Journal *of* Singularities
Volume 16 (2017), 180-193

received: 2 August 2016
in revised form: 29 June 2017
DOI: 10.5427/jsing.2017.16h

HOROSPHERICAL AND HYPERBOLIC DUAL SURFACES OF SPACELIKE CURVES IN DE SITTER SPACE

SHYUICHI IZUMIYA, ANA CLAUDIA NABARRO, AND ANDREA DE JESUS SACRAMENTO

Abstract. We define two surfaces, the horospherical surface and the hyperbolic dual surface of a spacelike curve in the de Sitter 3-space, in the Lorentzian-Minkowski 4-space. These surfaces are, respectively, in the lightcone 3-space and in the hyperbolic 3-space (other pseudo-spheres). We use techniques from singularity theory to obtain the generic shape of these surfaces and of their singular point sets. Furthermore, we give a relation between these surfaces from the viewpoint of the theory of Legendrian dualities between pseudo-spheres.

1. Introduction

Submanifolds in Lorentz-Minkowski space are investigated from various mathematical viewpoints and are of interest also in relativity theory. In recent years, using singularity theory, very important progress has been made and many investigations have been conducted to classify and characterize the singularities of submanifolds in Euclidean spaces or in semi-Euclidean spaces (see, for example, [1]-[9] and [11]). The first author introduced Legendrian dualities between three kinds of pseudo-spheres in Lorentz-Minkowski space [5, 6]. Curves in the pseudo-spheres and duality relations between the curves and some surfaces in pseudo-spheres are studied. For example, in [3, 4, 8], curves in the hyperbolic space $H^3(-1)$ in $\mathbb{R}^4_1$, in the de Sitter dual surface in S^3_1, and in the horospherical surface in the lightcone LC^*, are investigated. The results in this paper contribute to the study of the extrinsic geometry of curves in the above different ambient spaces.

We use Legendrian duality to investigate spacelike curves in the de Sitter space $S^3_1 \subset \mathbb{R}^4_1$ and two special surfaces related by duality. For a curve $\gamma : I \to S^3_1$ with nowhere vanishing curvature, we define its associated horospherical surface in the lightcone LC^* and its hyperbolic dual surface in the hyperbolic space $H^3(-1)$. For the study of the generic differential geometry of these surfaces and of their singular sets, we use singularity theory techniques, and in particular, classical deformation theory.

Our paper is organized as follows: Section 2 reviews basic definitions for the Minkowski 4-space and introduces a moving frame along γ together with Frenet-Serret type formulae. We also review the definition of the A_k-singularities and their discriminant sets. We define the hyperbolic focal surface and the horospherical surface of γ. In Sections 3 and 5, we define two families of height functions on γ, horospherical height functions and hyperbolic height functions. These functions measure the contact of γ with special hyperplanes. Differentiating these functions yields invariants related to each surface. We show that the horospherical surface of γ is the discriminant set of the family of horospherical height functions (Corollary 3.2) and that its hyperbolic dual surface is the discriminant set of the family of hyperbolic height functions (Corollary 5.3).

The second author was supported by FAPESP grant 2013/02794-4. The third author was supported by FAPESP grant 2010/20301-7.

Furthermore, using the theory of deformations, we give a classification and a characterization of the diffeomorphism-type of these surfaces (Theorems 3.4 and 5.5). It is easy to show that the discriminant sets of these families on timelike curves in S^3_1 are empty. For this reason, we consider only spacelike curves in S^3_1.

In Section 4, we investigate the geometric meaning of the invariants discussed in the previous sections. We prove results that give conditions (related to these invariants) for the curve γ to be on a parabolic de Sitter quadric and we give also conditions for γ to be part of a T-horoparabola or an S-horoparabola (Propositions 4.1 and 4.2). In Section 5, we give information about the geometry of the hyperbolic dual surface and of its singular set. We separate the cases where γ has spacelike normal vectors from those where γ has timelike normal vectors. We prove that, if the normal vector is timelike, then the hyperbolic dual surface of γ has no singular points. For this reason, in Section 5, we consider only the case when γ has spacelike normal vectors.

In Section 6, we show that γ can be part of an elliptic de Sitter quadric (Proposition 6.1) by using an invariant of the curve. When γ is not part of an elliptic de Sitter quadric, we characterize the contact of γ with an elliptic de Sitter quadric using the singularity types of the hyperbolic dual surface of γ (Proposition 6.2).

Finally, in Section 7, we recall the concepts of Legendrian dualities between pseudo-spheres in Lorentz-Minkowski space, introduced in [6]. Several duality relationships are presented in Theorem 7.1. These give a dual relation between the horospherical surface and the hyperbolic dual surface of γ.

2. Preliminaries

The *Minkowski space* $\mathbb{R}^4_1$ is the vector space $\mathbb{R}^4$ endowed with the pseudo-scalar product $\langle x, y\rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$, for any $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$ in $\mathbb{R}^4_1$ (see, e.g., [10]). We say that a non-zero vector $x \in \mathbb{R}^4_1$ is *spacelike* if $\langle x, x\rangle > 0$, *lightlike* if $\langle x, x\rangle = 0$ and *timelike* if $\langle x, x\rangle < 0$. We call $\gamma : I \to \mathbb{R}^4_1$, with $I \subset \mathbb{R}$ an open interval, a *spacelike* (resp. *timelike*) *curve* if $\gamma'(t)$ is a spacelike (resp. timelike) vector for any $t \in I$. We define, for $x \in \mathbb{R}^4_1$,

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x \text{ is spacelike,} \\ 0 & \text{if } x \text{ is lightlike,} \\ -1 & \text{if } x \text{ is timelike.} \end{cases}$$

We call $\operatorname{sign}(x)$ the *signature* of x. The norm of a vector $x \in \mathbb{R}^4_1$ is defined by $\| x \| = \sqrt{| \langle x, x\rangle |}$. We now consider the pseudo-spheres in $\mathbb{R}^4_1$. The *hyperbolic 3-space* is defined by

$$H^3(-1) = \{x \in \mathbb{R}^4_1 \mid \langle x, x\rangle = -1\},$$

the *de Sitter 3-space* by

$$S^3_1 = \{x \in \mathbb{R}^4_1 \mid \langle x, x\rangle = 1\},$$

and the *lightcone* by

$$LC^* = \{x \in \mathbb{R}^4_1 \setminus \{0\} \mid \langle x, x\rangle = 0\}.$$

For any $x = (x_0, x_1, x_2, x_3)$, $y = (y_0, y_1, y_2, y_3)$, $z = (z_0, z_1, z_2, z_3) \in \mathbb{R}^4_1$, the pseudo-product of x, y and z is defined by:

$$x \wedge y \wedge z = \begin{vmatrix} -e_0 & e_1 & e_2 & e_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \end{vmatrix},$$

where $\{e_0, e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^4$.

For a non-zero vector $v \in \mathbb{R}^4_1$ and a real number c, a *hyperplane* with *pseudo-normal vector* v is defined by

$$HP(v,c) = \{x \in \mathbb{R}^4_1 \mid \langle x, v\rangle = c\}.$$

We call $HP(v,c)$ a *spacelike*, a *timelike*, or a *lightlike* hyperplane if v is spacelike, timelike, or lightlike, respectively.

