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ABELIAN SINGULARITIES OF HOLOMORPHIC LIE-FOLIATIONS

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Abstract. We study holomorphic foliations with generic singularities and Lie group transverse structure outside of some invariant codimension one analytic subset. We introduce the concept of abelian singularity and prove that, for this type of singularities, the foliation is logarithmic. The Lie transverse structure is then used to extend the local (logarithmic) normal form from a neighborhood of the singularity, to the whole manifold.

1. Introduction

Foliations with Lie transverse structure are among the simplest constructive examples of foliations. They are however a natural object when one considers the possible applications of the theory of foliations in the classification of manifolds and dynamical systems. By a foliation with a Lie group transverse structure we mean a foliation that is given by an atlas of submersions taking values on a given Lie group G and with transition maps given by restrictions of left-translations on the group G. Such a foliation will be called a *G-foliation*. The theory of G-foliations is a well-developed subject and follows the original work of Blumenthal [2].

In the present work we study the possible Lie transverse structures associated to holomorphic foliations with singularities. This study initiated in [6] where we prove that a one-dimensional holomorphic foliation with generic singularities in dimension 3 and having a Lie transverse structure, outside of some analytic invariant subset of codimension one, is logarithmic.

As a consequence of our results, we conclude that, in dimension two, *the presence of generic singularities forces the transverse structure to be abelian.* The exact sense of the term generic is given below. We stress that our results are first steps in the comprehension of the possible Lie group for holomorphic foliations with singularities.

Abelian singularities. Let $\mathcal{F}$ be a germ of a one-dimensional foliation at the origin $0 \in \mathbb{C}^m$. We recall that $\mathcal{F}$ is *linearizable without resonances* if it is given in some neighborhood U of $0 \in \mathbb{C}^m$ by a holomorphic vector field X which is linearizable as

$$X = \sum_{j=1}^{m} \lambda_j z_j \frac{\partial}{\partial z_j}, \tag{1}$$

with eigenvalues $\lambda_1, \cdots, \lambda_m$ satisfying the following non-resonance hypothesis:
If $n_1, \cdots, n_m \in \mathbb{Z}$ are such that

$$\sum_{j=1}^{m} n_j \lambda_j = 0,$$

then $n_1 = n_2 = \cdots = n_m = 0$.

Now we consider a $(m-r)$-dimensional holomorphic foliation with singularities $\mathcal{F}$ in a connected open subset $V \subset \mathbb{C}^m$. Denote by $\operatorname{sing}(\mathcal{F}) \subset V$ the singular set of $\mathcal{F}$. The following definition is motivated by the two dimensional case (cf. Proposition 1):

Definition 1 (abelian singularity)**.** A $(m-r)$-dimensional singularity $p \in \text{sing}(\mathcal{F}) \subset \mathbb{C}^m$ is said to be *abelian* if $\mathcal{F}$ is given by a system of *commuting* vector fields $X_1, \cdots, X_{m-r}$ defined in a neighborhood U of p such that $X_1, \cdots, X_{m-r}$ vanish at p and are linearly independent off $\text{sing}(\mathcal{F}) \cap U$. The singularity $p \in \text{sing}(\mathcal{F})$ is *generic* if we can choose the system above such that:

(i) Each vector field is of the form $X_k = \sum\limits_{j=1}^{m} \lambda_j^k z_j \frac{\partial}{\partial z_j} + \text{h. o. t.}$.

(ii) The $m \times (m-r)$ matrix $A = (\lambda_j^k)$, where $j = 1, ..., m$ and $k = 1, ..., m-r$, is *nonresonant* in the following sense: the set of its $(m-r) \times (m-r)$ minor determinants is linearly independent over the integer numbers.

(iii) Some vector field X_j is *nonresonant* and analytically linearizable at the origin.

