Journal *of* Singularities
Volume 14 (2016), 52-73

received: 7 April 2015
in revised form: 9 March 2016
DOI: 10.5427/jsing.2016.14d

CLASSIFICATION OF FOLIATIONS ON $\mathbb{CP}^2$ OF DEGREE 3 WITH DEGENERATE SINGULARITIES

CLAUDIA R. ALCÁNTARA AND RAMÓN RONZÓN-LAVIE

Abstract. The aim of this work is to classify foliations on $\mathbb{CP}^2$ of degree 3 with degenerate singular points. For that we construct a stratification of the space of holomorphic foliations by locally closed, irreducible, non-singular algebraic subvarieties which parametrize foliations with a special degenerate singularity. We also prove that there are only two foliations with isolated singularities with automorphism group of dimension two, the maximum possible dimension. Finally we obtain the unstable foliations with only one singular point, that is, a singular point with Milnor number 13.

1. Introduction

The aim of this work is to classify holomorphic foliations on $\mathbb{CP}^2$ of degree 3 with certain degenerate singular point using Geometric Invariant Theory (GIT). This theory was developed principally by David Hilbert and David Mumford (see [6]). We obtain locally closed, irreducible, non-singular algebraic subvarieties which parametrize foliations of degree 3 with a special degenerate singularity. We also get the dimension and explicit generators for each stratum. Similar results for degree 2 are given in [2] and in [3], we have some general results for degree d.

Geometric Invariant Theory gives a method for constructing quotients for group actions on algebraic varieties. More specifically, we have a linear action by a reductive group on a projective variety and we can construct a good quotient if we remove the closed set of unstable points. When the projective variety parametrizes geometric objects, the unstable points are in some sense degenerate objects. For example, the unstable plane algebraic curves with respect to the action by projective transformations are curves with non-ordinary singularities with order greater than 2.

In this article the projective variety $\mathcal{F}_3$ is the space of holomorphic foliations on $\mathbb{CP}^2$ of degree 3 and the action is given by change of coordinates. For this action we obtain the closed set of unstable foliations. We will prove that a foliation is unstable if and only if it has a special degenerate singular point (see Theorem 8). In this closed set we construct the stratification studied by Kirwan (in [12]), Hesselink (in [9]) and Kempf (in [11]). The strata are locally closed, non-singular, irreducible algebraic subvarieties of $\mathcal{F}_3$. We characterize the generic foliation on every stratum according to the Milnor number and multiplicity of their singularities. We also obtain the dimension of the strata (see Theorem 7).

2000 *Mathematics Subject Classification.* Primary 37F75, 14L24.

Key words and phrases. holomorphic foliation, unstable point, degenerate singularity, stratification.

The first author was partially supported by CONACyT Grant 165891. The second author was partially supported by DAIP-UG Grant 191/2013.

As a corollary we describe the irreducible components of the closed set of unstable foliations. We find, up to change of coordinates, the only two foliations with isolated singularities with automorphism group of dimension 2 (see Theorem 10). Finally we classify unstable foliations on $\mathbb{CP}^2$ of degree 3 with only one singular point, that is with Milnor number 13 (see Theorem 11). This result is important because the classification of foliations on $\mathbb{CP}^2$ with only one singular point is known only for degree 2 (see [5] and [2]).

In sections 2 and 3 we recall the basic results about Geometric Invariant Theory and foliations that we need in the sequel. We compute in section 4 the unstable foliations of degree 3 using the numerical criterion of one parameter subgroups. The construction of the stratification of the space of foliations and the characterization of the generic foliation on every stratum is included in section 5. The last section is devoted to give some important corollaries of the construction.

2. Geometric Invariant Theory

In this section we recall basic facts about Geometric Invariant Theory. All the definitions and results can be found in [14] and [11].

Let V be a projective variety in $\mathbb{CP}^n$, and consider a reductive group G acting linearly on V.

Definition 1. *Let $x \in V \subset \mathbb{CP}^n$, and consider $\overline{x} \in \mathbb{C}^{n+1}$ such that $\overline{x} \in x$. Denote by $O(\overline{x})$ the orbit of $\overline{x}$ in the affine cone of V and by $O(x)$ the orbit of x. Then*
*(i) x is **unstable** if $0 \in \overline{O(\overline{x})}$.*
*(ii) x is **semi-stable** if $0 \notin \overline{O(\overline{x})}$. The set of semi-stable points will be denoted by V^{ss}.*
*(iii) x is **stable** if it is semi-stable, $O(x)$ is closed in V^{ss} and $\dim O(x) = \dim G$. The set of stable points will be denoted by V^s.*

The main result in GIT is the following:

Theorem 1. *(see page 74 in [14])*
(i) There exists a projective variety Y and a morphism $\phi : V^{ss} \to Y$, which is a good quotient.

(ii) There exists an open set $Y^s \subset Y$ such that $\phi^{-1}(Y^s) = V^s$ and the morphism $\phi_| : V^s \to Y^s$ is a good quotient and an orbit space.

It is very often difficult to find the unstable points for a given action, but there exists a very useful criterion due to Hilbert and Mumford. Let us describe it.

A 1-parameter subgroup (1-PS) of the group G is an algebraic morphism $\lambda : \mathbb{C}^* \to G$. Since the action on V is linear, this induces a diagonal representation of $\mathbb{C}^*$:

$$\begin{aligned} \mathbb{C}^* &\to GL(n+1, \mathbb{C}) \\ t &\mapsto \lambda(t) : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \\ &\qquad\qquad\qquad v \mapsto \lambda(t)v. \end{aligned}$$

Therefore there exists a basis $\{v_0, ..., v_n\}$ of $\mathbb{C}^{n+1}$ such that $\lambda(t)v_i = t^{r_i}v_i$, where $r_i \in \mathbb{Z}$.

Definition 2. *Let $x \in X$ and let $\lambda : \mathbb{C}^* \to G$ be a 1-PS of G. If $\bar{x} \in x$ and $\bar{x} = \sum_{i=0}^n a_i v_i$, then $\lambda(t)\bar{x} = \sum_{i=0}^n t^{r_i} a_i v_i$. We define the following function*

$$\mu(x, \lambda) := min\{r_i : a_i \neq 0\}.$$

The numerical criterion can now be stated.

Theorem 2. *(see Theorem 4.9 of [14])*
(i) x is stable if and only if $\mu(x,\lambda) < 0$ for every 1-PS, λ, of G.
(ii) x is unstable if and only if there exists a 1-PS, λ, of G such that $\mu(x,\lambda) > 0$.

Definition 3. *If $\mu(x,\lambda) > 0$ we will say that x is **λ-unstable**.*

The following is a useful tool for applying the criterion of 1-PS when $G = SL(n,\mathbb{C})$. We formulate the result for the case $n = 3$.

Lemma 1. *(see [14]) Every 1-parameter subgroup of $SL(3,\mathbb{C})$ has the form*

$$\lambda(t) = g\begin{pmatrix} t^{k_1} & 0 & 0 \\ 0 & t^{k_2} & 0 \\ 0 & 0 & t^{k_3} \end{pmatrix} g^{-1},$$

for some $g \in SL(3,\mathbb{C})$ and some integers k_1, k_2, k_3 such that $k_1 \geq k_2 \geq k_3$ and $k_1 + k_2 + k_3 = 0$.

3. Foliations on $\mathbb{CP}^2$ of degree d

This section provides the definitions and results that we need to know about holomorphic foliations on $\mathbb{CP}^2$ for the development of the paper.

Definition 4. *A holomorphic foliation X of $\mathbb{CP}^2$ of degree d is a non-trivial morphism of vector bundles:*

$$X : \mathcal{O}(1-d) \to \mathcal{T}\mathbb{CP}^2,$$

modulo multiplication by a nonzero scalar. The space of foliations of degree d is

$$\mathcal{F}_d := \mathbb{P}H^0(\mathbb{CP}^2, \mathcal{T}\mathbb{CP}^2(d-1)),$$

where $d \geq 0$.

Take homogeneous coordinates $(x : y : z)$ on $\mathbb{CP}^2$. Up to multiplication by a nonzero scalar there are two equivalent ways to describe a foliation of degree d (see [8]):

(1) By a homogeneous vector field:

$$X = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y} + R(x,y,z)\frac{\partial}{\partial z} = \begin{pmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{pmatrix}$$

where $P, Q, R \in \mathbb{C}[x,y,z]$ are homogeneous of degree d. And if we consider the radial foliation

$$E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$$

then X and $X + F(x,y,z)E$ represent the same foliation for all $F \in \mathbb{C}[x,y,z]$ homogeneous of degree $d-1$.

(2) By a homogeneous 1-form: $\Omega = L(x,y,z)dx + M(x,y,z)dy + N(x,y,z)dz$, such that $L, M, N \in \mathbb{C}[x,y,z]$ are homogeneous of degree $d+1$ and these satisfy the Euler's condition $xL + yM + zN = 0$.

With this we can see that the space of foliations on $\mathbb{CP}^2$ of degree d is a projective space of dimension $d^2 + 4d + 2$. We will use the description 1 for the rest of the paper.

We now define the notion of singular point for a foliation and two important invariants for this.

Definition 5. *A point $p = (a : b : c) \in \mathbb{CP}^2$ is **singular** for the above foliation X if*

$$(P(a,b,c), Q(a,b,c), R(a,b,c)) = (ka, kb, kc)$$

for some $k \in \mathbb{C}$. The set of singular points of X will be denoted by $Sing(X)$.

Definition 6. *Let*

$$\begin{pmatrix} f(y,z) \\ g(y,z) \end{pmatrix}$$

be a local generator of X in $p = (1 : b : c)$. Then

***the Milnor number of** p is $\mu_p(X) := \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2,p}}{<f,g>}$,*
***the multiplicity of** p is $m_p(X) := min\{ord_p(f), ord_p(g)\}$.*

Proposition 1. *(see [4]) Let X be a foliation of degree d with isolated singularities then*

$$d^2 + d + 1 = \sum_{p \in \mathbb{CP}^2} \mu_p(X).$$

From Lemma 1.2 in [7] we can deduce that

$$\{X \in \mathcal{F}_d : \text{there exists } p \in \mathbb{CP}^2 \text{ such that } \mu_p(X) \geq 2\}$$

is a divisor in $\mathcal{F}_d$, therefore we have the following:

Theorem 3. *The set $\{X \in \mathcal{F}_d :$ every singular point for X has Milnor number $1\}$ is open and non-empty in $\mathcal{F}_d$.*

Finally we give the definition of algebraic leaf for a foliation.

Definition 7. *A plane curve defined by a polynomial $F(x,y,z)$ is an **algebraic leaf for** X **or invariant by** X if and only if there exists a polynomial $H(x,y,z)$ such that:*

$$P(x,y,z)\frac{\partial F(x,y,z)}{\partial x} + Q(x,y,z)\frac{\partial F(x,y,z)}{\partial y} + R(x,y,z)\frac{\partial F(x,y,z)}{\partial z} = FH.$$

Theorem 4. *(see Theorem 1.1, p.158 in [10] and [13]) The set*

$$\{X \in \mathcal{F}_d : X \text{ has no algebraic leaves}\}$$

is open and non-empty in $\mathcal{F}_d$.