We have three types of models of quadric surfaces in S^3_1, which are given by intersections of S^3_1 with hyperplanes in $\mathbb{R}^4_1$, determined by the type of the hyperplane. A surface $S^3_1 \cap HP(v,c)$ is called an *elliptic de Sitter quadric*, a *hyperbolic de Sitter quadric* or a *parabolic de Sitter quadric* if $HP(v,c)$ is spacelike, timelike, or lightlike, respectively. We denote the parabolic de Sitter quadric by $QDP(v,c)$ and the elliptic de Sitter quadric by $QDE(v,c)$.

Let $\gamma : I \to S^3_1$ be a smooth and regular spacelike curve in S^3_1. We can parametrise it by arc length s, and write $t(s) = \gamma'(s)$ for the unit tangent vector. In this case, we call γ a *unit speed spacelike curve*. If $\langle t'(s), t'(s)\rangle \neq 1$, then $\| t'(s) + \gamma(s) \| \neq 0$, and we define the unit vector $n(s) = \dfrac{t'(s) + \gamma(s)}{\| t'(s) + \gamma(s) \|}$. We also define another unit vector by $e(s) = \gamma(s) \wedge t(s) \wedge n(s)$. Then we obtain a pseudo-orthonormal frame $\{\gamma(s), t(s), n(s), e(s)\}$ of $\mathbb{R}^4_1$ along γ. The Frenet-Serret type formulae for that frame are given by

$$\begin{cases} \gamma'(s) = t(s), \\ t'(s) = -\gamma(s) + k_g(s)\, n(s), \\ n'(s) = -\delta(\gamma(s))\, k_g(s)\, t(s) + \tau_g(s)\, e(s), \\ e'(s) = \tau_g(s) n(s), \end{cases}$$

where $\delta(\gamma(s)) = \operatorname{sign}(n(s))$ (which we shall write as simply δ), $k_g(s) = \| t'(s) + \gamma(s) \|$ and

$$\tau_g(s) = \frac{\delta(\gamma(s))}{k_g^2(s)} \det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s)).$$

The invariant k_g is called the *geodesic curvature* and τ_g the *geodesic torsion* of γ (see [7]).

Since $\langle t'(s)+\gamma(s), t'(s)+\gamma(s)\rangle = \langle t'(s), t'(s)\rangle - 1$, it follows that $\langle t'(s), t'(s)\rangle \neq 1$ is equivalent to $k_g(s) \neq 0$.

We define the following maps

$$HS^{\pm}_{\gamma} : I \times J \to LC^* \quad \text{and} \quad HD^{\pm}_{\gamma} : I \times J \to H^3(-1)$$

by

$$HS^{\pm}_{\gamma}(s,\mu) = \gamma(s) + \mu n(s) + \lambda e(s) \quad \text{and} \quad HD^{\pm}_{\gamma}(s,\mu) = \mu n(s) + \lambda e(s),$$

respectively, where $\lambda^2 - \mu^2 = \delta(\gamma(s))$.

In other words,

$$HS^{\pm}_{\gamma}(s,\mu) = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))} e(s) \quad \text{and} \quad HD^{\pm}_{\gamma}(s,\mu) = \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))} e(s),$$

with $\mu^2 + \delta(\gamma(s)) \geq 0$, i.e., $\mu \in J = \mathbb{R}$ for $n(s)$ spacelike and $\mu \in J = (-\infty, -1] \cup [1, \infty)$ for $n(s)$ timelike. We call $HS^{\pm}_{\gamma}$ the *horospherical surface of* γ and $HD^{\pm}_{\gamma}$ the *hyperbolic dual surface of* γ. We can suppose that λ and μ are one of cosh and sinh, depending on $\delta(\gamma(s))$.

Definition 2.1. *Let $F : \mathbb{R}^4_1 \to \mathbb{R}$ be a submersion and $\gamma : I \to S^3_1$ be a regular curve. We say that γ and $F^{-1}(0)$ (respectively $F^{-1}(0) \cap S^3_1$) have contact of order k at s_0, if the function $g(s) = F \circ \gamma(s)$ satisfies $g(s_0) = g'(s_0) = \cdots = g^{(k)}(s_0) = 0$ and $g^{(k+1)}(s_0) \neq 0$, i.e., g has an A_k-singularity at s_0.*

Let $G : \mathbb{R} \times \mathbb{R}^r, (s_0, \bar{x}) \to \mathbb{R}$ be a family of germs of functions. We call G an r-parameter deformation of f if $f(s) = G_{\bar{x}}(s)$. Suppose that f has an A_k-singularity ($k \geq 1$) at s_0. If we write

$$j^{(k-1)}\left(\frac{\partial G}{\partial x_i}(s, \bar{x})\right)(s_0) = \sum_{j=0}^{k-1} \alpha_{ji}(s - s_0)^j,$$

for $i = 1, \dots, r$, then G is a *versal deformation* if the $k \times r$ matrix of coefficients (α_{ji}) has rank k ($k \leq r$) (see [2]).

The *discriminant set* of G is the set

$$\mathcal{D}_G = \left\{ x \in (\mathbb{R}^r, \bar{x}) \;\Big|\; G = \frac{\partial G}{\partial s} = 0 \ at\ (s, x)\ for\ some\ s \in (\mathbb{R}, s_0) \right\}.$$

Theorem 2.2. [2] *Let $G : \mathbb{R} \times \mathbb{R}^r, (s_0, \bar{x}) \to \mathbb{R}$ be an r-parameter deformation of f, with f having an A_k-singularity at s_0. Suppose that G is a versal deformation. Then $\mathcal{D}_G$ is locally diffeomorphic to*

(1) *$C \times \mathbb{R}^{r-2}$, if $k = 2$, and*
(2) *$SW \times \mathbb{R}^{r-3}$, if $k = 3$,*

where $C = \{(x_1, x_2) \mid x_1^2 = x_2^3\}$ is the ordinary cusp and

$$SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$$

is the swallowtail surface.

We use families of height functions on curves in S^3_1 to study the horospherical surface and the hyperbolic dual surface. In fact, these surfaces are the discriminant sets of these families.

It is easy to show that the discriminant sets of the family of horospherical height functions and family of hyperbolic height functions on timelike curves in S^3_1 are empty. For this reason, we only consider spacelike curves in S^3_1.

3. Horospherical height functions

In this section, we introduce a family of height functions on a curve that is useful for the study of the horospherical surface. We prove that the horospherical surface is the discriminant set of this family.

For a spacelike curve $\gamma : I \to S^3_1$, we define a function $H : I \times LC^* \to \mathbb{R}$ by

$$H(s, v) = \langle \gamma(s), v \rangle - 1.$$

We call H a family of *horospherical height functions on* γ. We denote $h_v(s) = H(s, v)$ for any fixed $v \in LC^*$. The family of horospherical height functions measures the contact of γ with lightlike hyperplanes in $\mathbb{R}^4_1$. Generically, this contact can be of order k, where $k = 1, 2, 3$.

We obtain equivalent conditions for each A_k-singularity, $k = 1, 2, 3$ of h_v by the following result. For example, h_v has an A_2-singularity at s if and only if

$$v = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s), \quad \mu = \frac{1}{k_g(s)\delta(\gamma(s))}, \quad \text{and } \sigma(s) \neq 0.$$

Proposition 3.1. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve such that $k_g(s) \neq 0$. Then*

(1) *$h_v(s) = 0$ if and only if there exist real numbers μ, λ, η with*

$$\eta^2 + \delta(\gamma(s))\mu^2 - \delta(\gamma(s))\lambda^2 = -1$$

such that $v = \gamma(s) + \eta t(s) + \mu n(s) + \lambda e(s)$.