Remark 1. Regarding the notions above we have:

(1) A germ of a singular holomorphic vector field X at the origin $0 \in \mathbb{C}^m$ is in the *Poincaré domain* if the convex hull of its eigenvalues does not the origin $0 \in \mathbb{C}$. Otherwise it is in the *Siegel domain*. The so called Poincaré-Dulac theorem states that a Poincaré type singularity is analytically linearizable in the nonresonant case ([1]). In the generic case, a nonresonant Siegel type singularity is also linearizable ([5]).

(2) If $\mathcal{F}$ has dimension one then the singularity is generic if and only if it is generated by a generic vector field.

In this paper we consider the case where $\mathcal{F}$ has a G-transverse structure outside of some analytic codimension one subset Λ such that each irreducible component of Λ contains the origin $0 \in \mathbb{C}^m$. In this case, thanks to the linearization hypothesis, it is natural to assume that the germ of such a subset Λ at the origin is the germ of a union of coordinate hyperplanes.

A codimension r holomorphic foliation with singularities in a complex manifold V is *logarithmic* if it is given by a system of closed meromorphic one-forms with simple poles $\{\omega_1, ..., \omega_r\}$ in V. In this paper we prove:

Theorem 1. *Let $\mathcal{F}$ be a holomorphic foliation defined in an open connected neighborhood V of the origin $0 \in \mathbb{C}^m$, such that $\mathcal{F}$ has an abelian generic singularity at the origin. Assume that $\mathcal{F}$ has a G-transverse structure outside of some invariant codimension one analytic subset $\Lambda \subset V$, such that each irreducible component of Λ contains the origin. Then $\mathcal{F}$ is a logarithmic foliation.*

Remark 2. Theorem 1 contains the case of dimension two foliations (cf. Proposition 1) and of codimension one foliations (cf. [3]). We highlight the fact that the conclusion of Theorem 1 states that the foliation is logarithmic in the whole manifold V. From Lemma 1 we will see that the germ of singularity induced by the foliation at the origin, is already a germ of a logarithmic foliation. Thus, the main role of the Lie transverse structure is to extend this local (logarithmic) normal form from a neighborhood of the origin, to the manifold V.

2. Generic abelian singularities

In what follows we motivate and prove some results about the notion of abelian singularity. The next proposition motivates our approach.

Proposition 1. *Let $\{A_1, A_2\}$ be an integrable system of linear vector fields on $\mathbb{C}^m$. Assume that A_1 and A_2 are nonresonant. Then A_1 and A_2 commute. Indeed, A_1 and A_2 are simultaneously diagonalizable.*

Proof. Write $A = A_1 = (f_{ij})_{i,j=1}^m$. By hypothesis A_2 is nonresonant and therefore diagonalizable. Thus we may assume that A_2 is in the diagonal form D with eigenvalues $d_1, ..., d_m$. Also by hypothesis $[A, D] = c_1 A + c_2 D$, for some holomorphic functions c_1, c_2 defined in a neighborhood of the origin $0 \in \mathbb{C}^m$.

$$AD = \begin{pmatrix} f_{11}d_1 & f_{12}d_2 & \dots & f_{1n}d_m \\ f_{12}d_1 & f_{22}d_2 & \dots & f_{2n}d_m \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}d_1 & f_{n2}d_2 & \dots & f_{nn}d_m \end{pmatrix}$$

and

$$DA = \begin{pmatrix} f_{11}d_1 & f_{12}d_1 & \dots & f_{1n}d_1 \\ f_{21}d_2 & f_{22}d_2 & \dots & f_{2n}d_2 \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}d_m & f_{n2}d_m & \dots & f_{nn}d_m \end{pmatrix}$$

and

$$AD - DA = \begin{pmatrix} 0 & f_{12}(d_2 - d_1) & \dots & f_{1n}(d_m - d_1) \\ f_{21}(d_1 - d_2) & 0 & \dots & f_{2n}(d_m - d_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(d_1 - d_m) & f_{n2}(d_2 - d_m) & \dots & 0 \end{pmatrix}.$$