Generically a foliation on $\mathbb{CP}^2$ of degree d does not have degenerate singularities and does not have algebraic leaves. So it is important to classify foliations in the complement of these sets. In this article we say something about that for degree 3.

The group $PGL(3,\mathbb{C})$ of automorphisms of $\mathbb{CP}^2$ is a reductive group that acts linearly on $\mathcal{F}_d$ by change of coordinates:

$$PGL(3,\mathbb{C}) \times \mathcal{F}_d \to \mathcal{F}_d$$
$$(g, X) \mapsto gX = DgX \circ (g^{-1}).$$

In the computations we will use $SL(3,\mathbb{C})$ instead of $PGL(3,\mathbb{C})$, we will get the same results.

4. Unstable Foliations on $\mathbb{CP}^2$ of degree 3

As we saw before the space of foliations $\mathcal{F}_3$ is a projective space of dimension 23. In this section we apply the numerical criterion of one parameter subgroups to obtain the closed set of unstable foliations of degree 3. Remember that $X \in \mathcal{F}_d$ is unstable with respect to the action by change of coordinates if and only if there exists λ a 1-PS of $SL(3,\mathbb{C})$ such that $\mu(X,\lambda) > 0$ (see Theorem 2). For all λ a 1-PS of $SL(3,\mathbb{C})$ there exists $g \in SL(3,\mathbb{C})$ such that $D(t) := g\lambda(t)g^{-1}$ is a diagonal 1-PS, with the form:

$$D : \mathbb{C}^* \to SL(3,\mathbb{C}), \quad t \mapsto \begin{pmatrix} t^{k_1} & 0 & 0 \\ 0 & t^{k_2} & 0 \\ 0 & 0 & t^{k_3} \end{pmatrix},$$

for some intergers k_1, k_2, k_3 such that $k_1 \geq k_2 \geq k_3$ and $k_1 + k_2 + k_3 = 0$.

Since $\mu(gX,D) = \mu(X, g^{-1}Dg) = \mu(X,\lambda)$ (see remark 4.10 of [14]), every unstable foliation is in the orbit of an unstable point with respect to a diagonal 1-PS. Therefore, we will find the unstable foliations with respect to a diagonal one parameter subgroup and then we will take the set of orbits of these points.

Let us consider the basis for the vector space $H^0(\mathbb{CP}^2, \mathcal{T}\mathbb{CP}^2(2))$ given by

$$\{M\frac{\partial}{\partial x}, M\frac{\partial}{\partial y}, x^3\frac{\partial}{\partial z}, x^2y\frac{\partial}{\partial z}, xy^2\frac{\partial}{\partial z}, y^3\frac{\partial}{\partial z} : M \in \mathbb{C}[x,y,z] \text{ is a monic monomial of degree 3}\}.$$

This basis diagonalizes the action of $SL(3,\mathbb{C})$. Let $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$ be a foliation on $\mathbb{CP}^2$ of degree 3 where

$$\begin{aligned} P(x,y,z) &= \sum a_{\alpha,\beta}x^\alpha y^\beta z^{3-\alpha-\beta} \\ Q(x,y,z) &= \sum b_{\alpha,\beta}x^\alpha y^\beta z^{3-\alpha-\beta} \\ R(x,y,z) &= \sum c_{\alpha,\beta}x^\alpha y^\beta z^{3-\alpha-\beta}. \end{aligned}$$

Then we are looking for the points $X \in \mathcal{F}_3$ such that there exist $k_1, k_2, k_3 \in \mathbb{Z}$ with $k_1 \geq k_2 \geq k_3$ and $k_1 + k_2 + k_3 = 0$ and such that $\max\{-E_P, -E_Q, -E_R\} < 0$, where

$$\begin{aligned} E_P &= \min\{-k_1(\alpha-1) - k_2\beta - k_3\gamma : a_{\alpha,\beta} \neq 0\} \\ E_Q &= \min\{-k_1\alpha - k_2(\beta-1) - k_3\gamma : b_{\alpha,\beta} \neq 0\} \\ E_R &= \min\{-k_1\alpha - k_2\beta - k_3(\gamma-1) : c_{\alpha,\beta} \neq 0\}. \end{aligned}$$

From definition 2, $\mu(X,D) = -\max\{-E_P, -E_Q, -E_R\}$, where D is the diagonal 1-PS defined above.

Since $k_1 > 0$ and $k_3 < 0$, then we can define $q_i := \frac{k_i}{k_1}$. Therefore $q_1 + q_2 + q_3 = 0$, $1 \geq q_2 \geq q_3$ and $q_2 \in [-\frac{1}{2}, 1] \cap \mathbb{Q}$. We must find the conditions in the rational numbers q_i to have non-zero coefficients for the monomials of P, Q and R. It is easy to obtain the following conclusion.

Coefficients $a_{\alpha,\beta}$ $(\alpha-1)k_1+\beta k_2+\gamma k_3<0$
$\alpha=2,\beta=0,\gamma=1,\ q_2\in(0,1]$
$\alpha=1,\beta=2,\gamma=0,\ q_2\in[-\frac{1}{2},0)$
$\alpha=1,\beta=1,\gamma=1,\ q_2\in[-\frac{1}{2},1]$
$\alpha=1,\beta=0,\gamma=2,\ q_2\in[-\frac{1}{2},1]$
$\alpha=0,\beta=3,\gamma=0,\ q_2\in[-\frac{1}{2},\frac{1}{3})$
$\alpha=0,\beta=2,\gamma=1,\ q_2\in[-\frac{1}{2},1]$
$\alpha=0,\beta=1,\gamma=2,\ q_2\in[-\frac{1}{2},1]$
$\alpha=0,\beta=0,\gamma=3,\ q_2\in[-\frac{1}{2},1]$

Coefficients $b_{\alpha,\beta}$ $\alpha k_1+(\beta-1)k_2+\gamma k_3<0$
$\alpha=2,\beta=0,\gamma=1,\ q_2\in(\frac{1}{2},1]$
$\alpha=1,\beta=1,\gamma=1,\ q_2\in(0,1]$
$\alpha=1,\beta=0,\gamma=2,\ q_2\in(-\frac{1}{3},1]$
$\alpha=0,\beta=3,\gamma=0,\ q_2\in[-\frac{1}{2},0)$
$\alpha=0,\beta=2,\gamma=1,\ q_2\in[-\frac{1}{2},1]$
$\alpha=0,\beta=1,\gamma=2,\ q_2\in[-\frac{1}{2},1]$
$\alpha=0,\beta=0,\gamma=3,\ q_2\in[-\frac{1}{2},1]$

Coefficients $c_{\alpha,\beta}$ $\alpha k_1+\beta k_2+(\gamma-1)k_3<0$
$\alpha=0,\beta=3,\gamma=0,\ q_2\in[-\frac{1}{2},-\frac{1}{4})$

From this, we see that $a_{3,0}$, $a_{2,1}$, $b_{3,0}$, $b_{2,1}$, $b_{1,2}$, $c_{3,0}$, $c_{2,1}$, $c_{1,2}=0$ and we can have $a_{1,1}$, $a_{1,0}$, $a_{0,2}$, $a_{0,1}$, $a_{0,0}$, $b_{0,2}$, $b_{0,1}$, $b_{0,0}\neq 0$. Now we do a partition of $\left[-\frac{1}{2},1\right]$ to have the subspaces of unstable foliations with respect to a diagonal 1-PS.

$$q_2\in\left[-\frac{1}{2},-\frac{1}{3}\right]\Rightarrow a_{2,0},b_{2,0},b_{1,1},b_{1,0}=0$$
$$q_2\in\left(-\frac{1}{3},-\frac{1}{4}\right)\Rightarrow a_{2,0},b_{2,0},b_{1,1}=0$$
$$q_2\in\left[-\frac{1}{4},0\right)\Rightarrow a_{2,0},b_{2,0},b_{1,1},c_{0,3}=0$$
$$q_2=0\Rightarrow a_{2,0},a_{1,2},b_{2,0},b_{1,1},b_{0,3},c_{0,3}=0$$
$$q_2\in\left(0,\frac{1}{3}\right)\Rightarrow a_{1,2},b_{2,0},b_{0,3},c_{0,3}=0$$
$$q_2\in\left[\frac{1}{3},\frac{1}{2}\right]\Rightarrow a_{1,2},a_{0,3},b_{2,0},b_{0,3},c_{0,3}=0$$
$$q_2\in\left(\frac{1}{2},1\right]\Rightarrow a_{1,2},a_{0,3},b_{0,3},c_{0,3}=0.$$

Consider the seven subspaces with the corresponding coefficients equal to zero. In these sets we have 3 maximal subspaces of $H^0(\mathbb{CP}^2,\mathcal{T}\mathbb{CP}^2(2))$. That we describe below:

$$\begin{aligned}V_1:=\Big\langle & xy^2\frac{\partial}{\partial x},xyz\frac{\partial}{\partial x},xz^2\frac{\partial}{\partial x},y^3\frac{\partial}{\partial x},y^2z\frac{\partial}{\partial x},yz^2\frac{\partial}{\partial x},z^3\frac{\partial}{\partial x},\\ & xz^2\frac{\partial}{\partial y},y^3\frac{\partial}{\partial y},y^2z\frac{\partial}{\partial y},yz^2\frac{\partial}{\partial y},z^3\frac{\partial}{\partial y},y^3\frac{\partial}{\partial z}\Big\rangle_{\mathbb{C}}\end{aligned}$$

$$\begin{aligned}V_2:=\Big\langle & x^2z\frac{\partial}{\partial x},xyz\frac{\partial}{\partial x},xz^2\frac{\partial}{\partial x},y^2z\frac{\partial}{\partial x},yz^2\frac{\partial}{\partial x},z^3\frac{\partial}{\partial x},x^2z\frac{\partial}{\partial y},xyz\frac{\partial}{\partial y},xz^2\frac{\partial}{\partial y},\\ & y^2z\frac{\partial}{\partial y},yz^2\frac{\partial}{\partial y},z^3\frac{\partial}{\partial y}\Big\rangle_{\mathbb{C}}\end{aligned}$$

$$V_3 := \Big\langle x^2z\frac{\partial}{\partial x}, xyz\frac{\partial}{\partial x}, xz^2\frac{\partial}{\partial x}, y^3\frac{\partial}{\partial x}, y^2z\frac{\partial}{\partial x}, yz^2\frac{\partial}{\partial x}, z^3\frac{\partial}{\partial x}, xyz\frac{\partial}{\partial y}, xz^2\frac{\partial}{\partial y},$$
$$y^2z\frac{\partial}{\partial y}, yz^2\frac{\partial}{\partial y}, z^3\frac{\partial}{\partial y}\Big\rangle_{\mathbb{C}},$$

Then,

$\{X \in H^0(\mathbb{CP}^2, \mathcal{T}\mathbb{CP}^2(2)) : \text{there exists } D \text{ a 1-PS diagonal such that } \mu(X,D) > 0\} = V_1 \cup V_2 \cup V_3$.