(2) $h_v(s) = h_v'(s) = 0$ *if and only if there exist real numbers* μ, λ *such that*

$$v = \gamma(s) + \mu n(s) + \lambda e(s)$$

with $\lambda^2 - \mu^2 = \delta(\gamma(s))$.

(3) $h_v(s) = h_v'(s) = h_v''(s) = 0$ *if and only if* $v = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s)$ *with* $\mu = \dfrac{1}{k_g(s)\delta(\gamma(s))}$.

(4) $h_v(s) = h_v'(s) = h_v''(s) = h_v^{(3)}(s) = 0$ *if and only if* $v = \gamma(s)+\mu n(s)\pm\sqrt{\mu^2 + \delta(\gamma(s))}e(s)$, $\mu = \dfrac{1}{k_g(s)\delta(\gamma(s))}$ *and* $\sigma(s) = 0$, *where*

$$\sigma(s) = (k_g' \pm k_g\tau_g(-\delta)\sqrt{1 + k_g^2\delta})(s).$$

(5)

(i) If $n(s)$ *is timelike with* $k_g(s) = 1$ *then* $h_v(s) = h_v'(s) = \cdots = h_v^{(4)}(s) = 0$ *if and only if* $v = \gamma(s)+\mu n(s)\pm\sqrt{\mu^2 + \delta(\gamma(s))}e(s)$, $\mu = \dfrac{1}{k_g(s)\delta(\gamma(s))}$, $\sigma(s) = 0$ *and* $k_g''(s)+\tau_g^2(s) = 0$.

(ii) If $n(s)$ *is timelike with* $k_g(s) \neq 1$ *or if* $n(s)$ *is spacelike, then*

$$h_v(s) = h_v'(s) = \cdots = h_v^{(4)}(s) = 0$$

if and only if

$$v = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s), \quad mu = \frac{1}{k_g(s)\delta(\gamma(s))}, \quad \text{and} \quad \sigma(s) = \sigma'(s) = 0.$$

Proof. Since $h_v(s) = \langle\gamma(s), v\rangle - 1$, by using the Frenet-Serret type formulae, we have

(a) $h_v'(s) = \langle t(s), v\rangle$,
(b) $h_v''(s) = \langle -\gamma(s) + k_g(s)n(s), v\rangle$,
(c) $h_v^{(3)}(s) = \langle(-1 - k_g^2(s)\delta(\gamma(s)))t(s) + k_g'(s)n(s) + k_g(s)\tau_g(s)e(s), v\rangle$, and
(d) $h^{(4)}(s) = \langle(1+k_g^2(s)\delta(\gamma(s)))\gamma(s)-3\delta(\gamma(s))k_g'(s)k_g(s)t(s)+(-k_g(s)+k_g''(s)+k_g(s)\tau_g^2(s)-k_g^3(s)\delta(\gamma(s)))n(s) + (2k_g'(s)\tau_g(s) + k_g(s)\tau_g'(s))e(s), v\rangle$.

The proof follows by simple calculations using (a)-(d). □

Corollary 3.2. *The horospherical surface of* γ *is the discriminant set* $\mathcal{D}_H$ *of the family of horospherical height functions* H.

Proof. The proof follows from the definition of the discriminant set given in Section 2 and by Proposition 3.1 (2). □

Following Proposition 3.1, we define the invariant

$$\sigma(s) = \left(k_g' \pm k_g\tau_g(-\delta)\sqrt{1 + k_g^2\delta}\right)(s)$$

of the curve γ. We will study the geometric meaning of this invariant in Section 4.

In the next result, we show that the family of horospherical height functions on a curve in S_1^3 is a versal deformation of an A_k-singularity, $k = 2, 3$, of its members.

Proposition 3.3. *With the same assumptions as in Proposition 3.1, let* $H : I \times LC^* \to \mathbb{R}$ *be the family of horospherical height functions on* γ. *If* h_v *has an* A_2*-singularity at* s_0, *then* H *is a versal deformation of* h_v. *If* h_v *has an* A_3*-singularity at* s_0 *and* $n(s_0)$ *is timelike with* $k_g(s_0) \neq 1$ *(which is a generic condition) or if* $n(s_0)$ *is spacelike, then* H *is a versal deformation of* h_v.

Proof. The family of horospherical height functions is given by

$$H(s,v) = -v_0x_0(s) + v_1x_1(s) + v_2x_2(s) + v_3x_3(s) - 1,$$

where $v = (v_0, v_1, v_2, v_3)$, $\gamma(s) = (x_0(s), x_1(s), x_2(s), x_3(s))$ is the curve parametrised by arc length, $v_0 = \sqrt{v_1^2 + v_2^2 + v_3^2}$ and $x_0(s) = \sqrt{x_1^2(s) + x_2^2(s) + x_3^2(s) - 1}$.

Writing $H(s,v) = H(s, v_1, v_2, v_3)$, we have

$$\frac{\partial H}{\partial v_i} = x_i(s) - \frac{v_i}{v_0}x_0(s),$$

for $i = 1,2,3$. Therefore, the 2-jet of $\dfrac{\partial H}{\partial v_i}$ at s_0, is given by

$$x_i(s_0) - \frac{v_i}{v_0}x_0(s_0) + \left(x_i'(s_0) - \frac{v_i}{v_0}x_0'(s_0)\right)(s-s_0) + \frac{1}{2}\left(x_i''(s_0) - \frac{v_i}{v_0}x_0''(s_0)\right)(s-s_0)^2.$$

We assume first that h_v has an A_3-singularity at $s = s_0$, and we show that the determinant of the 3×3 matrix

$$A = \begin{pmatrix} x_1(s_0) - \frac{v_1}{v_0}x_0(s_0) & x_2(s_0) - \frac{v_2}{v_0}x_0(s_0) & x_3(s_0) - \frac{v_3}{v_0}x_0(s_0) \\ x_1'(s_0) - \frac{v_1}{v_0}x_0'(s_0) & x_2'(s_0) - \frac{v_2}{v_0}x_0'(s_0) & x_3'(s_0) - \frac{v_3}{v_0}x_0'(s_0) \\ x_1''(s_0) - \frac{v_1}{v_0}x_0''(s_0) & x_2''(s_0) - \frac{v_2}{v_0}x_0''(s_0) & x_3''(s_0) - \frac{v_3}{v_0}x_0''(s_0) \end{pmatrix}$$

is nonzero. Denote

$$a = \begin{pmatrix} x_0(s_0) \\ x_0'(s_0) \\ x_0''(s_0) \end{pmatrix}, b_i = \begin{pmatrix} x_i(s_0) \\ x_i'(s_0) \\ x_i''(s_0) \end{pmatrix},$$

for $i = 1,2,3$. Then

$$\det A = \frac{v_0}{v_0}\det(b_1\ b_2\ b_3) - \frac{v_1}{v_0}\det(a\ b_2\ b_3) - \frac{v_2}{v_0}\det(b_1\ a\ b_3) - \frac{v_3}{v_0}\det(b_1\ b_2\ a).$$

On the other hand,

$$(\gamma \wedge \gamma' \wedge \gamma'')(s_0) = (-\det(b_1\ b_2\ b_3), -\det(a\ b_2\ b_3), -\det(b_1\ a\ b_3), -\det(b_1\ b_2\ a)).$$