On the other hand

$$c_1 A + c_2 D = \begin{pmatrix} c_1 f_{11} + c_2 d_1 & c_1 f_{12} & \dots & c_1 f_{1n} \\ c_1 f_{21} & c_1 f_{22} + c_2 d_2 & \dots & c_1 f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 f_{n1} & c_1 f_{n2} & \dots & c_1 f_{nn} + c_2 d_m \end{pmatrix}.$$

From $AD - DA = c_1 A + c_2 D$ we obtain:

$$c_1 f_{ij} = f_{ij}(d_j - d_i),\ c_1 f_{ji} = f_{ji}(d_i - d_j)$$

Assume $f_{ij} \neq 0$ for some i, j. Then $c_1 = d_j - d_i$. Notice that if also $f_{ji} \neq 0$ then $c_1 = d_i - d_j$ and therefore $d_i = d_j$, contradiction. Therefore, $f_{ij} \neq 0 \implies f_{ji} = 0$.

Given now an index $k \in \{1, ..., n\}$, as before we have $f_{ik} = 0$ or $f_{ki} = 0$. If $f_{ik} \neq 0$ we get $c_1 = d_k - d_i$ and therefore $d_k - d_i = d_i - d_j$ and thus $d_k - 2d_i + d_j = 0$, contradiction. If $f_{ki} \neq 0$ then $c_1 = d_i - d_k$ and thus $d_i - d_j = d_i - d_k$, that implies $d_j = d_k$, again a contradiction provided that $k \neq j$. We conclude that $f_{ik} = 0$ for all $k \neq i$ and $f_{ki} = 0$, $\forall k \neq j$. This means that, except for the elements f_{ii} on the diagonal of A, at most one element f_{ij} is different from zero. Since by hypothesis $A_1 = A$ is also nonresonant and diagonalizable, we conclude that A is also in the diagonal form and therefore A and D commute. □

A germ of a codimension one holomorphic foliation singularity at the origin is given in a neighborhood V of the origin $0 \in \mathbb{C}^m$ by an integrable holomorphic one-form ω. We can write $\omega = \omega_\nu + \omega_{\nu+1} + \cdots$ as a sum of homogeneous one-forms, where ω_ν is the first nonzero jet of ω. According to Cerveau-Mattei [3], under *generic* conditions on the coefficients of ω_ν, the foliation is given by an integrable system of $n-1$ commuting vector fields, all of them with non-degenerate linear part at the origin. By generic we mean, for an open dense Zariski subset of the affine space of coefficients of ω_ν (see [3] in a more precise description).

In general, abelian singularities are *linearizable*, i.e., defined by simultaneously linearizable commuting vector fields, as the following proposition shows:

Proposition 2. *An abelian singularity is analytically linearizable provided that it is defined by commuting vector fields one of which has an analytically linearizable nonresonant singularity.*

Proof. It is enough to prove that given two commuting vector fields X and Y in a neighborhood U of $0 \in \mathbb{C}^m$, and such that X has an analytically linearizable nonresonant singularity at $0 \in \mathbb{C}^m$ then X and Y are simultaneously linearizable in a neighborhood of the origin. In fact, in a suitable local chart $x = (x_1, \cdots x_m)$ we have

$$X = \sum_{j=1}^{m} \lambda_j x_j \frac{\partial}{\partial x_j}, \qquad Y = \sum_{i=1}^{m} b_i(x) \frac{\partial}{\partial x_i}, \qquad [X,Y] = \sum_{i=1}^{m} \left(\sum_{j=1}^{m} \lambda_j x_j \frac{\partial b_i(x)}{\partial x_j} - \lambda_i b_i(x) \right) \frac{\partial}{\partial x_i}.$$

Since $[X,Y] = 0$ we get

$$\sum_{j=1}^{m} \lambda_j x_j \frac{\partial b_i(x)}{\partial x_j} - \lambda_i b_i(x) = 0,$$

for $i = 1, 2, \cdots, m$.