Therefore we can state:

Theorem 5. *The closed set of unstable foliations on $\mathbb{CP}^2$ of degree 3 is*

$$\mathcal{F}_3^{un} = SL(3,\mathbb{C})\mathbb{P}V_1 \cup SL(3,\mathbb{C})\mathbb{P}V_2 \cup SL(3,\mathbb{C})\mathbb{P}V_3.$$

5. The Stratification of $\mathcal{F}_3$

In the previous section we exhibit the closed set of unstable foliations on $\mathbb{CP}^2$ of degree 3. In this section we will use properties of the singularities of the foliations to construct locally closed, non-singular subvarieties of $\mathcal{F}_3^{un}$. Firstly we will explain the stratification described in the following Theorem by F. Kirwan and then apply it to $\mathcal{F}_3$.

Theorem 6. *(see Theorem 13.5 in [12]) Let V be a non-singular projective variety with a linear action by a reductive group G. Then there exists a stratification*

$$\{S_\beta : \beta \in \mathcal{B}\}$$

of V such that the unique open stratum is V^{ss} and every stratum S_β in the set of unstable points is non-singular, locally closed and isomorphic to $G \times_{P_\beta} Y_\beta^{ss}$, where Y_β^{ss} is a non-singular locally-closed subvariety of V and P_β is a parabolic subgroup of G.

Throughout the text we will use the same notation as in §12 of [12].

Definition 8. *Let $Y(G)$ be the set of one parameter subgroups $\lambda : \mathbb{C}^* \to G$. Define in $Y(G) \times \mathbb{N}$ the equivalence relation: (λ_1, n_1) is related with (λ_2, n_2) if and only if $\lambda_1(t^{n_2}) = \lambda_2(t^{n_1})$ for all $t \in \mathbb{C}^*$. **A virtual one parameter subgroups of** G is an equivalence class of this relation, the set of these classes will be denoted by $M(G)$.*

The indexing set $\mathcal{B}$ of the stratification is a finite subset of $M(G)$ and this may be described in terms of the weights of the representation of G which defines the action. For the construction we must consider on $M(G)$ a norm q which is the square of an inner product $\langle\ ,\ \rangle$. This norm gives the partial order $>$ on $\mathcal{B}$.

On the other hand, the representation of $\mathbf{D}$ on $\mathbb{C}^{n+1}$, where $\mathbf{D}$ is a maximal torus of G, splits as a sum of scalar representations given by characters $\alpha_0, ..., \alpha_n$. These characters are elements of the dual of $M(\mathbf{D})$ but we can identify them with elements of $M(\mathbf{D})$ using $\langle\ ,\ \rangle$.

Definition 9. *Once we have the indexing set $\mathcal{B}$ we can describe the objects that appear in Theorem 6. Let $\beta \in \mathcal{B}$, we define:*

$$Z_\beta = \{(x_0 : ... : x_n) \in V : x_j = 0 \text{ if } \langle \alpha_j, \beta\rangle \neq q(\beta)\},$$
$$Y_\beta = \{(x_0 : ... : x_n) \in V : x_j = 0 \text{ if } \langle \alpha_j, \beta\rangle < q(\beta)$$
$$\text{and } x_j \neq 0 \text{ for some } j \text{ with } \langle \alpha_j, \beta\rangle = q(\beta)\},$$

the map $p_\beta : Y_\beta \to Z_\beta, (x_0, ..., x_n) \mapsto (x'_0, ..., x'_n)$ *as* $x'_j = x_j$ *if* $\langle \alpha_j, \beta\rangle = q(\beta)$ *and* $x'_j = 0$ *otherwise.*

Consider $Stab(\beta)$, the stabilizer of β under the adjoint action of G. There exists a unique connected reductive subgroup G_β of $Stab_\beta$ such that $M(G_\beta) = \{\lambda \in M(Stab_\beta) : \langle\lambda, \beta\rangle = 0\}$ (see 12.21 in [12]). With this group we can define

$$Z_\beta^{ss} = \{x \in Z_\beta : x \text{ is semistable under the action of } G_\beta \text{ on } Z_\beta\}$$

and $Y_\beta^{ss} = p_\beta^{-1}(Z_\beta^{ss})$.

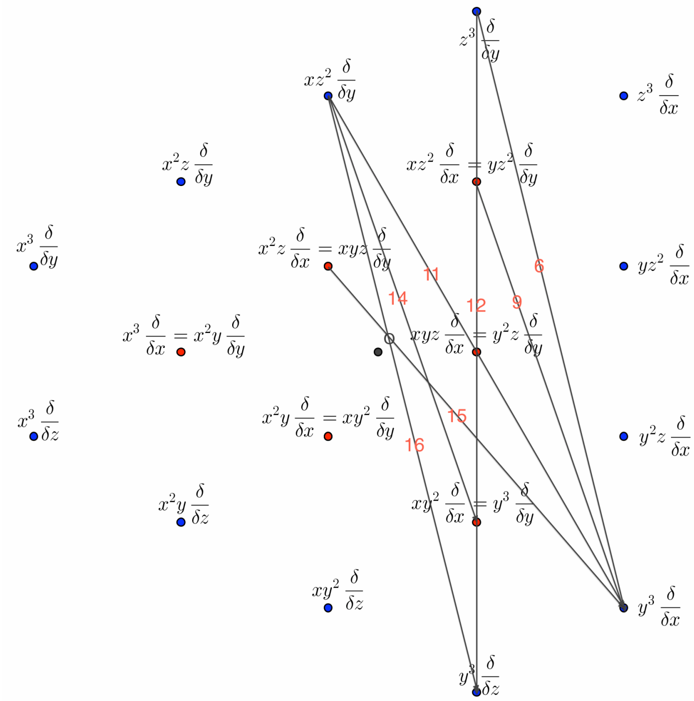
Finally the parabolic group of β is: if $x \in Y_\beta^{ss}$ then $P_\beta = \{g \in G : gx \in Y_\beta^{ss}\}$.

Remark 1. *Since* S_β *is isomorphic to* $G \times_{P_\beta} Y_\beta^{ss}$*, it has dimension* $\dim Y_\beta^{ss} + \dim G - \dim P_\beta$*.*

5.1. **The representation of $\mathcal{F}_3$.** Norbert A'Campo and Vladimir Popov give in [15] a computer program such that given a reductive group and one of its representation, the output is the finite subset $\mathcal{B}$ of virtual 1-parameter subgroups for the above stratification. For a more detailed construction of the virtual 1-parameter subgroups in the case of the action by change of coordinates of $SL(3, \mathbb{C})$ in $\mathcal{F}_d$ we refer to section 3 of [3]. For $\mathcal{F}_3$ the virtual 1-parameter subgroups for the stratification are:

$$\beta_1 := \left(\frac{5}{3}, \frac{2}{3}, -\frac{7}{3}\right), \beta_2 := \left(\frac{5}{3}, -\frac{1}{3}, -\frac{4}{3}\right), \beta_3 := \left(\frac{3}{2}, 0, -\frac{3}{2}\right), \beta_4 := \left(\frac{5}{3}, -\frac{5}{6}, -\frac{5}{6}\right),$$
$$\beta_5 := \left(\frac{55}{42}, -\frac{11}{42}, -\frac{22}{21}\right), \beta_6 := \left(\frac{7}{6}, -\frac{1}{3}, -\frac{5}{6}\right), \beta_7 := \left(\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\right), \beta_8 := (1, 0, -1),$$
$$\beta_9 := \left(\frac{20}{21}, -\frac{4}{21}, -\frac{16}{21}\right), \beta_{10} := \left(\frac{2}{3}, \frac{1}{6}, -\frac{5}{6}\right), \beta_{11} := \left(\frac{1}{2}, 0, -\frac{1}{2}\right),$$
$$\beta_{12} := \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right), \beta_{13} := \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right), \beta_{14} := \left(\frac{5}{21}, -\frac{1}{21}, -\frac{4}{21}\right),$$
$$\beta_{15} := \left(\frac{2}{21}, \frac{1}{42}, -\frac{5}{42}\right), \beta_{16} := \left(\frac{7}{78}, -\frac{1}{39}, -\frac{5}{78}\right).$$

Now we consider the induced representation $H^0(\mathbb{CP}^2, T\mathbb{CP}^2(2))$ of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. The weight diagrama for this irreducible representation is the following (the number i denoted the virtual 1-PS β_i):



From this information we can easily obtain the sets Z_i and Y_i described in definition 9. In Y_i with $i \in \{1, 2, 3, 4, 5, 7, 8, 10, 13\}$ every foliation has a curve of singularities, we can study these foliations as foliations of degree 2, so we are going to discard these strata.

To obtain the strata with foliations with isolated singularities we must find

$$Z_i^{ss} := \{x \in Z_i : \mu(x, \lambda) \leq \langle \lambda, \beta_i \rangle \text{ for all } \lambda \in M(\mathrm{Stab}(\beta_i))\}.$$

See definition 12.10 of [12]. For this we will use the following results.

Lemma 2. *(see [2, p. 430]) Let $X \in Z_i$ such that the virtual one parameter subgroup (n_0, n_1, n_2) corresponding to β_i satisfies $n_0 > n_1 > n_2$. Then $X \in Z_i^{ss}$ if and only if β_i is the closest point to zero in C_X with respect to $\mathbf{D}$, where C_X is the convex hull formed with the weights of X.*

The only virtual 1-PS where $n_1 = n_2$ in β_{12}, for finding Z_{12}^{ss} we need further analysis. We must recall that $\mathrm{Stab}(\beta_{12})$ is the stabilizer of β_{12} under the adjoint action of $SL(3, \mathbb{C})$ on $M(SL(3, \mathbb{C}))$ (see 12.21 in [12]), i.e.,

$$\begin{aligned}\mathrm{Stab}(\beta_{12}) &= \left\{ g \in SL(3,\mathbb{C}) : g \begin{pmatrix} \frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix} g^{-1} = \begin{pmatrix} \frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \in SL(3,\mathbb{C}) \right\}.\end{aligned}$$

We know that if $\lambda \in M(\mathrm{Stab}(\beta_{12}))$ then there exists $g \in \mathrm{Stab}(\beta_{12})$ such that $g\lambda g^{-1}$ has the form $Diag(t^{k_1}, t^{k_2}, t^{k_3})$, where $k_1 \geq k_2 \geq k_3$; therefore:

$$Z_{12}^{ss} = \{x \in Z_{12} : \mu(gx,\lambda) \leq \langle \lambda, \beta_{12}\rangle, \text{ for all } \lambda = Diag(t^{k_1},t^{k_2},t^{k_3}), \text{ where } k_1 \geq k_2 \geq k_3 \text{ and } g \in \mathrm{Stab}(\beta_{12})\}.$$