Therefore,

$$\begin{aligned} \det A &= \left\langle \left(\frac{v_0}{v_0}, \frac{v_1}{v_0}, \frac{v_2}{v_0}, \frac{v_3}{v_0}\right), (\gamma \wedge \gamma' \wedge \gamma'')(s_0) \right\rangle \\ &= \frac{1}{v_0}\langle \gamma(s_0) + \mu n(s_0) \pm \sqrt{\mu^2 + \delta}e(s_0), k_g(s_0)e(s_0)\rangle \\ &= \pm\frac{1}{v_0}(-\delta)\sqrt{k_g^2(s_0)\delta + 1}. \end{aligned}$$

In the case where $n(s_0)$ is a spacelike vector, we have $\det A = \mp\frac{1}{v_0}\sqrt{k_g^2(s_0)+1} \neq 0$ and therefore H is a versal deformation of h_v at $s = s_0$. If $n(s_0)$ is a timelike vector, then we have

$$\det A = \pm\frac{1}{v_0}\sqrt{1 - k_g^2(s_0)}$$

and therefore $\det A \neq 0$ under the condition that $k_g(s_0) \neq 1$, so H is a versal deformation of h_v at $s = s_0$.

When $k = 2$, we require the rank of B to equal 2, where B is the matrix

$$B = \begin{pmatrix} x_1(s_0) - \frac{v_1}{v_0}x_0(s_0) & x_2(s_0) - \frac{v_2}{v_0}x_0(s_0) & x_3(s_0) - \frac{v_3}{v_0}x_0(s_0) \\ x_1'(s_0) - \frac{v_1}{v_0}x_0'(s_0) & x_2'(s_0) - \frac{v_2}{v_0}x_0'(s_0) & x_3'(s_0) - \frac{v_3}{v_0}x_0'(s_0) \end{pmatrix}.$$

Since B consists of the first and second lines of A, we have that if $n(s_0)$ is a spacelike vector, then rank of B is 2 because $\det A \neq 0$. If $n(s_0)$ is a timelike vector, the rank of B is 2 if $k_g(s_0) \neq 1$. For the case $k_g(s_0) = 1$, the rank of B is 2 if $\frac{2(x_0(s_0) - v_0)}{v_0} \neq 0$. Then it is enough to show that $x_0(s_0) \neq v_0$. As $k_g(s_0) = 1$, we have by Proposition 3.1 (2) that

$$v(s_0) = \gamma(s_0) - n(s_0).$$

Therefore $v_0 = x_0(s_0) - n_0(s_0)$, where $n(s_0) = (n_0(s_0), n_1(s_0), n_2(s_0), n_3(s_0))$. Without loss of generality, we can suppose $n_0(s_0) \neq 0$, so the rank of B is 2. □

Using Theorem 2.2 and Proposition 3.3, we can obtain the diffeomorphism type of the horospherical surface.

Theorem 3.4. *With the same assumptions as in Proposition 3.1, let $HS_\gamma^\pm$ be the horospherical surface of γ. Then we have the following:*

(1) *The singular values of $HS_\gamma^\pm$ are given by*

$$h_\mu^\pm S_\gamma(s) = \gamma(s) + \frac{1}{k_g(s)\delta(\gamma(s))}n(s) \pm \sqrt{\frac{1}{k_g^2(s)} + \delta(\gamma(s))}e(s).$$

(2) *$HS_\gamma^\pm$ is, at (s_0, μ_0), locally diffeomorphic to a cuspidal edge if and only if*

$$\mu_0 = \frac{1}{k_g(s_0)\delta(\gamma(s_0))} \quad \text{and} \quad \sigma(s_0) \neq 0.$$

(3) *$HS_\gamma^\pm$ is, at (s_0, μ_0), locally diffeomorphic to a swallowtail surface if and only if*

$$\mu_0 = \frac{1}{k_g(s_0)\delta(\gamma(s_0))}, \quad \sigma(s_0) = 0, \quad \text{and} \quad \sigma'(s_0) \neq 0,$$

for $n(s_0)$ timelike with $k_g(s_0) \neq 1$, or for $n(s_0)$ spacelike.

Proof. Consider the horospherical surface given by $HS_\gamma^\pm(s,\mu) = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s)$. Then

$$\frac{\partial HS_\gamma^\pm}{\partial s}(s,\mu) = (1 - \mu\delta(\gamma(s))k_g(s))t(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s) \quad \text{and}$$
$$\frac{\partial HS_\gamma^\pm}{\partial \mu}(s,\mu) = n(s) \pm \frac{\mu}{\sqrt{\mu^2 + \delta(\gamma(s))}}e(s).$$

The vectors

$$\left\{\frac{\partial HS_\gamma^\pm}{\partial s}(s_0,\mu_0), \frac{\partial HS_\gamma^\pm}{\partial \mu}(s_0,\mu_0)\right\}$$

are linearly dependent if and only if

$$\mu_0 = \frac{1}{k_g(s_0)\delta(\gamma(s_0))}.$$

Then the singular values of $HS_\gamma^\pm$ are given by $h_{\mu_0}^\pm S_\gamma(s_0) = HS_\gamma^\pm(s_0, \mu_0)$ and assertion (1) follows. By Corollary 3.2, the discriminant set $\mathcal{D}_H$ of the family of horospherical height functions H of

γ is the horospherical surface of γ. It also follows from assertions (3) and (4) of Proposition 3.1 that h_v has an A_2-singularity (respectively, an A_3-singularity) at $s = s_0$ if and only if

$$\mu_0 = \frac{1}{k_g(s_0)\delta(\gamma(s_0))} \quad \text{and} \quad \sigma(s_0) \neq 0$$

(respectively,

$$\mu_0 = \frac{1}{k_g(s_0)\delta(\gamma(s_0))}, \quad \sigma(s_0) = 0, \quad \text{and} \quad \sigma'(s_0) \neq 0).$$

By Theorem 2.2 and Proposition 3.3, we have assertions (2) and (3). We observe that, in (3), if $n(s_0)$ is timelike, it is necessary to suppose that $k_g(s_0) \neq 1$ in order to obtain Proposition 3.3. □

4. Invariants and special geometry of the horosphe-rical surface

We study the geometric meaning of the invariant $\sigma(s)$ defined in the previous section. Let v be a lightlike vector, w be a spacelike vector, and z be a timelike vector. We call the de Sitter space curve, given by the intersections of the parabolic de Sitter quadric $QDP(v, 1)$ with $HP(w, 0)$ (resp. $HP(z, 0)$), T-horoparabolas (resp. S-horoparabolas).

Given a unit speed spacelike curve γ in S^3_1, the unit normal vector n can be a timelike vector or a spacelike vector. We prove the following results that give conditions depending on the invariants, for the curve γ to be in a parabolic de Sitter quadric. In addition, we also give conditions for γ to be part of a T-horoparabola or a S-horoparabola. These facts are related to the invariants $\sigma(s)$ and $\tau_g(s)$. It is convenient to divide the discussion into two cases: $n(s)$ is timelike (Proposition 4.1) and $n(s)$ is spacelike (Proposition 4.2).

We observe that for a curve in hyperbolic 3-space (see [8]), there is only one case because $n(s)$ is always spacelike.