We write b_i in its Laurent series expansion in the variable x

$$b_i = \sum_{|(l_1,\ldots,l_m)| \neq 0} b^i_{l_1 \cdots l_m} x_1^{l_1} \cdots x_m^{l_m}$$

$$x_j \frac{\partial b_i}{\partial x_j} = \sum_{|(l_1,\ldots,l_m)| \neq 0} l_j b^i_{l_1 \cdots l_m} x^{l_1} \cdots x^{l_m}.$$

By hypothesis X is nonresonant. Therefore $\sum_{j=1}^{m} l_j \lambda_j - \lambda_i \neq 0$ and $Y = \sum_{j=1}^{m} \mu_j x_j \frac{\partial}{\partial x_j}$. □

Let now X be a linearizable vector field in neighborhood U of the origin where X can be written as in (1). We may introduce closed meromorphic one-forms $\omega_1, \cdots, \omega_{m-1}$ on U, linearly independent and holomorphic on $U \setminus \Lambda$, and such that $\omega_l(X) = 0, l = 1, ..., m-1$ by

$$\omega_l = \sum_{j=1}^{m} \alpha_j^l \frac{dz_j}{z_j} \tag{2}$$

where $l = 1, \cdots, m-1$ and the vectors $\vec{\alpha_l} := (\alpha_1^l, ..., \alpha_m^l) \in \mathbb{C}^m$ are suitably chosen in the hyperplane $z_1\lambda_1 + ... + z_m\lambda_m = 0$ in $\mathbb{C}^m$. We extend this fact by defining a *nonresonant linearizable abelian singularity* as an abelian singularity which is defined by $m-r$ simultaneously analytically linearizable nonresonant vector fields. Using this we prove:

Lemma 1. *A nonresonant linearizable abelian singularity is a germ of a logarithmic singularity.*

Proof. In fact, the singularity is given by a system of vector fields $X_k(y) = A_k y$, where $A_k \in \mathrm{GL}(m, \mathbb{C})$ is a diagonal matrix for each $k = 1, 2, \cdots, m-r$. If

$$A_k = \begin{pmatrix} \lambda_1^k & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & \lambda_m^k \end{pmatrix}$$

we define r one-forms $\omega_1, \cdots, \omega_r$ on $U \setminus \Lambda$ as in (2). Condition $\omega_l(X_k) = 0$ is equivalent to the following system of equations

$$\sum_{j=1}^{m} \alpha_j^l \lambda_j^k = 0, \quad l = 1, \cdots, r. \tag{3}$$

Set $\vec{\lambda}_k = \left(\lambda_1^k, \cdots, \lambda_m^k\right) \in \mathbb{C}^m$ and let $P_k \subset \mathbb{C}^m$ be the hyperplane given by

$$P_k = \left\{ (z_1, \cdots, z_m) \in \mathbb{C}^m \quad : \quad \sum_{j=1}^{m} \lambda_j^k z_j = 0 \right\}.$$

Then (3) is equivalent to $\vec{\lambda}_k \in P_k$. Because the vector fields X_k are linearly independent off the singular set of the foliation, which is of codimension ≥ 2, the vectors $\vec{\lambda}_1 \cdots, \vec{\lambda}_{m-r}$ are linearly independent in $\mathbb{C}^m$ and therefore the hyperplanes $P_1, \cdots, P_{m-r}$ intersect transversely at a linear subspace $Q = P_1 \cap \cdots \cap P_r \subset \mathbb{C}^m$ of dimension $m - r$. Since $\dim(Q) = m - r$, we can choose linearly independent vectors $\vec{\alpha_l} := (\alpha_1^l, ..., \alpha_m^l) \in \mathbb{C}^m$, $l = 1, ..., r$ so that the corresponding one-forms $\omega_1, ..., \omega_r$ defined by $\omega_l = \sum_{j=1}^m \alpha_j^l \frac{dz_j}{z_j}$ satisfy $\omega_l(X_k) = 0$ and the system $\{\omega_1, \cdots, \omega_r\}$ has maximal rank outside the set $\{\omega_1 \wedge \cdots \wedge \omega_r = 0\}$. Therefore $\mathcal{F}$ is logarithmic. □