Where, from the weight diagram:

$$Z_{12} = \mathbb{P}\Big\langle xy^2\frac{\partial}{\partial x}, xyz\frac{\partial}{\partial x}, xz^2\frac{\partial}{\partial x}, y^3\frac{\partial}{\partial y}, y^2z\frac{\partial}{\partial y}, yz^2\frac{\partial}{\partial y}, z^3\frac{\partial}{\partial y}, y^3\frac{\partial}{\partial z}\Big\rangle_{\mathbb{C}},$$

and we have $\langle \lambda(t) = Diag(t^{k_1},t^{k_2},t^{k_3}), \beta_{12}\rangle = \frac{2}{3}k_1 - \frac{1}{3}k_2 - \frac{1}{3}k_3$. For $X \in Z_{12}$ we obtain

$$\lambda(t)\cdot X = \begin{pmatrix} a_{1,2}t^{-2k_2}xy^2 + a_{1,1}t^{-k_2-k_3}xyz + a_{1,0}t^{-2k_3}xz^2 \\ b_{0,3}t^{-2k_2}y^3 + b_{0,2}t^{-k_2-k_3}y^2z + b_{0,1}t^{-2k_3}yz^2 + b_{0,0}t^{k_2-3k_3}z^3 \\ c_{0,3}t^{k_3-3k_2}y^3 \end{pmatrix},$$

therefore $\mu(X,\lambda) = \min\{-2k_2, -k_2-k_3, -2k_3, k_2-3k_3, k_3-3k_2\}$. With the conditions

$$\begin{array}{rcl} -2k_2 \leq \frac{2}{3}k_1 - \frac{1}{3}k_2 - \frac{1}{3}k_3 & \Leftrightarrow & k_2 \geq k_3 \\ -k_2-k_3 \leq \frac{2}{3}k_1 - \frac{1}{3}k_2 - \frac{1}{3}k_3 & \Leftrightarrow & 0 \geq 0 \\ -2k_3 \leq \frac{2}{3}k_1 - \frac{1}{3}k_2 - \frac{1}{3}k_3 & \Leftrightarrow & k_2 \leq k_3 \\ k_2-3k_3 \leq \frac{2}{3}k_1 - \frac{1}{3}k_2 - \frac{1}{3}k_3 & \Leftrightarrow & k_2 \leq k_3 \\ k_3-3k_2 \leq \frac{2}{3}k_1 - \frac{1}{3}k_2 - \frac{1}{3}k_3 & \Leftrightarrow & k_2 \geq k_3. \end{array}$$

we conclude that $Z_{12}^{ss} = \{X \in Z_{12} : (a_{1,2}, a_{1,1}, b_{0,3}, b_{0,2}, c_{0,3}) \neq 0, (a_{1,1}, a_{1,0}, b_{0,2}, b_{0,1}, b_{0,0}) \neq 0\}$. Now we can give the full list of linear subspaces of $\mathcal{F}_3$ for the construction of the strata.

$$Z_6^{ss} = \mathbb{P}\left\{\begin{pmatrix} a_{0,3}y^3 \\ b_{0,0}z^3 \\ 0 \end{pmatrix} \in Z_6 : a_{0,3} \neq 0, b_{0,0} \neq 0\right\}$$

$$Z_9^{ss} = \mathbb{P}\left\{\begin{pmatrix} a_{1,0}xz^2 + a_{0,3}y^3 \\ b_{0,1}yz^2 \\ 0 \end{pmatrix} : a_{0,3} \neq 0, (a_{1,0}, b_{0,1}) \neq 0\right\}$$

$$Z_{11}^{ss} = \mathbb{P}\left\{\begin{pmatrix} a_{1,1}xyz + a_{0,3}y^3 \\ b_{1,0}xz^2 + b_{0,2}y^2z \\ 0 \end{pmatrix} : b_{1,0} \neq 0, (a_{1,1}, a_{0,3}, b_{0,2}) \neq 0\right\}$$

$$Z_{12}^{ss} = \mathbb{P}\left\{\begin{pmatrix} a_{1,2}xy^2 + a_{1,1}xyz + a_{1,0}xz^2 \\ b_{0,3}y^3 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3 \\ c_{0,3}y^3 \end{pmatrix} : \right.$$
$$\left. (a_{1,2}, a_{1,1}, b_{0,3}, b_{0,2}, c_{0,3}) \neq 0, (a_{1,1}, a_{1,0}, b_{0,2}, b_{0,1}, b_{0,0}) \neq 0\right\}$$

$$Z_{14}^{ss} = \mathbb{P}\left\{\begin{pmatrix} a_{1,2}xy^2 \\ b_{1,0}xz^2 + b_{0,3}y^3 \\ 0 \end{pmatrix} : b_{1,0} \neq 0, (a_{1,2}, b_{0,3}) \neq 0\right\}$$

$$Z_{15}^{ss} = \mathbb{P}\left\{\begin{pmatrix} a_{2,0}x^2z + a_{0,3}y^3 \\ b_{1,1}xyz \\ 0 \end{pmatrix} : a_{0,3} \neq 0, (a_{2,0}, b_{1,1}) \neq 0\right\}$$

$$Z_{16}^{ss} = \mathbb{P}\left\{ \begin{pmatrix} 0 \\ b_{1,0}xz^2 \\ c_{0,3}y^3 \end{pmatrix} : b_{1,0} \neq 0, c_{0,3} \neq 0 \right\},$$

and

$$Y_6^{ss} = \mathbb{P}\left\{ \begin{pmatrix} a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ b_{0,0}z^3 \\ 0 \end{pmatrix} : a_{0,3} \neq 0, b_{0,0} \neq 0 \right\}$$

$$Y_9^{ss} = \mathbb{P}\left\{ \begin{pmatrix} a_{1,0}xz^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ b_{0,1}yz^2 + b_{0,0}z^3 \\ 0 \end{pmatrix} : a_{0,3} \neq 0, (a_{1,0}, b_{0,1}) \neq 0 \right\}$$

$$Y_{11}^{ss} = \mathbb{P}\left\{ \begin{pmatrix} a_{1,1}xyz + a_{1,0}xz^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ b_{1,0}xz^2 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3 \\ 0 \end{pmatrix} : \right.$$
$$\left. b_{1,0} \neq 0, (a_{1,1}, a_{0,3}, b_{0,2}) \neq 0 \right\}$$

$$Y_{12}^{ss} = \mathbb{P}\left\{ \begin{pmatrix} \sum_{j=0}^{2} a_{1,j}xy^jz^{2-j} + \sum_{j=0}^{3} a_{0,j}y^jz^{3-j} \\ \sum_{j=0}^{3} b_{0,j}y^jz^{3-j} \\ c_{0,3}y^3 \end{pmatrix} : \right.$$
$$\left. (a_{1,2}, a_{1,1}, b_{0,3}, b_{0,2}, c_{0,3}) \neq 0, (a_{1,1}, a_{1,0}, b_{0,2}, b_{0,1}, b_{0,0}) \neq 0 \right\}$$

$$Y_{14}^{ss} = \mathbb{P}\left\{ \begin{pmatrix} a_{1,2}xy^2 + a_{1,1}xyz + a_{1,0}xz^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ b_{1,0}xz^2 + b_{0,3}y^3 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3 \\ 0 \end{pmatrix} : \right.$$
$$\left. b_{1,0} \neq 0, (a_{1,2}, b_{0,3}) \neq 0 \right\}$$

$$Y_{15}^{ss} = \mathbb{P}\left\{ \begin{pmatrix} a_{2,0}x^2z + a_{1,1}xyz + a_{1,0}xz^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ b_{1,1}xyz + b_{1,0}xz^2 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3 \\ 0 \end{pmatrix} : \right.$$
$$\left. a_{0,3} \neq 0, (a_{2,0}, b_{1,1}) \neq 0 \right\}$$

$$Y_{16}^{ss} = \mathbb{P}\left\{ \begin{pmatrix} a_{1,2}xy^2 + a_{1,1}xyz + a_{1,0}xz^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ b_{1,0}xz^2 + b_{0,3}y^3 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3 \\ c_{0,3}y^3 \end{pmatrix} : \right.$$
$$\left. b_{1,0} \neq 0, c_{0,3} \neq 0 \right\}.$$

6. Strata of the space of foliations of degree 3 and its singularities

In this section we calculate the Milnor number and the multiplicity of a common singularity in the generic foliation in every stratum. We also obtain the dimension of the strata.

Note that the point $p = (1 : 0 : 0)$ is a singularity for every foliation in Y_i^{ss} for all $i = 6$, 9, 11, 12, 14, 15, 16. Along this section we use the following notation: given a foliation,

$$X = \begin{pmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{pmatrix} \in Y_i^{ss},$$

we consider the corresponding local polynomial vector field around $(0,0)$:

$$X_0 = (Q(1,y,z) - yP(1,y,z))\frac{\partial}{\partial y} + (R(1,y,z) - zP(1,y,z))\frac{\partial}{\partial z}.$$

We define $f_i(y,z) := Q(1,y,z) - yP(1,y,z)$, $g_i(y,z) := R(1,y,z) - zP(1,y,z)$ and $I_0(f,g)$ will be the intersection index of f and g at $(0,0)$.

6.1. **Stratum 6.** As we saw before if $X \in Y_6^{ss}$ then $a_{0,3}$ and $b_{0,0}$ are different from zero and

$$\begin{aligned} f_6(y,z) = Q(1,y,z) - yP(1,y,z) &= b_{0,0}z^3 - a_{0,3}y^4 - a_{0,2}y^3z - a_{0,1}y^2z^2 - a_{0,0}yz^3 \\ g_6(y,z) = -zP(1,y,z) &= -a_{0,3}y^3z - a_{0,2}y^2z^2 - a_{0,1}yz^3 - a_{0,0}z^4. \end{aligned}$$

Note that $b_{0,0}z^3$ and $P(1,y,z)$ does not have common tangent lines, therefore

$$\begin{aligned} \mu_p(X) &= I_0(f_6(y,z), g_6(y,z)) = I_0(f_6(y,z), z) + I_0(f_6(y,z), P(1,y,z)) \\ &= I_0(-a_{0,3}y^4, z) + 9 = 13 \\ m_p(X) &= 3. \end{aligned}$$

Finally, the 2-jet of $\begin{pmatrix} f_6 \\ g_6 \end{pmatrix}$ is trivial and the 3-jet is $\begin{pmatrix} z^3 \\ 0 \end{pmatrix}$, if we suppose $b_{0,0} = 1$. On the other hand, if X is a foliation of degree 3 with $m_{(1:0:0)}(X) = 3$, $\mu_{(1:0:0)}(X) = 13$ and with 3-jet $\begin{pmatrix} z^3 \\ 0 \end{pmatrix}$, it is easy to see that $X \in Y_6^{ss}$. In this case the corresponding parabolic subgroup P_6 is the subgroup of upper triangular matrices, therefore $\dim S_6 = \dim Y_6^{ss} + \dim SL(3,\mathbb{C}) - \dim P_6 = 7$ (see Remark 1).