Proposition 4.1. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve such that $n(s)$ is a timelike vector field along γ, $k_g(s) \leq 1$, and $k_g(s) \neq 0$. Consider the singular values $h^{\pm}_{\mu} S_\gamma(s)$ of the horospherical surface.*

(1) *Suppose that $k_g(s) \equiv 1$. Then the following conditions are equivalent:*
  (a) *$h^{\pm}_{\mu} S_\gamma(s)$ is a constant vector,*
  (b) *$\tau_g(s) \equiv 0$,*
  (c) *γ is a part of a T-horoparabola.*

(2) *Suppose that the set $\{s \in I \mid k_g(s) = 1\}$ consists of isolated points. The following conditions are equivalent:*
  (a) *$h^{\pm}_{\mu} S_\gamma(s)$ is a constant vector $v_0 \in LC^*$,*
  (b) *$\sigma(s) \equiv 0$,*
  (c) *γ is located on a parabolic de Sitter quadric $QDP(v_0, 1)$.*

Proof. The proof is similar to that for a curve in hyperbolic space in [8]. Consider the singular values $h^{\pm}_{\mu} S_\gamma(s)$ of the surface that we denote by

$$v(s) = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 - 1}e(s) \ \text{ with } \ \mu = -\frac{1}{k_g(s)}.$$

Suppose that $k_g(s) \equiv 1$. Then $v(s) = \gamma(s) - n(s)$, and $v'(s) = -\tau_g(s)e(s)$. Therefore, $v(s)$ is constant if and only if $\tau_g(s) \equiv 0$, so statements (a) and (b) of (1) are equivalent. If $v(s)$ is constant, then $\tau_g(s) \equiv 0$ and, as $e'(s) = \tau_g(s)n(s)$, this means that $e(s)$ is constant. Thus, the hyperplane $HP(e(s), 0)$ generated by $\gamma(s)$, $t(s)$ and $n(s)$, is constant. In this case, the parabolic de Sitter quadric $QDP(v(s), 1)$ is also constant. Thus, the image of γ is a part of a horoparabola given by $QDP(v(s), 1) \cap HP(e(s), 0)$. Therefore, (a) implies (c). If γ is a part of a

T-horoparabola, then it is a de Sitter plane curve and, hence, $\tau_g(s) \equiv 0$; so (c) implies (b). This completes the proof of (1).

Suppose now that $k_g(s) \neq 1$. Since $\mu(s) = -\dfrac{1}{k_g(s)}$, we have

$$v(s) = \gamma(s) - \frac{1}{k_g(s)} n(s) \pm \frac{\sqrt{1 - k_g^2(s)}}{k_g(s)} e(s).$$

Thus

$$v'(s) = \left(\frac{k_g' \pm k_g \tau_g \sqrt{1 - k_g^2}}{k_g^2} \right)(s) n(s) - \left(\frac{\sqrt{1 - k_g^2} k_g \tau_g \pm k_g'}{k_g^2 \sqrt{1 - k_g^2}} \right)(s) e(s).$$

Therefore, $v'(s) \equiv 0$ if and only if $\sigma(s) \equiv 0$, so the statements (a) and (b) of (2) are equivalent at any point $s \in I$.

We now consider the family of horospherical height functions $H(s, v)$ on γ. If γ is located on the parabolic de Sitter quadric $QDP(v_0, 1)$, then $H(s, v_0) \equiv 0$. By Proposition 3.1 (4), we have $(k_g' \pm k_g \tau_g \sqrt{1 - k_g^2})(s) \equiv 0$. Therefore, (c) implies (b). If v is a constant vector v_0, then $\langle \gamma(s), v_0 \rangle = 1$ for all $s \in I$ and thus $\gamma(s) \in QDP(v_0, 1)$ for all $s \in I$. Therefore, γ is located on a parabolic de Sitter quadric. □

Proposition 4.2. *Let $\gamma : I \to S_1^3$ be a unit speed spacelike curve such that $n(s)$ is a spacelike vector field along γ and $k_g(s) \neq 0$. Consider the singular values $h_\mu^\pm S_\gamma(s)$ of the horospherical surface. The following conditions are equivalent:*

(a) *$h_\mu^\pm S_\gamma(s)$ is a constant vector $v_0 \in LC^*$,*
(b) *$\sigma(s) \equiv 0$,*
(c) *γ is located on a parabolic de Sitter quadric $QDP(v_0, 1)$ for some v_0.*

Furthermore, if $\gamma \subset QDP(v_0, 1)$ and $\tau_g(s) \equiv 0$, then γ is part of a S-horoparabola.

Proof. The proof is analogous to that of Proposition 4.1 (2). □

5. Hyperbolic height functions

We introduce here a family of functions on a curve which is useful to study the singularities of the hyperbolic dual surface of a spacelike unit speed curve γ. First, we explain why we consider only spacelike curves with spacelike normal vector fields. Let $\gamma : I \to S_1^3$ be a unit speed spacelike curve. We suppose, as we did previously, $\langle t'(s), t'(s) \rangle \neq 1$ (generic condition), equivalently $k_g(s) \neq 0$, in order to define $n(s) = \dfrac{t'(s) + \gamma(s)}{\| t'(s) + \gamma(s) \|}$. Then $n(s)$ is a spacelike normal vector field or a timelike normal vector field along γ.

Proposition 5.1. *Let $\gamma : I \to S_1^3$ be a unit speed spacelike curve such that $k_g(s) \neq 0$ for all $s \in I$.*

(1) *Suppose that $n(s)$ is a spacelike normal vector field along γ. Then the hyperbolic dual surface $HD_\gamma^\pm$ of γ is singular at (s_0, μ_0) if and only if $\mu_0 = 0$. That is, the singular values of the hyperbolic dual surface are given by $h_{\mu_0}^\pm D_\gamma(s) = HD_\gamma^\pm(s, 0)$ with $s \in I$ and $\mu_0 = 0$.*
(2) *If $n(s)$ is a timelike normal vector field along γ, then the hyperbolic dual surface $HD_\gamma^\pm$ of γ does not have singular points.*

Proof. Consider the hyperbolic dual surface of γ,

$$HD_\gamma^{\pm}(s,\mu) = \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s).$$

Then, we have

$$\frac{\partial HD_\gamma^{\pm}}{\partial s}(s,\mu) = -\delta(\gamma(s))\mu k_g(s)t(s) \pm \sqrt{\mu^2+\delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s) \quad \text{and}$$
$$\frac{\partial HD_\gamma^{\pm}}{\partial \mu}(s,\mu) = n(s) \pm \frac{\mu}{\sqrt{\mu^2+\delta(\gamma(s))}}e(s).$$

If $n(s)$ is a spacelike normal vector field, the proof of (1) is similar to that of Theorem 3.4 (1). However, if $n(s)$ is a timelike normal vector field, the hyperbolic dual surface is not defined for $\mu_0 = 0$. Therefore assertion (2) holds. □

Since we are interested in studying the singularities of the hyperbolic dual surface of a spacelike curve, then it follows from Proposition 5.1 (2) that we need only to consider spacelike curves with spacelike normal vector fields $n(s)$.

We define a family of functions $H : I \times H^3(-1) \to \mathbb{R}$ on γ given by $H(s,v) = \langle \gamma(s), v\rangle$. We call H the family of *hyperbolic height functions on* γ and denote $h_v(s) = H(s,v)$ for any fixed $v \in H^3(-1)$. By Definition 2.1, the hyperbolic height function measures the contact of γ with spacelike hyperplanes. Generically, the order of this contact can be k, $k = 1, 2, 3$.