3. Proof of Theorem 1

In this section we prove Theorem 1. The starting point in our study is the following characterization of G-foliations given by the classical theorem of Darboux-Lie ([2, 4]):

Darboux-Lie theorem. *Let $\mathcal{F}$ be a G-foliation on V. Then there are one-forms $\theta_1, ..., \theta_r$ in V such that: $\{\theta_1, ..., \theta_r\}$ is a rank r integrable system which defines $\mathcal{F}$ and the forms satisfy the Maurer-Cartan equation:*

$$d\theta_i = \sum_{j,k} c_{jk}^i \theta_j \wedge \theta_k. \tag{4}$$

The numbers $\{c_{ij}^k\}$ are the structure constants of a Lie algebra basis of G.
If $\mathcal{F}, V$ and G are complex (holomorphic) then the θ_j can be taken holomorphic.

The proof of Theorem 1 is also based on the following two lemmas:

Lemma 2. *Let $\{\omega_1, ..., \omega_r\}$ be a maximal rank system of logarithmic one-forms, say*

$$\omega_l = \sum_{j=1}^{m} \alpha_j^l \frac{dz_j}{z_j},$$

defined in an open connected neighborhood U of the origin $0 \in \mathbb{C}^m$. Assume that the coefficients matrix $B = (\alpha_j^l)_{j,l}$ is nonresonant in the following sense: the set of its $(m-r) \times (m-r)$ minor determinants is linearly independent over the integer numbers. Let $f : U \to \mathbb{C}$ be a holomorphic function such that $df \wedge \omega_1 \wedge ... \wedge \omega_r = 0$ in U. Then f is constant in U.

Proof. We have $\omega_1 \wedge ... \wedge \omega_r = \sum\limits_{j_1,...,j_r} \alpha_{j_1}^1 ... \alpha_{j_r}^r \frac{dz_{j_1} \wedge ... \wedge dz_{j_r}}{z_{j_1} ... z_{j_r}} = \sum\limits_{j_1 < ... < j_r} \Delta(j_1, ..., j_r) \frac{dz_{j_1} \wedge ... \wedge dz_{j_r}}{z_{j_1} ... z_{j_r}}$ where $\Delta(j_1, ..., j_r)$ is the $r \times r$ minor determinant of the matriz $A = (\alpha_j^l)_{j,l}$ obtained by considering the lines $j_1 < ... < j_r$.

Write $f(z_1,...,z_m) = \sum_{i_1,...,i_m} f_{i_1,...,i_m} z_1^{i_1}...z_m^{i_m}$. Then

$$df = \sum_{\ell=1}^{m} \sum_{i_1,...,i_m} i_\ell f_{i_1,...,i_m} z_1^{i_1}..z_\ell^{i_\ell-1}....z_m^{i_m} dz_\ell.$$

Therefore

$$df \wedge \omega_1 \wedge ... \wedge \omega_r = \sum_{\ell=1}^{m} \sum_{i_1,...,i_m} i_\ell f_{i_1,...,i_m} z_1^{i_1}..z_\ell^{i_\ell-1}....z_m^{i_m} dz_\ell \wedge \sum_{j_1<...<j_r} \Delta(j_1,...,j_r) \frac{dz_{j_1} \wedge ... \wedge dz_{j_r}}{z_{j_1}...z_{j_r}}$$