6.2. **Stratum 9.** If $X \in Y_9^{ss}$ then $a_{0,3} \neq 0$ and $(a_{1,0}, b_{0,1}) \neq (0,0)$; therefore

$$\begin{aligned} f_9(y,z) = Q(1,y,z) - yP(1,y,z) &= (b_{0,1} - a_{1,0})yz^2 + b_{0,0}z^3 \\ &\quad - a_{0,3}y^4 - a_{0,2}y^3z - a_{0,1}y^2z^2 - a_{0,0}yz^3 \\ g_9(y,z) = -zP(1,y,z) &= -a_{1,0}z^3 - a_{0,3}y^3z - a_{0,2}y^2z^2 - a_{0,1}yz^3 - a_{0,0}z^4, \end{aligned}$$

and

$$\begin{aligned} \mu_p(X) &= I_0(f_9(y,z), g_9(y,z)) = I_0(f_9(y,z), z) + I_0(f_9(y,z), P(1,y,z)) \\ &= I_0(a_{0,3}y^4, z) + 2I_0(z, a_{0,3}y^3) + I_0(b_{0,1}y + b_{0,0}z, P(1,y,z)) \\ &= 10 + I_0(b_{0,1}y + b_{0,0}z, P(1,y,z)). \end{aligned}$$

Note that $I_0(b_{0,1}y + b_{0,0}z, P(1,y,z))$ is

$$\begin{cases} 2 & a_{1,0} \neq 0, b_{0,1} \neq 0 \\ 3 & (b_{0,1} = 0, b_{0,0} \neq 0) \text{ or } (a_{1,0} = 0 \text{ and } b_{0,1}y + b_{0,0}z \text{ is not tangent for } P(1,y,z)) \\ \infty & (b_{1,0}, b_{0,0}) = 0 \text{ or } (a_{1,0} = 0 \text{ and } b_{0,1}y + b_{0,0}z \text{ is tangent for } P(1,y,z)) \end{cases}$$

If $a_{1,0} \neq 0$ it is clear that the multiplicity of the singular point is 3. If $a_{1,0} = 0$ then $b_{0,1} \neq 0$ and also the multiplicity is 3. Finally, the 3-jet is $\begin{pmatrix} (b_{0,1} - a_{1,0})yz^2 + b_{0,0}z^3 \\ -a_{1,0}z^3 \end{pmatrix}$. Then in the open set where $a_{1,0} \neq 0, b_{0,1} \neq 0$ every foliation has a singular point with multiplicity 3 and Milnor number 12. In this case the corresponding parabolic subgroup is the subgroup of upper triangular matrices, therefore $\dim S_9 = 9$.

6.3. **Stratum 11.** Remember that if $X \in Y_{11}^{ss}$ then $b_{1,0}$ and $(a_{1,1}, a_{0,3}, b_{0,2})$ are different from zero, and

$$\begin{aligned} f_{11}(y,z) &= Q(1,y,z) - yP(1,y,z) \\ &= b_{1,0}z^2 + (b_{0,2} - a_{1,1})y^2z + (b_{0,1} - a_{1,0})yz^2 + b_{0,0}z^3 \\ &\quad - a_{0,3}y^4 - a_{0,2}y^3z - a_{0,1}y^2z^2 - a_{0,0}yz^3, \\ g_{11}(y,z) &= -zP(1,y,z) = -a_{1,1}yz^2 - a_{1,0}z^3 - a_{0,3}y^3z - a_{0,2}y^2z^2 - a_{0,1}yz^3 - a_{0,0}z^4. \end{aligned}$$

If $a_{0,3} = 0$, then $z = 0$ is a curve of singularities. Suppose $a_{0,3} \neq 0$. Note that

$$\begin{aligned} I_0(f_{11}, g_{11}) = I_0(-a_{0,3}y^4, z) + I_0(z, -a_{0,3}y^3) + \\ I_0(b_{1,0}z + b_{0,2}y^2 + b_{0,1}yz + b_{0,0}z^2, P(1,y,z)) \\ = 7 + I_0(b_{1,0}z + b_{0,2}y^2 + b_{0,1}yz + b_{0,0}z^2, P(1,y,z)). \end{aligned}$$

And

$$\begin{aligned} & I_0(b_{1,0}z + b_{0,2}y^2 + b_{0,1}yz + b_{0,0}z^2, a_{1,1}yz + a_{1,0}z^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3) \\ &= I_0\Bigg(b_{1,0}z + b_{0,2}y^2 + b_{0,1}yz + b_{0,0}z^2, \left(a_{0,3} - \frac{a_{1,1}}{b_{1,0}} b_{0,2} \right) y^3 \\ &\qquad + \left(a_{0,2} - \frac{a_{1,0}}{b_{1,0}} b_{0,2} - \frac{a_{1,1}}{b_{1,0}} b_{0,1} \right) y^2z + \left(a_{0,1} - \frac{a_{1,0}}{b_{1,0}} b_{0,1} - \frac{a_{1,1}}{b_{1,0}} b_{0,0} \right) yz^2 \\ &\qquad + \left(a_{0,0} - \frac{a_{1,0}}{b_{1,0}} b_{0,0} \right) z^3 \Bigg) \\ &= \begin{cases} 3 & \text{if } a_{0,3}b_{1,0} \neq a_{1,1}b_{0,2} \\ 4 & \text{if } a_{0,3}b_{1,0} = a_{1,1}b_{0,2} \text{ and } a_{0,2}b_{1,0} \neq a_{1,1}b_{0,1} + a_{1,0}b_{0,2} \\ 5 & \text{if } [\ldots] \text{ and } a_{0,1}b_{1,0} \neq a_{1,1}b_{0,0} + a_{1,0}b_{0,1} \\ 6 & \text{if } [\ldots], a_{0,1}b_{1,0} = a_{1,1}b_{0,0} + a_{1,0}b_{0,1} \text{ and } a_{0,0}b_{1,0} \neq a_{1,0}b_{0,0} \\ \infty & \text{if } [\ldots], a_{0,1}b_{1,0} = a_{1,1}b_{0,0} + a_{1,0}b_{0,1} \text{ and } a_{0,0}b_{1,0} = a_{1,0}b_{0,0} \end{cases} \end{aligned}$$

where $[\ldots]$ is $a_{0,3}b_{1,0} = a_{1,1}b_{0,2}, a_{0,2}b_{1,0} = a_{1,1}b_{0,1} + a_{1,0}b_{0,2}$.

We conclude that in the open set of Y_{11} where $a_{0,3}b_{1,0} \neq a_{1,1}b_{0,2}$ every foliation has a singularity with Milnor number 10. Since $b_{1,0} \neq 0$ then the multiplicity for the singular point $(1:0:0)$ is equal to 2 and the 2-jet is $\begin{pmatrix} z^2 \\ 0 \end{pmatrix}$. The corresponding parabolic subgroup is the subgroup of upper triangular matrices, therefore $\dim S_{11} = 12$.

6.4. **Stratum 12.** If $X \in Y_{12}^{ss}$ then we have that

$$(a_{1,2}, a_{1,1}, b_{0,2}, b_{0,3}, c_{0,3}) \quad \text{and} \quad (a_{1,1}, a_{1,0}, b_{0,0}, b_{0,1}, b_{0,2})$$

are different from zero and

$$\begin{aligned} f_{12}(y,z) &= Q(1,y,z) - yP(1,y,z) \\ &= (b_{0,3} - a_{1,2})y^3 + (b_{0,2} - a_{1,1})y^2z + (b_{0,1} - a_{1,0})yz^2 + b_{0,0}z^3 \\ &\quad - a_{0,3}y^4 - a_{0,2}y^3z - a_{0,1}y^2z^2 - a_{0,0}yz^3 \\ g_{12}(y,z) &= c_{0,3}y^3 - zP(1,y,z) \\ &= c_{0,3}y^3 - a_{1,2}y^2z - a_{1,1}yz^2 - a_{1,0}z^3 \\ &\quad - a_{0,3}y^3z - a_{0,2}y^2z^2 - a_{0,1}yz^3 - a_{0,0}z^4. \end{aligned}$$

These polynomials are homogenous in two variables, then generically we have

$$I_p(f_{12}, g_{12}) = 9.$$

If $(a_{1,2}, a_{1,1}, c_{0,3}) \neq 0$ then $g_{12} \neq 0$ and $m_p(X) = 3$. If $(a_{1,2}, a_{1,1}, c_{0,3}) = 0$ then $(b_{0,3}, b_{0,2}) \neq 0$ and we have also $m_p(X) = 3$. In this case the parabolic subgroup is

$$P_{12} = \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23} \\ 0 & \alpha_{32} & \alpha_{33} \end{pmatrix} \in SL(3,\mathbb{C}) \right\},$$

therefore $\dim S_{12} = 13$.
Moreover, the set

$$\{X \in \mathcal{F}_3 : \text{there exists } p \text{ such that } m_p(X) = 3,\ \mu_p(X) = 9\},$$

is an open set in S_{12} because a foliation with these properties for the point $(1:0:0)$ is unstable and it does not be in another stratum.

6.5. **Stratum 14.** If $X \in Y_{14}^{ss}$ then $b_{1,0}$ and $(a_{1,2}, b_{0,3})$ are different from zero, and

$$\begin{aligned} f_{14}(y,z) &= Q(1,y,z) - yP(1,y,z) \\ &= b_{1,0}z^2 + (b_{0,3} - a_{1,2})y^3 + (b_{0,2} - a_{1,1})y^2z + (b_{0,1} - a_{1,0})yz^2 + b_{0,0}z^3 \\ &\quad - a_{0,3}y^4 - a_{0,2}y^3z - a_{0,1}y^2z^2 - a_{0,0}yz^3 \\ g_{14}(y,z) &= -zP(1,y,z) = -a_{1,2}y^2z - a_{1,1}yz^2 - a_{1,0}z^3 \\ &\quad - a_{0,3}y^3z - a_{0,2}y^2z^2 - a_{0,1}yz^3 - a_{0,0}z^4. \end{aligned}$$

Note that

$$\begin{aligned} I_0(f_{14}, g_{14}) &= I_0((b_{0,3} - a_{1,2})y^3 - a_{0,3}y^4, z) \\ &\quad + I_0(f_{14}, -a_{1,2}y^2 - a_{1,1}yz - a_{1,0}z^2 - a_{0,3}y^3 - a_{0,2}y^2z - a_{0,1}yz^2 - a_{0,0}z^3). \end{aligned}$$