We have the following result about the singularities of h_v.

Proposition 5.2. *Let $\gamma : I \to S_1^3$ be a unit speed spacelike curve such that $n(s)$ is a spacelike vector field along γ and $k_g(s) \neq 0$ for all $s \in I$. Then we have the following:*

(1) *$h_v(s) = 0$ if and only if there exist real numbers μ, λ, η with $\eta^2 + \mu^2 - \lambda^2 = -1$ such that $v = \eta t(s) + \mu n(s) + \lambda e(s)$.*
(2) *$h_v(s) = h_v'(s) = 0$ if and only if there exist real numbers μ, λ such that $v = \mu n(s) + \lambda e(s)$ with $\lambda^2 - \mu^2 = 1$.*
(3) *$h_v(s) = h_v'(s) = h_v''(s) = 0$ if and only if $v = \pm e(s)$.*
(4) *$h_v(s) = h_v'(s) = h_v''(s) = h_v^{(3)}(s) = 0$ if and only if $v = \pm e(s)$ and $\tau_g(s) = 0$.*
(5) *$h_v(s) = h_v'(s) = \cdots = h_v^{(4)}(s) = 0$ if and only if $v = \pm e(s)$ and $\tau_g(s) = \tau_g'(s) = 0$.*

Proof. Since $h_v(s) = \langle \gamma(s), v\rangle$, we have

(a) $h_v'(s) = \langle t(s), v\rangle$,
(b) $h_v''(s) = \langle -\gamma(s) + k_g(s)n(s), v\rangle$,
(c) $h_v^{(3)}(s) = \langle (-1 - k_g^2(s))t(s) + k_g'(s)n(s) + k_g(s)\tau_g(s)e(s), v\rangle$,
(d) $h^{(4)}(s) = \langle (1+k_g^2(s))\gamma(s) - 3k_g'(s)k_g(s)t(s) + (-k_g(s) + k_g''(s) + k_g(s)\tau_g^2(s) - k_g^3(s))n(s) + (2k_g'(s)\tau_g(s) + k_g(s)\tau_g'(s))e(s), v\rangle$.

The proof follows by simple calculations using (a)-(d). □

Corollary 5.3. *The hyperbolic dual surface of γ is the discriminant set $\mathcal{D}_H$ of the family of hyperbolic height functions H.*

Proof. The proof follows from the definition of the discriminant set given in Section 2 and Proposition 5.2 (2). □

Proposition 5.4. *Let $\gamma : I \to S_1^3$ be a unit speed spacelike curve such that $n(s)$ is a spacelike vector field along γ, $k_g \neq 0$. Then the family H of hyperbolic height functions on γ is a versal deformation of the A_2 and A_3-singularities of h_v.*

Proof. The method of the proof is similar to that of Proposition 3.3. □

We can now obtain the diffeomorphism-type of the hyperbolic dual surface.

Theorem 5.5. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve such that $n(s)$ is a spacelike vector field along γ and $k_g(s) \neq 0$ for all $s \in I$. Consider the hyperbolic dual surface $HD^{\pm}_{\gamma}$ of γ.*

(1) *The singular values of $HD^{\pm}_{\gamma}$ are given by $h^{\pm}_{\mu}D_{\gamma}(s) = \pm e(s)$.*
(2) *$HD^{\pm}_{\gamma}$ is, at (s_0, μ_0), locally diffeomorphic to a cuspidal edge if and only if $\mu_0 = 0$ and $\tau_g(s_0) \neq 0$.*
(3) *$HD^{\pm}_{\gamma}$ is, at (s_0, μ_0), locally diffeomorphic to a swallowtail surface if and only if $\mu_0 = 0$, $\tau_g(s_0) = 0$ and $\tau'_g(s_0) \neq 0$.*

Proof. By Corollary 5.3, the discriminant set $\mathcal{D}_H$ of the family of hyperbolic height functions H on γ is the hyperbolic dual surface of γ. It follows from Proposition 5.2 (3) and (4) that h_v has an A_2-singularity (respectively, an A_3-singularity) at s_0 if and only if $\mu_0 = 0$ and $\tau_g(s_0) \neq 0$ (respectively, $\mu_0 = 0$, $\tau_g(s_0) = 0$ and $\tau'_g(s_0) \neq 0$). By Theorem 2.2 and Proposition 5.4, this completes the proof. □

6. Invariant and special geometry of the hyperbolic dual surface

In this section, we investigate the geometric properties of a hyperbolic dual surface $HD^{\pm}_{\gamma}$ at its singularities by using the invariant τ_g of γ. The de Sitter focal surfaces of hyperbolic space curves are studied in [3].

Proposition 6.1. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve such that $n(s)$ is a spacelike vector field along γ and $k_g(s) \neq 0$ for all $s \in I$. Consider the singular values $h^{\pm}_{\mu}D_{\gamma}(s)$ of the hyperbolic dual surface. The following conditions are equivalent:*

(a) *$h^{\pm}_{\mu}D_{\gamma}(s)$ is a constant vector $v_0 \in H^3(-1)$,*
(b) *$\tau_g(s) \equiv 0$,*
(c) *γ is part of the elliptic de Sitter quadric $QDE(v_0, 0)$.*

Proof. If the hyperbolic dual surface is singular at (s, μ), then $\mu = 0$. Therefore,

$$h^{\pm}_{\mu}D_{\gamma}(s) = HD^{\pm}_{\gamma}(s, \mu) = \pm e(s) \quad \text{and} \quad \frac{\partial HD^{\pm}_{\gamma}}{\partial s}(s, \mu) = \pm\tau_g(s)n(s) \equiv 0$$

if and only if $\tau_g(s) \equiv 0$. This means that assertion (a) is equivalent to assertion (b). Suppose that $\tau_g(s) \equiv 0$ then $h^{\pm}_{\mu}D_{\gamma}(s) = \pm e(s) = \pm v_0$ is constant. Since $\langle \gamma(s), \pm e(s) \rangle = 0$, then $\gamma(s) \in S^3_1 \cap HP(e(s), 0)$, where $v_0 = e(s)$ that is a timelike vector. Therefore, assertion (b) implies assertion (c).

On the other hand, suppose that $Im\gamma \subset QDE(v, 0) = S^3_1 \cap HP(v, 0)$, where v is a timelike fixed vector. Then we have $h_v(s) = \langle \gamma(s), v \rangle = 0$ for all $s \in I$. By Proposition 5.2, (4), $\tau_g(s) \equiv 0$. This completes the proof. □

Proposition 6.1 characterizes the case when γ is contained in the elliptic de Sitter quadric: $\tau_g(s) \equiv 0$. If $\tau_g(s) \not\equiv 0$ the result below shows that the degeneracy of the singularities of $HD^{\pm}_{\gamma}$ characterize the contact of the γ with elliptic de Sitter quadrics.