and then

$$df \wedge \omega_1 \wedge ... \wedge \omega_r = \sum_{\ell=1}^{m} \sum_{i_1,...,i_m} \sum_{j_1<...<j_r} i_\ell f_{i_1,...,i_m} \Delta(j_1,...,j_r)\, z_1^{i_1}..z_\ell^{i_\ell}....z_m^{i_m} \frac{dz_\ell}{z_\ell} \wedge \frac{dz_{j_1} \wedge ... \wedge dz_{j_r}}{z_{j_1}...z_{j_r}}$$

$$df \wedge \omega_1 \wedge ... \wedge \omega_r = \sum_{i_1,...,i_m} \sum_{j_1<...<j_r,\, \ell=1}^{\ell=m} (-1)^\ell i_\ell \Delta(j_1,...,j_r)\, f_{i_1,...,i_m}\, z_1^{i_1}..z_\ell^{i_\ell}....z_m^{i_m} \frac{dz_{j_1} \wedge ... \wedge dz_{j_r} \wedge dz_\ell}{z_{j_1}...z_{j_r}\, z_\ell}$$

Then $df \wedge \wedge\omega_1 \wedge ... \wedge \omega_r = 0$ implies

$$f_{i_1,...,i_m}\Big(\sum_{\ell \in \{1,...,m\} \setminus \{j_1,...,j_r\}} [(-1)^\ell i_\ell \Delta(j_1,...,j_r)] \Big) = 0$$

for all $j_1 < ... < j_r$ and for all $i_1,...,i_m$. Therefore, if $f_{i_1,...,i_m} \neq 0$ then we have

$$\sum_{\ell \in \{1,...,m\} \setminus \{j_1,...,j_r\}} (-1)^\ell i_\ell \Delta(j_1,...,j_r) = 0.$$

By the nonresonance hypothesis this occurs only for $(i_1,...,i_m) = (0,...,0)$.

□

Lemma 3. *Let $B = (\alpha_j^k)_{j,k}$ be a $r \times m$ matrix and let $A = (\lambda_j^k)$ a $m \times (m-r)$ matrix, such that $BA = 0$. Denote by $\Delta(B;\{k_1,...,k_r\})$ the $r \times r$ minor determinant obtained by choosing the columns $(k_1,...,k_r)$ in the matrix B and by $\Delta(A;\{k_1,...,k_r\}^c)$ the $(m-r)\times(m-r)$ minor determinant obtained by deleting in A the lines $k_1,...,k_r$. Then for any pair of choices $(k_1,...,k_r)$ and $(\tilde{k}_1,..,\tilde{k}_r)$ we have*

$$\frac{\operatorname{sign}(\sigma(k_1,...,k_r))}{\operatorname{sign}(\sigma(\tilde{k}_1,...,\tilde{k}_r))} \Delta(B;\{k_1,...,k_r\}\}\Delta(A;\{\tilde{k}_1,...,\tilde{k}_r\}^c) = \Delta(B;\{\tilde{k}_1,...,\tilde{k}_r\})\Delta(A;\{k_1,...,k_r\}^c)$$

where $\operatorname{sign}(\sigma(k_1,...,k_r))$ is the sign of the permutation $(k_1,...,k_r,j_1,...,j_{m-r})$, where

$$\{j_1 < ... < j_r\} = \{1,...,m\} \setminus \{k_1,...,k_r\}.$$

Assume that each such minor determinant is nonzero. Then we have

$$\operatorname{sign}(\sigma(k_1,...,k_r)) \frac{\Delta(B;\{k_1,...,k_r\}\}}{\Delta(A;\{k_1,...,k_r\}^c)} = \operatorname{sign}(\sigma(\tilde{k}_1,...,\tilde{k}_r)) \frac{\Delta(B;\{\tilde{k}_1,...,\tilde{k}_r\})}{\Delta(A;\{\tilde{k}_1,...,\tilde{k}_r\}^c)}$$

In particular, if A is nonresonant then B is also nonresonant.