If we suppose that $a_{1,2} \neq 0$ and $b_{0,3} \neq a_{1,2}$ then the Milnor number of $(1:0:0)$ is 7. If $a_{1,2} \neq 0, b_{0,3} = a_{1,2}$ and $a_{0,3} \neq 0$ we have $\mu_{(1:0:0)}(X) = 8$. On the other hand, $a_{1,2} \neq 0, b_{0,3} = a_{1,2}$

and $a_{0,3}=0$ implies that we have a curve of singularities. Supposing $a_{1,2}=0$, we obtain $b_{0,3}\neq a_{1,2},0$, and with this we have

$$\begin{aligned}
&I_0(f_{14},a_{1,1}yz+a_{1,0}z^2+a_{0,3}y^3+a_{0,2}y^2z+a_{0,1}yz^2+a_{0,0}z^3)\\
&\quad=I_0\Bigg(b_{1,0}z^2+b_{0,3}y^3+b_{0,2}y^2z+b_{0,1}yz^2+b_{0,0}z^3,a_{1,1}yz+\left(a_{1,0}-\frac{a_{0,3}}{b_{0,3}}b_{1,0}\right)z^2\\
&\qquad\quad+\left(a_{0,2}-\frac{a_{0,3}}{b_{0,3}}b_{0,2}\right)y^2z+\left(a_{0,1}-\frac{a_{0,3}}{b_{0,3}}b_{0,1}\right)yz^2+\left(a_{0,0}-\frac{a_{0,3}}{b_{0,3}}b_{0,0}\right)z^3\Bigg)\\
&\quad=3+I_0\Bigg(b_{1,0}z^2+b_{0,3}y^3+b_{0,2}y^2z+b_{0,1}yz^2+b_{0,0}z^3,a_{1,1}y+\left(a_{1,0}-\frac{a_{0,3}}{b_{0,3}}b_{1,0}\right)z\\
&\qquad\qquad+\left(a_{0,2}-\frac{a_{0,3}}{b_{0,3}}b_{0,2}\right)y^2+\left(a_{0,1}-\frac{a_{0,3}}{b_{0,3}}b_{0,1}\right)yz+\left(a_{0,0}-\frac{a_{0,3}}{b_{0,3}}b_{0,0}\right)z^2\Bigg).
\end{aligned}$$

We can verify that if $a_{1,1}\neq 0$ then the last expression is equal to 5. When $a_{1,1}=0$ we can see that

$$\begin{aligned}
&I_0(f_{14},a_{1,1}yz+a_{1,0}z^2+a_{0,3}y^3+a_{0,2}y^2z+a_{0,1}yz^2+a_{0,0}z^3)\\
&\quad=I_0\left(b_{1,0}z^2+b_{0,3}y^3+b_{0,2}y^2z+b_{0,1}yz^2+b_{0,0}z^3,a'_{0,3}y^3+a'_{0,2}y^2z+a'_{0,1}yz^2+a'_{0,0}z^3\right),
\end{aligned}$$

where $a'_{i,j}$ denotes $a_{i,j}-\frac{a_{1,0}}{b_{1,0}}b_{i,j}$.

In conclusion,

$$\begin{aligned}
&I_0(b_{1,0}z^2+b_{0,3}y^3+b_{0,2}y^2z+b_{0,1}yz^2+b_{0,0}z^3,a'_{0,3}y^3+a'_{0,2}y^2+a'_{0,1}yz+a'_{0,0}z^2)\\
&\qquad=\begin{cases}6 & \text{if } a'_{0,3}\neq 0\\ 7 & \text{if } a'_{0,3}=0 \text{ and } a'_{0,2}\neq 0\\ 8 & \text{if } (a'_{0,3},a'_{0,2})=0 \text{ and } a'_{0,1}\neq 0\\ 9 & \text{if } (a'_{0,3},a'_{0,2},a'_{0,1})=0 \text{ and } a'_{0,0}\neq 0\\ \infty & \text{if } (a'_{0,3},a'_{0,2},a'_{0,1},a'_{0,0})=0.\end{cases}
\end{aligned}$$

Therefore,

$$\mu_{(1:0:0)}(X)=\begin{cases}7 & \text{if } b_{0,3}\neq a_{1,2}\neq 0\\ 8 & \text{if } (b_{0,3}=a_{1,2}\neq 0 \text{ and } a_{0,3}\neq 0) \text{ or if } (a_{1,2}=0 \text{ and } a_{1,1}\neq 0)\\ 9 & \text{if } (a_{1,2},a_{1,1})=0 \text{ and } a'_{0,3}\neq 0\\ 10 & \text{if } (a_{1,2},a_{1,1},a'_{0,3})=0 \text{ and } a'_{0,2}\neq 0\\ 11 & \text{if } (a_{1,2},a_{1,1},a'_{0,3},a'_{0,2})=0 \text{ and } a'_{0,1}\neq 0\\ 12 & \text{if } (a_{1,2},a_{1,1},a'_{0,3},a'_{0,2},a'_{0,1})=0 \text{ and } a'_{0,0}\neq 0\\ \infty & \text{if } (a_{1,2},a_{1,1},a'_{0,3},a'_{0,2},a'_{0,1},a'_{0,0})=0\end{cases}$$

Since $b_{1,0}\neq 0$ we get $m_{(1:0:0)}(X)=2$ with 2-jet $\begin{pmatrix}z^2\\0\end{pmatrix}$. Since the parabolic subgroup is the subgroup of upper triangular matrices we obtain $\dim S_{14}=14$. Moreover, the set

$\{X\in\mathcal{F}_3$: there exists p such that $m_p(X)=2$, $\mu_p(X)=7$ and 2-jet linearly equivalent to $\begin{pmatrix}z^2\\0\end{pmatrix}\}$, is an open set in S_{14}, because a foliation with these properties for the point $(1:0:0)$ is unstable and it does not be in another stratum.

6.6. **Stratum 15.** If $X \in Y_{15}^{ss}$ we have that $a_{0,3}$ and $(a_{2,0}, b_{1,1})$ are different from zero, and

$$\begin{aligned} f_{15}(y,z) &= Q(1,y,z) - yP(1,y,z) \\ &= (b_{1,1} - a_{2,0})yz + b_{1,0}z^2 + (b_{0,2} - a_{1,1})y^2 z + (b_{0,1} - a_{1,0})yz^2 + b_{0,0}z^3 \\ &\quad - a_{0,3}y^4 - a_{0,2}y^3 z - a_{0,1}y^2 z^2 - a_{0,0}yz^3 \\ g_{15}(y,z) &= -zP(1,y,z) \\ &= -a_{2,0}z^2 - a_{1,1}yz^2 - a_{1,0}z^3 - a_{0,3}y^3 z - a_{0,2}y^2 z^2 - a_{0,1}yz^3 - a_{0,0}z^4. \end{aligned}$$

Note that

$$\begin{aligned} I_0(f_{15}, g_{15}) &= I_0(-a_{0,3}y^4, z) \\ &\quad + I_0(f_{15}, -a_{1,2}y - a_{1,1}z - a_{1,0}z^2 - a_{0,3}y^3 - a_{0,2}y^2 z - a_{0,1}yz^2 - a_{0,0}z^3) \\ &= 4 + I_0(b_{1,1}yz + b_{1,0}z^2 + b_{0,2}y^2 z + b_{0,1}yz^2 + b_{0,0}z^3, \\ &\qquad a_{2,0}z + a_{1,1}yz + a_{1,0}z^2 + a_{0,3}y^3 + a_{0,2}y^2 z + a_{0,1}yz^2 + a_{0,0}z^3) \\ &= 4 + I_0(z, a_{0,3}y^3) + I_0(b_{1,1}y + b_{1,0}z + b_{0,2}y^2 + b_{0,1}yz + b_{0,0}z^2, \\ &\qquad a_{2,0}z + a_{1,1}yz + a_{1,0}z^2 + a_{0,3}y^3 + a_{0,2}y^2 z + a_{0,1}yz^2 + a_{0,0}z^3). \end{aligned}$$

It is clear that if $b_{1,1}$ and $a_{2,0}$ are different from zero then the intersection index of f_{15} and g_{15} is 8. Suppose that $a_{2,0} \neq 0$ and $b_{1,1} = 0$, then

$$\begin{aligned} I_0(b_{1,0}z + b_{0,2}y^2 + b_{0,1}yz + b_{0,0}z^2, a_{2,0}z + a_{1,1}yz + a_{1,0}z^2 + \\ a_{0,3}y^3 + a_{0,2}y^2 z + a_{0,1}yz^2 + a_{0,0}z^3) \\ = \begin{cases} 2 & \text{if } b_{0,2} \neq 0 \\ 3 & \text{if } b_{0,2} = 0 \text{ and } b_{1,0} \neq 0 \\ 4 & \text{if } (b_{0,2}, b_{1,0}) = 0 \text{ and } b_{0,1} \neq 0 \\ 6 & \text{if } (b_{0,2}, b_{1,0}, b_{0,1}) = 0 \text{ and } b_{0,0} \neq 0 \\ \infty & \text{if } (b_{0,2}, b_{1,0}, b_{0,1}, b_{0,0}) = 0 \end{cases} \end{aligned}$$

Now suppose $a_{2,0} = 0$ and $b_{1,1} \neq 0$. We define

$$\begin{aligned} &L(y,z) = b_{1,1}y + b_{1,0}z \quad M(y,z) = a_{1,1}yz + a_{1,0}z^2 \\ &N(y,z) = b_{0,2}y^2 + b_{0,1}yz + b_{0,0}z^2 \quad F(y,z) = a_{0,3}y^3 + a_{0,2}y^2 z + a_{0,1}yz^2 + a_{0,0}z^3. \end{aligned}$$

We have

$$I_0(L+N, M+F) = 2 \qquad \text{if } L \nmid M.$$

Now suppose $L \mid M$, we have two cases: if $L \nmid N$,

$$
\begin{aligned}
I_0(L+N, M+F) &= I_0\left(L+N, F-\frac{M}{L}N\right) \\
&= \begin{cases} 3 & \text{if } L^2 \nmid (LF-MN) \\ 4 & \text{if } L^2 \mid (LF-MN) \text{ and } L^3 \nmid (LF-MN) \\ 5 & \text{if } L^3 \mid (LF-MN) \text{ and } L^4 \nmid (LF-MN) \\ 6 & \text{if } L^4 \mid (LF-MN) \end{cases}.
\end{aligned}
$$

On the other hand, if $L \mid N$:

$$
\begin{aligned}
I_0\left(L+N, F-\frac{M}{L}N\right) &= I_0\left(L, F-\frac{M}{L}N\right) + I_0\left(1+\frac{N}{L}, F-\frac{M}{L}N\right) \\
= I_0\left(L, F-\frac{M}{L}N\right) &= \begin{cases} 3 & \text{if } L \nmid (LF-MN) \\ \infty & \text{if } L \mid (LF-MN) \end{cases}
\end{aligned}
$$

We conclude that in S_{15} we have a nonempty open set which consists of foliations with a singularity with multiplicity 2 and Milnor number 8. But we can have in this stratum foliations with a singularity with multiplicity 2 and Milnor number 9, 10, 11, 12 and 13 or with a curve of singularities. The 2-jet around the singular point $(1:0:0)$ of X is $\begin{pmatrix} (b_{1,1}-a_{2,0})yz + b_{1,0}z^2 \\ -a_{2,0}z^2 \end{pmatrix}$. Since the parabolic subgroup is the subgroup of upper triangular matrices we obtain $\dim S_{15} = 14$.

