) = (d^k\widetilde{\kappa}/du^k)(u_0)$. By the induction, we have the first assertion.

By the reversing arguments, we can prove the converse assertion up to congruent. □

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