Proof. The proof is standard Linear Algebra. Indeed, we first write $BA = 0$ as above in the following linear homogeneous system of equations

$$\sum_{j=1}^{m} \alpha_j^l \lambda_j^k = 0, \quad l \in \{1, \cdots, r\}, \, k \in \{1, \cdots, m-r\}. \tag{5}$$

From now on it is just Gaussian elimination process. We give a sketch for the case $r = 2$ and $m = 4$. The general case is proved in the same way.

Write

$$A = \begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \\ \lambda_4 & \mu_4 \end{pmatrix}$$

and

$$B = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}.$$

From $BA = 0$ we get

$$a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 = 0 \tag{6}$$
$$a_1\mu_1 + a_2\mu_2 + a_3\mu_3 + a_4\mu_4 = 0 \tag{7}$$
$$b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_4\lambda_4 = 0 \tag{8}$$
$$b_1\mu_1 + b_2\mu_2 + b_3\mu_3 + b_4\mu_4 = 0 \tag{9}$$

Multiplying equation (5) by b_2 and equation (7) by $-a_2$ and then summing up these resulting equations we eliminate λ_2 in the first and the third equations obtaining:

$$(b_2a_1 - a_2b_1)\lambda_1 + (b_2a_3 - a_2b_3)\lambda_3 + (b_2a_4 - a_2b_4)\lambda_4 = 0$$

Eliminating in a similar way μ_2 in the second and fourth equations we obtain

$$(b_2a_1 - a_2b_1)\mu_1 + (b_2a_3 - a_2b_3)\mu_3 + (b_2a_4 - a_2b_4)\mu_4 = 0$$

Using these two equations and eliminating the term $b_2a_3 - a_2b_3$ we obtain

$$\begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_3 & \mu_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = - \begin{vmatrix} \lambda_3 & \lambda_4 \\ \mu_3 & \mu_4 \end{vmatrix} \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix}$$

Notice that, during the Gaussian elimination process, no division is performed. Thus, we do not need to make considerations regarding whether the coefficients are zero or not. □

Proof of Theorem 1. By Proposition 2 and Lemma 1, in a small neighborhood $U \subset V$ of the origin the foliation is defined by a system of logarithmic one-forms $\omega_1, ..., \omega_r$ where $\omega_l = \sum\limits_{j=1}^{m} \alpha_j^l \frac{dz_j}{z_j}$ and simultaneously by linear vector fields $X_1, ..., X_{m-r}$ of the form

$$X_k(z_1, .., z_m) = \sum_{i=1}^{m} \lambda_i^k \, z_i \frac{\partial}{\partial z_i}.$$

Since $\omega_l(X_k) = 0$ we have

$$\sum_{j=1}^{m} \alpha_j^l \lambda_j^k = 0, \quad l \in \{1, \cdots, r\}, \, k \in \{1, \cdots, m-r\}. \tag{10}$$

Let $B = (\alpha_j^l)_{j,l}$ be the matrix of coefficients of the forms ω_l and $A = (\lambda_j^k)$ the matrix of coefficients of the vector fields X_k. From equation (10) we have $BA = 0$. Since A is nonresonant, by Lemma 3, B is also nonresonant.

On the other hand, by hypothesis the foliation is a Lie-foliation in $V \setminus \Lambda$. Let therefore $\{\theta_1, \cdots, \theta_r\}$ be a system of holomorphic one-forms in $V \setminus \Lambda$ defining $\mathcal{F}$ and satisfying the Maurer-Cartan equation as stated in Darboux-Lie theorem. Since $\{\omega_l\}_{l=1,\dots,r}$ and $\{\theta_l\}_{l=1,\dots,r}$ define the same foliation outside a codimension 1 analytical subset, given by the union of Λ with the singular locus of $\mathcal{F}$ (which has codimension ≥ 2), it is clear that there is a holomorphic map $F\colon U \setminus \Lambda \to \mathrm{GL}(r, \mathbb{C})$ given by $F(z) = (f_{ij})_{i,j=1}^r$ such that