6.7. **Stratum 16.** If $X \in Y_{16}^{ss}$ then $b_{1,0} \neq 0$ and $c_{0,3} \neq 0$. We have

$$
\begin{aligned}
f_{15}(y,z) &= Q(1,y,z) - yP(1,y,z) \\
&= b_{1,0}z^2 + (b_{0,3}-a_{1,2})y^3 + (b_{0,2}-a_{1,1})y^2z + (b_{0,1}-a_{1,0})yz^2 + b_{0,0}z^3 \\
&\quad - a_{0,3}y^4 - a_{0,2}y^3z - a_{0,1}y^2z^2 - a_{0,0}yz^3 \\
g_{15}(y,z) &= c_{0,3}y^3 - zP(1,y,z) \\
&= c_{0,3}y^3 - a_{1,2}y^2z - a_{1,1}yz^2 - a_{1,0}z^3 - a_{0,3}y^3z - a_{0,2}y^2z^2 - a_{0,1}yz^3 - a_{0,0}z^4,
\end{aligned}
$$

and the polynomials $c_{0,3}y^3 - a_{1,2}y^2z - a_{1,1}yz^2 - a_{1,0}z^3$ and $b_{1,0}z^2$ do not have common factors. As a result the Milnor number of $(1:0:0)$ is 6. And the multiplicity of this point is 2 with 2-jet $\begin{pmatrix} z^2 \\ 0 \end{pmatrix}$. The corresponding parabolic subgroup is the subgroup of upper triangular matrices, therefore $\dim S_{16} = 15$.
In the following theorem we summarize the above.

Theorem 7. *The spaces $S_i = SL(3,\mathbb{C})Y_i^{ss}$ for $i \in \{1,\dots,16\}$, are locally closed, irreducible non-singular algebraic subvarieties of $\mathcal{F}_3$. They form a stratification of the closed set of unstable foliations $\mathcal{F}_3^{un}$, and $\overline{S_i} \subset \bigcup_{j\leq i} S_j$. Moreover, these varieties satisfy the following:*

Stratum	*Characterization of the generic foliation*
$S_1,\ S_2,\ S_3,$ $S_4,\ S_5,\ S_7,$ $S_8,\ S_{10},\ S_{13}$	Every foliation has a curve of singularities
S_6 $\dim S_6 = 7$	$\left\{X \in \mathcal{F}_3 : \exists p \text{ with } m_p(X) = 3,\ \mu_p(X) = 13 \text{ and 3-jet linearly equivalent to } \begin{pmatrix} z^3 \\ 0 \end{pmatrix}\right\} = S_6$
S_9 $\dim S_9 = 9$	$\{X \in \mathcal{F}_3 : \exists p \text{ with } m_p(X) = 3,\ \mu_p(X) = 12\} \cap S_9$ is open in S_9
S_{11} $\dim S_{11} = 12$	$\{X \in \mathcal{F}_3 : \exists p \text{ with } m_p(X) = 2,\ \mu_p(X) = 10 \text{ and 2-jet linearly equivalent to } \begin{pmatrix} z^2 \\ 0 \end{pmatrix}\} \cap S_{11}$ is open in S_{11}
S_{12} $\dim S_{12} = 13$	Contains $\{X \in \mathcal{F}_3 : \exists p \text{ with } m_p(X) = 3,\ \mu_p(X) = 9\}$ as an open set
S_{14} $\dim S_{14} = 14$	*Contains* $\left\{X \in \mathcal{F}_3 : \exists p \text{ with } m_p(X) = 2,\ \mu_p(X) = 7 \text{ and 2-jet linearly equivalent to } \begin{pmatrix} z^2 \\ 0 \end{pmatrix}\right\}$ *as an open set*
S_{15} $\dim S_{15} = 14$	$\{X \in \mathcal{F}_3 : \exists p \text{ with } m_p(X) = 2,\ \mu_p(X) = 8\} \cap S_{15}$ *is open in* S_{15}
S_{16} $\dim S_{16} = 15$	$\{X \in \mathcal{F}_3 : \exists p \text{ with } m_p(X) = 2,\ \mu_p(X) = 6 \text{ and 2-jet linearly equivalent to } \begin{pmatrix} z^2 \\ 0 \end{pmatrix}\} = S_{16}$

In [3] we also have studied the strata S_6, S_9 and S_{16}. As a consequence of the above we can mention the following general result.

Theorem 8. *Let $X \in \mathcal{F}_3$ with isolated singularities. Then X is unstable if and only if:*

(1) *X has a singular point with multiplicity* 3 *or*
(2) *X has a singular point with multiplicity* 2 *and 2-jet linearly equivalent to $z^2\frac{\partial}{\partial y}$ or*
(3) *$X \in S_{15}$.*

Moreover, the irreducible components of $\mathcal{F}_3^{un}$ are the closure of the locally closed subvarieties S_{15} and S_{16}. The first one has dimension 14 *and the second one has dimension* 15.

Proof. The first affirmation is consequence of the results in the table. For the second one we use proposition 4.2 of [9], it says that S_j is irreducible and $\overline{S_j} = SL(3,\mathbb{C})\overline{Y_j}$ for all j. Since $\overline{Y_{16}} \subset \bigcup_{j\neq 15} Y_j$ and $S_{15} \not\subset S_{16}$, we have that $\mathcal{F}_3^{un} = \overline{S_{15}} \cup \overline{S_{16}}$ is the decomposition in irreducible components.

□

Remark 2. *Note that $V_3 = Y_{15}$ and $V_1 = Y_{16}$ in theorem 5.*

6.8. **Semistable non-stable foliations on $\mathbb{CP}^2$ of degree 3.** In this subsection we describe the semistable non-stable foliations on $\mathbb{CP}^2$ of degree 3.

Theorem 9. *The set of semistable non-stable foliations on $\mathbb{CP}^2$ of degree* 3 *with isolated singularities is*

$$SL(3,\mathbb{C})\mathbb{P}\left\{\begin{pmatrix} P(y,z) \\ b_{1,1}xyz + b_{1,0}xz^2 + \sum_{j=0}^{3} b_{0,j}y^j z^{3-j} \\ c_{1,0}xz^2 + c_{0,2}y^2z + c_{0,1}yz^2 + b_{0,0}z^3 \end{pmatrix} : P(y,z) \in \mathbb{C}_3[y,z], (b_{0,3}, c_{0,2}) \neq 0, (b_{1,1}, c_{1,0}) \neq 0\right\}.$$

Proof. Let $X \in \mathcal{F}_3^{ss} - \mathcal{F}_3^s$ then X is not in any strata and it satisfies one of the following properties:

(1) $\dim O(X) < 8$: in this case, by Theorem 1.2 of [1], the foliation is λ-invariant for some 1-PS λ and it is not in any Z_j. Then the foliation is, up to change of coordinates, such that the line with its weights pass through zero, if we see the representation we conclude that X is:
$$\begin{pmatrix} 0 \\ b_{1,1}xyz + b_{0,3}y^3 \\ c_{1,0}xz^2 + c_{0,2}y^2z \end{pmatrix},$$
where $(b_{0,3}, c_{0,2}) \neq 0, (b_{1,1}, c_{1,0}) \neq 0$, since X has isolated singularities we can suppose $c_{1,0} = 1$.

(2) $O(X)$ is not closed in $\mathcal{F}_3^{ss}$: then there exists $Y \in \mathcal{F}_3^{ss} \cap (\overline{O(X)} - O(X))$. Since $\dim O(Y) < 8$, we conclude that Y is the above foliation, therefore X has its weights in one hyperplane given by the weights of Y, therefore X is, up to change of coordinates, in

$$\mathbb{P}\left\{ \begin{pmatrix} P(y,z) \\ b_{1,1}xyz + b_{1,0}xz^2 + \sum_{j=0}^{3} b_{0,j}y^jz^{3-j} \\ c_{1,0}xz^2 + c_{0,2}y^2z + c_{0,1}yz^2 + b_{0,0}z^3 \end{pmatrix} : P(y,z) \in \mathbb{C}_3[y,z], (b_{0,3}, c_{0,2}) \neq 0, (b_{1,1}, c_{1,0}) \neq 0 \right\}.$$

□

7. Corollaries

7.1. **The dimension of the orbits.** Generically the orbit of a foliation on $\mathbb{CP}^2$ has dimension 8, for example, a stable foliation satisfies this property. It theorem 1.2 of [1] we classify foliations with isolated singular points such that the dimension of the orbit is less than or equal to 7. We can see in proposition 2.3 of [5] that the dimension of an orbit of a foliation with isolated singularities of degree d is greater than or equal to 6. In the same paper the authors describe the two unique foliations of degree 2, up to change of coordinates, such that the orbit has dimension 6. For the case of foliations of degree 3 we have the same situation.

Theorem 10. *There are, up to change of coordinates, two foliations on* $\mathbb{CP}^2$ *of degree* 3 *with isolated singularities with automorphism group of dimension* 2*:* $y^3\frac{\partial}{\partial x} + z^3\frac{\partial}{\partial y}$ *and* $y^3\frac{\partial}{\partial x} + z^3\frac{\partial}{\partial z}$.

Proof. In proposition 2.5 of [5] we can see that if $Aut(X)$ has dimension 2 then it is isomorphic to the group of affine transformations of the line. Therefore by theorem 1.2 of [1], X is λ-invariant for some 1-PS λ, and X is also invariant by $(\mathbb{C},+)$. This last affirmation implies, by the same theorem, that X is unstable with a singular point with Milnor number ≥ 12. An unstable foliation invariant by a 1-PS is, up to change of coordinates, in $Z_{15} \cup Z_{14} \cup Z_{12} \cup Z_{11} \cup Z_9 \cup Z_6$. It is easy to see that the unique foliations with a singular point with Milnor number ≥ 12 are in Z_9 and Z_6, they are $y^3\frac{\partial}{\partial x} + z^3\frac{\partial}{\partial y}$ and $y^3\frac{\partial}{\partial x} + z^3\frac{\partial}{\partial z}$.

□

7.2. **Foliations on $\mathbb{CP}^2$ of degree 3 with one singular point.** The classification of foliations on $\mathbb{CP}^2$ with one singular point is known only for degree 2 (see [5] and [2]). In this section we describe all the unstable foliations on $\mathbb{CP}^2$ of degree 3 with one singular point, that means with a singular point with Milnor number 13. To obtain the result we need the following lemma.