Theorem 6.2. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve such that $n(s)$ is a spacelike vector field along γ, $k_g \neq 0$ and $\tau_g \not\equiv 0$. For $v_0 = HD^{\pm}_{\gamma}(s_0, \mu_0)$, we have the following:*

(1) *γ has at least 2-point contact with $QDE(v_0, 0)$ at s_0 if and only if $\mu_0 = 0$, equivalently, the hyperbolic dual surface of γ is singular at (s_0, μ_0).*
(2) *γ has 2-point contact with $QDE(v_0, 0)$ at s_0 if and only if $\mu_0 = 0$ and $\tau_g(s_0) \neq 0$, equivalently, the hyperbolic dual surface of γ is locally diffeomorphic to a cuspidal edge.*

(3) *γ has 3-point contact with $QDE(v_0,0)$ at s_0 if and only if $\mu_0=0$, $\tau_g(s_0)=0$ and $\tau_g'(s_0)\neq 0$, equivalently, the hyperbolic dual surface of γ is locally diffeomorphic to a swallowtail surface.*

Proof. For $v_0=HD^{\pm}_{\gamma}(s_0,\mu_0)$, we define a map $\widetilde{h}_{v_0}:S^3_1\to\mathbb{R}$ by $\widetilde{h}_{v_0}(x)=\langle x,v_0\rangle$. Thus, we have $\left(\widetilde{h}_{v_0}\right)^{-1}(0)=QDE(v_0,0)$. In this case, $g(s)=\widetilde{h}_{v_0}\circ\gamma(s)=h_{v_0}(s)$ and then the proof follows from Definition 2.1, Proposition 5.2 and Theorem 5.5.

□

7. Dual relations on horospherical and hyperbolic dual surfaces

We require some properties of contact manifolds and Legendrian submanifolds for the duality results in this section, and we now review these concepts (for more details see, for example, [1]).

Let N be a $(2m+1)$-dimensional smooth manifold and K be a field of tangent hyperplanes on N. Locally, K is defined as the kernel of a 1-form θ. We say that the tangent hyperplane field K is *non-degenerate* if $\theta\wedge(d\theta)^m\neq 0$ at any point on N. The pair (N,K) is called a *contact manifold* if K is a non-degenerate hyperplane field. In this case, we call K a *contact structure* and θ a *contact form*. A submanifold $i:L\subset N$ of a contact manifold (N,K) is *Legendrian* if $\dim L=m$ and $di_x(T_xL)\subset K_{i(x)}$ at any $x\in L$, where i is an immersion. A smooth fibre bundle $\pi:E\to M$ is a *Legendrian fibration* if its total space E is furnished with a contact structure and the fibers of π are Legendrian submanifolds. For a Legendrian submanifold $i:L\subset E$, $\pi\circ i:L\to M$ is called a *Legendrian map*. We call the image of the Legendrian map $\pi\circ i$ a *wavefront* set of i, which is denoted by $W(i)$.

The duality concepts we use here are those introduced in [6] and [5] (the Legendrian dualities between pseudo-spheres in Lorentz-Minkowski space), where five Legendrian double fibrations are considered on the subsets Δ_i, $i=1,\dots,5$ of the product of two of the pseudo-spheres $H^n(-1)$, S^n_1 and LC^*. Here we use only $i=1,2,3$. We define one-forms $\langle dv,w\rangle=w_0dv_0+\sum_{i=1}^n w_idv_i$, $\langle v,dw\rangle=v_0dw_0+\sum_{i=1}^n v_idw_i$ on $\mathbb{R}^{n+1}_1\times\mathbb{R}^{n+1}_1$, and consider the following three Legendrian double fibrations.

(1) (*a*) $H^n(-1)\times S^n_1\supset\Delta_1=\{(v,w)\mid\langle v,w\rangle=0\}$,
(*b*) $\pi_{11}:\Delta_1\to H^n(-1)$, $\pi_{12}:\Delta_1\to S^n_1$,
(*c*) $\theta_{11}=\langle dv,w\rangle|_{\Delta_1}$, $\theta_{12}=\langle v,dw\rangle|_{\Delta_1}$.

(2) (*a*) $H^n(-1)\times LC^*\supset\Delta_2=\{(v,w)\mid\langle v,w\rangle=-1\}$,
(*b*) $\pi_{21}:\Delta_2\to H^n(-1)$, $\pi_{22}:\Delta_2\to LC^*$,
(*c*) $\theta_{21}=\langle dv,w\rangle|_{\Delta_2}$, $\theta_{22}=\langle v,dw\rangle|_{\Delta_2}$.

(3) (*a*) $LC^*\times S^n_1\supset\Delta_3=\{(v,w)\mid\langle v,w\rangle=1\}$,
(*b*) $\pi_{31}:\Delta_3\to LC^*$, $\pi_{32}:\Delta_3\to S^n_1$,
(*c*) $\theta_{31}=\langle dv,w\rangle|_{\Delta_3}$, $\theta_{32}=\langle v,dw\rangle|_{\Delta_3}$.

Here, $\pi_{i1}(v,w)=v$, $\pi_{i2}(v,w)=w$ are the canonical projections. We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over Δ_i which is denoted by K_i, $(i=1,2,3)$. It has been shown in [6] that each (Δ_i,K_i) $(i=1,2,3)$ is a contact manifold and π_{i1} and π_{i2} $(i=1,2,3)$ are Legendrian fibrations. Moreover, the contact manifolds (Δ_1,K_1), (Δ_2,K_2) and (Δ_3,K_3) are contact-diffeomorphic to each other.

For a given Legendrian embedding $\mathcal{L}_i : U \to \Delta_i$, $i = 1, 2, 3$, we say that $\pi_{i1}(\mathcal{L}_i(U))$ is the Δ_i-*dual* of $\pi_{i2}(\mathcal{L}_i(U))$ and vice-versa (see [4]). In the next result, to show duality, we have to show that the immersion $\mathcal{L}_i : U \to \Delta_i$, $i = 1, 2, 3$ is a Legendrian immersion , i.e., $\dim U = m$ and $(d\mathcal{L}_i)_x(T_x(U)) \subset K_{\mathcal{L}_i(x)}$ for all $x \in L$ (see also [6]). Equivalently, $\mathcal{L}_i$ is a Legendrian immersion if $\dim U = m$ and ${\mathcal{L}_i}^*\theta_{i1} = 0$ (see, e.g., [9]). Therefore, we can show that a submanifold is Legendrian using the second definition.

We have the following relations on horospherical and hyperbolic dual surfaces. We observe that here $n = 3$, $m = 2$ and $\dim \Delta_i = 5$, $i = 1, 2, 3$. (For hyperbolic curves γ, the are duality results in [4] for hyperbolic focal surface and de Sitter focal surface of γ).

Theorem 7.1. *Let $\gamma : I \to S^3_1$ be a unit speed spacelike curve such that $k_g(s) \neq 0$ for all $s \in I$. Then*

1. *γ is Δ_1-dual of $HD^{\pm}_{\gamma}$.*
2. *γ is Δ_3-dual of $HS^{\pm}_{\gamma}$.*
3. *$HD^{\pm}_{\gamma}$ is Δ_2-dual of $HS^{\pm}_{\gamma}$.*

Proof. (1) Define the mapping $\mathcal{L}_1 : I \times J \to \Delta_1$ by $\mathcal{L}_1(s, \mu) = (HD^{\pm}_{\gamma}(s, \mu), \gamma(s))$, where

$$M = \pi_{11}(\mathcal{L}_1(I \times J)) = HD^{\pm}_{\gamma}(s, \mu) = \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s)$$

and

$$M^* = \pi_{12}(\mathcal{L}_1(I \times J)) = \gamma(s).$$

Then $\langle HD^{\pm}_{\gamma}(s, \mu), \gamma(s)\rangle = 0$, so the mapping is well-defined, i.e., $\mathcal{L}_1(s, \mu) \in \Delta_1$. We have

$$\begin{aligned} \frac{\partial \mathcal{L}_1}{\partial s}(s, \mu) &= (-\delta(\gamma(s))\mu k_g(s)t(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s), t(s)) \\ \frac{\partial \mathcal{L}_1}{\partial \mu}(s, \mu) &= (n(s) \pm \frac{\mu}{\sqrt{\mu^2 + \delta(\gamma(s))}}e(s), 0), \end{aligned}$$

and so $\mathcal{L}_1$ is an immersion. Since $\mathcal{L}_1^*\theta_{12} = \langle HD^{\pm}_{\gamma}(s, \mu), t(s)\rangle ds = 0$, then, by definition, $\mathcal{L}_1(I \times J)$ is a Legendrian submanifold in Δ_1.