$$\theta_i = \sum_{l=1}^{r} f_{il}\omega_l. \tag{11}$$

Since each ω_l is closed we have from the above equation

$$d\theta_i = \sum_{l=1}^{r} df_{il} \wedge \omega_l. \tag{12}$$

From equations (4) and (11) we have

$$d\theta_i = \sum_{j,k} c_{jk}^i \theta_j \wedge \theta_k = \sum_{l<t} \Big(\sum_{j,k} c_{jk}^i \left(f_{jl} f_{kt} - f_{jt} f_{kl} \right) \Big) \omega_l \wedge \omega_t. \tag{13}$$

Claim 1. *We have* $df_{i1} \wedge \omega_1 \wedge \cdots \wedge \omega_{m-1} = 0$.

Proof. Indeed, from equation (13) above we have

$$d\theta_i \wedge \omega_2 \wedge \cdots \wedge \omega_r = 0.$$

From this last equation and equation (12) we obtain

$$df_{i1} \wedge \omega_1 \wedge \cdots \wedge \omega_{m-1} = 0.$$

□

Similarly we prove that

$$df_{ij} \wedge \omega_1 \wedge \cdots \wedge \omega_{m-1} = 0, \; \forall i, j. \tag{14}$$

Since the matrix B of the coefficients of the forms ω_l is nonresonant, by Lemma 2 each f_{ij} is constant in a neighborhood of the origin in U. On the other hand, each one-form θ_j is defined in $V \setminus \Lambda$, and each irreducible component of Λ contains the origin. Therefore, by classical Levi-Hartogs' extension theorem (applied to each irreducible component of Λ) each one-form θ_i extends to Λ as a meromorphic one-form Θ_i in V. We claim:

Claim 2. *Each extension* Θ_i *is a closed meromorphic one-form with simple poles in* V*. Moreover the polar set* $(\Theta_i)_\infty$ *is contained in* Λ*.*

Proof. First we observe that the extension Θ_i is closed by the Identity Principle (also note that since Λ is a thin set, $V \setminus \Lambda$ is connected). In order to see that the poles of Θ_i are contained in Λ it is enough to observe that Θ_i ans θ_i coincide in $V \setminus \Lambda$, where θ_i is holomorphic. Finally, to see that each irreducible component of Λ is also contained in the polar set of each Θ_i it is enough to use the fact that this is true in a neighborhood of the origin and, by hypothesis, each irreducible component of Λ contains the origin. □

Since each Θ_i is a simple poles closed meromorphic one-form in V, the foliation $\mathcal{F}$ is logarithmic in V. This ends the proof of Theorem 1. □

References

[1] V. I. Arnold, *Geometrical methods in the theory of ordinary differential equations.* Springer, 1983.
[2] Blumenthal, R., 'Transversely homogeneous foliations', Ann. Inst. Fourier t. 29, no.4 (1979), 143–158.
[3] Cerveau, D. and Mattei, J.-F., 'Formes intégrables holomorphes singulières´. Astérisque, vol.97 (1982).
[4] Godbillon, C., Feuilletages. Études géométriques. With a preface by G. Reeb. Progress in Mathematics, 98. Birkhäuser Verlag, Basel, 1991.
[5] Yu. Ilyashenko & S. Yakovenko. *Lectures on analytic differential equations.* Graduate Studies in Mathematics. American Mathematical Society (December 27, 2007).
[6] Mafra, A. and Scardua, B., 'One-dimensional Lie foliations with generic singularities in complex dimension three´, J. Aust. Math. Soc. 91 (2011), no. 1, 13-28.
[7] Siu, Y., Techniques of Extension of Analytic Objects. Marcel Dekker, New York, 1974.

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