Lemma 3. *Let X be a foliation on $\mathbb{CP}^2$ of degree d. If X has a singular point p with multiplicity d and Milnor number greater than d^2, then X has an invariant line that passes through p.*

Proof. We can suppose that X is a foliation on $\mathbb{CP}^2$ of degree d such that $m_{(1:0:0)}(X) = d$ and $\mu_{(1:0:0)}(X) > d^2$. Then

$$X = \begin{pmatrix} xP_{d-1} + P_d \\ Q_d \\ R_d \end{pmatrix}$$

where $P_k, Q_k, R_k \in \mathbb{C}[y, z]$ are homogeneous of degree k. In the chart U_0, the foliation is

$$(Q_d - yP_{d-1})\frac{\partial}{\partial y} + (R_d - zP_{d-1})\frac{\partial}{\partial z},$$

since $\mu_{(1:0:0)}(X) > d^2$ then there exists a line $L = \alpha y - \beta z$ such that $Q_d - yP_{d-1} = LF$ and $R_d - zP_{d-1} = LG$ for some $F, G \in \mathbb{C}[y, z]$. Therefore $\alpha Q_d - \beta R_d = L(P_{d-1} + \alpha F - \beta G)$, and this means that L is invariant for X and it passes through $(1 : 0 : 0)$. $\square$

Theorem 11. *The unstable foliations on $\mathbb{CP}^2$ of degree 3 with one singular point are:*

(1) *The stratum S_6, which has dimension 7.*

(2) *The subspace of S_9:*

$$SL(3,\mathbb{C})\left\{ \begin{pmatrix} a_{1,0}xz^2 + y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ b_{0,1}yz^2 + b_{0,0}z^3 \\ 0 \end{pmatrix} : b_{0,1} = 0, a_{1,0}b_{0,0} \neq 0 \textit{ or} \right.$$

$$\left. a_{1,0} = 0, b_{01} \neq 0 \textit{ and } b_{0,1}y + b_{0,0}z \nmid y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \right\},$$

of dimension 8.

(3) *The subspace of S_{11}:*

$$SL(3,\mathbb{C})\left\{ \begin{pmatrix} a_{1,1}xyz + a_{1,0}xz^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ xz^2 + b_{0,2}y^2z + b_{0,1}yz^2 + b_{0,0}z^3 \\ 0 \end{pmatrix} : (a_{1,1}, a_{0,3}, b_{0,2}) \neq 0, \right.$$

$$\left. a_{0,0} \neq a_{1,0}b_{0,0}; a_{1,0} = a_{1,1}b_{0,0} + a_{1,0}b_{0,1}; a_{0,2} = a_{1,1}b_{0,1} + a_{1,0}b_{0,2}; a_{0,3} = a_{1,1}b_{0,2} \right\},$$

of dimension 9.

(4) *The subspace of S_{12}:*

$$SL(3,\mathbb{C})\left\{ \begin{pmatrix} x(\alpha_1 y - \beta_1 z)(\alpha_2 y - \beta_2 z) + y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ \alpha(\alpha_1 y - \beta_1 z)^2(\alpha_2 y - \beta_2 z) \\ 0 \end{pmatrix} : \right.$$

$$\left. \textit{It has isolated singularities, } (\alpha_1, \alpha_2) \neq 0 \textit{ and } \alpha_1\alpha_2 = 0\textit{; or } \alpha\alpha_1 = 1 \textit{ and } \beta_1, \beta_2 \in \mathbb{C}^* \right\},$$

of dimension 10.

(5) *The subspace of S_{15}:*

$$SL(3,\mathbb{C})\left\{ \begin{pmatrix} x^2z + a_{1,1}xyz + a_{1,0}xz^2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ b_{0,0}z^3 \\ 0 \end{pmatrix} : a_{0,3} \neq 0 \right\},$$

of dimension 10.

In S_6, S_9 and S_{12} the singular point has multiplicity 3 *and in S_{11}, S_{15} has multiplicity* 2.

Proof. It remains only to find the foliations with one singular point in S_{12}, the other cases were studied in the construction of the strata.

To obtain the dimension of these spaces, we must observe that if X is a foliation in any of the described linear subspaces of Y_9^{ss}, Y_{11}^{ss}, or Y_{15}^{ss} then for $g \in SL(3,\mathbb{C})$ we have that gX is in the same linear subspace if and only if g is in the corresponding parabolic subgroup. Therefore the dimension of the space is the dimension of the linear subspace plus 3.

Now let $X \in Y_{12}^{ss}$ such that $(1:0:0)$ is the unique singularity. By the above lemma, X has an invariant line $\alpha y - \beta z$. There exists $g \in P_{12}$ such that z is invariant for $gX \in Y_{12}^{ss}$. For that we can suppose:

$$X = \begin{pmatrix} xL_1(y,z)L_2(y,z) + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ L_3(y,z)L_4(y,z)L_5(y,z) \\ 0 \end{pmatrix}$$

where $L_k(y,z) = \alpha_k y - \beta_k z$ and $\alpha_k, \beta_k, a_{0,j}, \in \mathbb{C}$ for $k = 1,...,5$ and $j = 0,1,2,3$. The Milnor number of $(1:0:0)$ is

$$\begin{aligned} &I\big(L_3L_4L_5 - y(L_1L_2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3), \\ &\qquad\qquad\qquad\qquad z(L_1L_2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3)\big) \\ &= I(z, L_3L_4L_5 - y(L_1L_2 + a_{0,3}y^3)) + I(L_3L_4L_5, L_1L_2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3), \end{aligned}$$

and this is 13 if and only if

$$\begin{aligned} &I(z, L_3L_4L_5 - y(L_1L_2 + a_{0,3}y^3)) = 4, \\ &I(L_3L_4L_5, L_1L_2 + a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3) = 9, \end{aligned}$$

and this happens if and only if $a_{0,3} \neq 0$, $z|(L_3L_4L_5 - yL_1L_2)$, $L_3L_4L_5 = \alpha L_1^2L_2$ for some $\alpha \neq 0$, and $L_1^2L_2 \nmid (a_{0,3}y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3)$. Since $z|L_1L_2(\alpha L_1 - y)$ then we have the following cases: $\alpha_1 = 0$, $\alpha_2 = 0$ or $\alpha\alpha_1 = 1$. The condition for X to be in Y_{12}^{ss} says that if $\alpha_1 = 0$ then $\alpha_2 \neq 0$ and if $\alpha_2 = 0$ then $\alpha_1 \neq 0$. If $\alpha\alpha_1 = 1$ then $\beta_1, \beta_2 \in \mathbb{C}^*$. We get

$$X = \begin{pmatrix} x(\alpha_1 y - \beta_1 z)(\alpha_2 y - \beta_2 z) + y^3 + a_{0,2}y^2z + a_{0,1}yz^2 + a_{0,0}z^3 \\ \alpha(\alpha_1 y - \beta_1 z)^2(\alpha_2 y - \beta_2 z) \\ 0 \end{pmatrix}.$$

Then the dimension of the projectivization of the linear space where X lives is 7. When we move the invariant line through $(1:0:0)$ we obtain a family of foliations of dimension 8 and when we take the action by $SL(3,\mathbb{C})$ module the parabolic subgroup P_{12} we obtain a space of dimension 10. ☐

To finish the classification of foliations on $\mathbb{CP}^2$ of degree 3 with one singularity with Milnor number 13 remains to find the semistable foliations with this property. For foliations on $\mathbb{CP}^2$ of degree 2 we know that there exists only one semistable foliation, up to change of coordinates (see Theorem 5.9 of [2]), with only one singular point. In this case the singularity is a saddle-node, that means multiplicity 1 non-nilpotent. For degree 3 the situation is different, for example, we have the following foliations:

$$X_1 = z^3 \frac{\partial}{\partial x} + (x^2 z + xy^2)\frac{\partial}{\partial y} - (xyz + y^3)\frac{\partial}{\partial z}$$

$$X_2 = y^2 z \frac{\partial}{\partial x} + (xyz + z^3)\frac{\partial}{\partial y} - y^3 \frac{\partial}{\partial z}.$$

Both are semistable foliations of degree 3 with only one singularity, the first one has a nilpotent singularity and in second one the singularity has multiplicity 2. In general is very difficult to find foliations on $\mathbb{CP}^2$ of degree d with one singular point. It is clear that using this stratification we can get all the unstable foliations. We think that using recursively this construction it is possible to find also the semi-stable foliations of degree d with a singularity with Milnor number d^2+d+1.

Acknowledgments. The authors would like to thank the reviewers for their comments that help to improve this paper.

References

[1] Alcántara, C. R. *Singular points and automorphisms of unstable foliations of CP2* Bol. Soc. Mat. Mexicana (3) 16 (2010), no. 1, 39-51.

[2] Alcántara, C. R. *Foliations on* $\mathbb{CP}^2$ *of degree* 2 *with degenerate singularities*. Bull. Braz. Math. Soc. (N.S.) 44 (2013), no. 3, 421-454.

[3] Alcántara, C. R. *Stratification of the space of foliations on CP2.* J. Symbolic Comput. 72 (2016), 147-160. DOI: 10.1016/j.jsc.2015.02.003

[4] Brunella, M. *Birational Geometry of Foliations.* IMPA Monographs, 1. Springer, Cham, 2015.

[5] Cerveau, D.; Déserti, J.; Garba Belko, D.; Meziani, R. *Géométrie classique de certains feuilletages de degré deux.* Bull Braz Math Soc, New Series 41(2), (2010), 161-198.

[6] Fogarty, J.; Kirwan, F.; Mumford, D. *Geometric Invariant Theory.* Springer-Verlag, 1994.

[7] Gómez-Mont, X.; Kempf, G. *Stability of Meromorphic Vector Fields in Projective Spaces.* Comment. Math. Helvetici 64 (1989) 462-473. DOI: 10.1007/BF02564687

[8] Gómez-Mont, X.; Ortiz-Bobadilla, L. *Sistemas dinámicos holomorfos en superficies.* Aportaciones Matemáticas: Notas de Investigación [Mathematical Contributions: Research Notes], 3. Sociedad Matemática Mexicana, México, 1989.

[9] Hesselink, W. H. *Desingularizations of varieties of nullforms.* Invent. Math. 55 (1979), no. 2, 141-163.

[10] Jouanolou, J.P. *Equations de Pfaff Algébriques.* Springer-Verlag, 1979.

[11] Kempf, G. *Instability in Invariant Theory.* Annals of Mathematics 108 (1978) 299-316. DOI: 10.2307/1971168

[12] Kirwan, F. *Cohomology of Quotients in Symplectic and Algebraic Geometry.* Mathematical Notes 31 (1984), 299-316.

[13] Lins Neto, A.; Soares, M.G. *Algebraic Solutions of one-dimensional foliations.* J. Differ. Geom. 43 (1996), 652-673.

[14] Newstead, P. *Introduction to Moduli Problems and Orbit Spaces.* Springer-Verlag, 1978.

[15] Popov, V. *The Cone of Hilbert Nullforms.* arχiv: 1009.6107v1

Departamento de Matemáticas, Universidad de Guanajuato, Callejón Jalisco s/n, A.P. 402, C.P. 36000, Guanajuato, Gto. México.

E-mail address: claudiagto@gmail.com, claudia@cimat.mx; ronzon@cimat.mx