(2) We also define the mapping $\mathcal{L}_3 : I \times J \to \Delta_3$ by $\mathcal{L}_3(s, \mu) = (HS^{\pm}_{\gamma}(s, \mu), \gamma(s))$, where $HS^{\pm}_{\gamma}(s, \mu) = \gamma(s) + \mu n(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}e(s)$. Thus, $\langle HS^{\pm}_{\gamma}(s, \mu), \gamma(s)\rangle = 1$, i.e., $\mathcal{L}_3(s, \mu) \in \Delta_3$ and the proof follows as in (1).

(3) We now define the mapping $\mathcal{L}_2 : I \times J \to \Delta_2$ by $\mathcal{L}_2(s, \mu) = (HD^{\pm}_{\gamma}(s, \mu), HS^{\pm}_{\gamma}(s, \mu))$. Then we have

$$\langle HD^{\pm}_{\gamma}(s, \mu), HS^{\pm}_{\gamma}(s, \mu))\rangle = \mu^2\delta(\gamma(s)) + (\mu^2 + \delta(\gamma(s)))(-\delta(\gamma(s))) = -1.$$

Thus, $\mathcal{L}_2(s, \mu) \in \Delta_2$, so the mapping is well-defined. Since

$$\begin{aligned} \frac{\partial \mathcal{L}_2}{\partial s}(s, \mu) &= (-\delta(\gamma(s))\mu k_g(s)t(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s), \\ &\qquad (1 - \delta(\gamma(s))\mu k_g(s))t(s) \pm \sqrt{\mu^2 + \delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s)) \\ \frac{\partial \mathcal{L}_2}{\partial \mu}(s, \mu) &= (n(s) \pm \frac{\mu}{\sqrt{\mu^2 + \delta(\gamma(s))}}e(s),\ n(s) \pm \frac{\mu}{\sqrt{\mu^2 + \delta(\gamma(s))}}e(s)), \end{aligned}$$

$\mathcal{L}_2$ is an immersion, because $-\delta(\gamma(s))\mu k_g(s) \neq 0$ or $1-\delta(\gamma(s))\mu k_g(s) \neq 0$. Moreover,

$$\begin{aligned}
&\mathcal{L}_2^*\theta_{21} = \left\langle d(HD_\gamma^\pm(s,\mu)), HS_\gamma^\pm(s,\mu)\right\rangle \\
&= \left\langle \frac{\partial HD_\gamma^\pm}{\partial s}(s,\mu)ds + \frac{\partial HD_\gamma^\pm}{\mu}(s,\mu)d\mu, HS_\gamma^\pm(s,\mu)\right\rangle \\
&= \left\langle -\mu\delta(\gamma(s))k_g(s)t(s) \pm \sqrt{\mu^2+\delta(\gamma(s))}\tau_g(s)n(s) + \mu\tau_g(s)e(s), \gamma(s)\right\rangle ds + \\
&\left\langle \tau_g(s)(\mu e(s) \pm \sqrt{\mu^2+\delta(\gamma(s))}\, n(s)) - \mu\delta(\gamma(s))k_g(s)t(s), \mu n(s) \pm \sqrt{\mu^2+\delta(\gamma(s))}e(s)\right\rangle ds \\
&+ \left\langle n(s) \pm \frac{\mu}{\sqrt{\mu^2+\delta(\gamma(s))}}e(s), \gamma(s) + \mu n(s) \pm \sqrt{\mu^2+\delta(\gamma(s))}e(s)\right\rangle d\mu = 0.
\end{aligned}$$

Therefore, $\mathcal{L}_2(I\times J)$ is a Legendrian submanifold in Δ_2.

□

Acknowledgments We would like to thank the referees and Farid Tari for useful suggestions.

References

[1] Arnold, V. I., Gusein-Zade, S. and Varchenko, A. N. *Singularities of Differentiable Maps-Vol. I: Classification of critical points caustics and wave fronts.* Translated from the Russian by Ian Porteous based on a previous translation by Mark Reynolds. Reprint of the 1985 edition. Modern Birkhuser Classics. Birkhäuser/Springer, New York, 2012.

[2] Bruce, J. W. and Giblin, P. J. *Curves and Singularities: a geometrical introduction to singularity theory.* Cambridge University Press, Cambridge, 1992. DOI: 10.1017/CBO9781139172615

[3] Hayashi, R., Izumiya, S. and Sato, T. Focal surfaces and evolutes of curves in hyperbolic space. *Commun. Korean Math. Soc. 32*, 1 (2017), 147–163. DOI: 10.4134/CKMS.c160024

[4] Hayashi, R., Izumiya, S. and Sato, T. Duals of curves in hyperbolic space. *Note di Matematica 33*, 2 (2013), 97–106.

[5] Izumiya, S. Time-like hypersurfaces in de Sitter space and Legendrian singularities. *J. Math. Sci. (N.Y.) 144*, 1 (2007), 3789–3803. DOI: 10.1007/s10958-007-0232-0

[6] Izumiya, S. Legendrian dualities and spacelike hypersurfaces in the lightcone. *Mosc. Math. J. 9*, 2 (2009), 325–357.

[7] Izumiya, S., Jiang, Y. and Sato, T. Lightcone dualities for curves in the lightcone unit 3-sphere. *J. Math. Phys. 54*, 6 (2013), 063511, 15 pp.

[8] Izumiya, S., Pei, D. and Sano, T. Horospherical surfaces of curves in hyperbolic space. *Publ. Math. Debrecen 64*, 1-2 (2004), 1–13.

[9] Ohnita, Y. On deformation of 3-dimensional certain minimal Legendrian submanifolds. In *Proceedings of the 13th International Workshop on Differential Geometry and Related Fields [Vol. 13]* (2009), Natl. Inst. Math. Sci. (NIMS), Taejŏn, pp. 71–87.

[10] Ratcliffe, J. G. *Foundations of hyperbolic manifolds.* Graduate texts in Mathematics, 149. Springer, New York, 2006.

[11] Tari, F. Caustics of surfaces in the Minkowski 3-space. *Q. J. Math. 63*, 1 (2012), 189–209. DOI: 10.1093/qmath/haq030

Shyuichi Izumiya, Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
E-mail address: izumiya@math.sci.hokudai.ac.jp

Ana Claudia Nabarro, Departamento de Matemática, ICMC Universidade de São Paulo, Campus de São Carlos, Caixa Postal 668, CEP 13560-970, São Carlos-SP, Brazil
E-mail address: anaclana@icmc.usp.br

Andrea de Jesus Sacramento, Departamento de Matemática, ICMC Universidade de São Paulo, Campus de São Carlos, Caixa Postal 668, CEP 13560-970, São Carlos-SP, Brazil
E-mail address: anddyunesp@yahoo.com.br