APPARENT CONTOURS OF STABLE MAPS OF SURFACES WITH BOUNDARY INTO THE PLANE

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Dedicated to Professor Takashi Nishimura on the occasion of his 60th birthday.

Abstract. Let $M$ be a connected compact surface with boundary. A $C^\infty$ map $M \to \mathbb{R}^2$ is admissible if it is non-singular on a neighborhood of the boundary. For a $C^\infty$ stable map $f: M \to \mathbb{R}^2$, denote by $c(f)$ and $n(f)$, $i(f)$ the number of cusps and nodes, connected components of the set of singular points respectively. In this paper, we introduce the notion of admissibly homotopic among $C^\infty$ maps $M \to \mathbb{R}^2$, and we will determine the minimal number $c + n$ for each admissibly homotopy class.

1. Introduction

Let $M$ be a connected compact surface with boundary $\partial$ and $P$ a surface without boundary. Denote by $C^\infty(M, P)$ the set of $C^\infty$ maps $M \to P$ equipped with the Whitney $C^\infty$ topology. A $C^\infty$ map $f: M \to P$ is called a $C^\infty$ stable map, (or stable map for short), if there exists a neighborhood $N(f) \subset C^\infty(M, P)$ of $f$ such that every map $g \in N(f)$ is $C^\infty$ right-left equivalent\(^1\) to $f$. A $C^\infty$ map $f: M \to P$ is stable if and only if $f$ has fold, cusp and $B_2$ as its singularities, and $f^{-1}(S(f) \cup \partial) \setminus (C(f) \cup B(f))$ is an immersion with normal crossings, where $C(f)$ and $B(f)$ denote the set of cusp points and $B_2$ points of $f$ respectively, see Proposition 2.2 for details.

Note that if a $C^\infty$ map $f: M \to P$ is stable, then $f|_{\partial}: \partial \to P$ is stable. Note also that a $B_2$ point is a fold point (or regular point) if we ignore the boundary (resp. we restrict $f$ to boundary).

A $C^\infty$ map $f: M \to P$ is called admissible if it is submersive on an open neighborhood of the boundary. Note that a $C^\infty$ stable map $f: M \to P$ is admissible if and only if it has no $B_2$ points.

For a $C^\infty$ stable map $f: M \to P$, denote by $c(f)$ and $n(f)$, $i(f)$ the numbers of cusps and nodes, connected components of the set singular points of $f$ respectively.

Denote by $M_k$ a connected compact surface with exactly $k$ boundary components. A connected compact and orientable (or non-orientable) surface of genus $g$ with exactly $k$ boundary components is denoted by $\Sigma_{g,k}$ (resp. $N_{g,k}$). The 2-dimensional sphere and the plane are denoted by $S^2$ and $\mathbb{R}^2$ respectively.

For a $C^\infty$ map $f: M \to P$, define the set of singular points of $f$ as

$$S(f) = \{p \in M \mid \text{rank } d_pf < 2\}.$$ 

We call $f(S(f))$ the apparent contour (or contour for short) of $f$ and denote it by $\gamma(f)$. For a closed surface $M$, the apparent contour of a stable map $M \to P$ ($P = \mathbb{R}^2$, $S^2$) relates the topology of $M$ as classical result of Thom [11] and a formula obtained by Pignoni [9] show.

\(^1\)Two maps $f, g \in C^\infty(M, P)$ are $C^\infty$ right-left equivalent if there exist a diffeomorphism $\Phi: M \to M$ preserving the boundary and a diffeomorphism $\psi: P \to P$ such that $f \circ \Phi = \psi \circ g$. 
Pignoni [9] introduced the notion of a minimal contour of a closed surface: The contour \( \gamma(f) \) of a stable map \( f: M \rightarrow \mathbb{R}^2 \) is called a minimal contour of \( M \) if the number \( c(f) + n(f) \) is the smallest among stable maps \( g: M \rightarrow \mathbb{R}^2 \) which satisfy \( i(g) = 1 \). Then, Demoto [2] introduced the notion of a minimal contour of a \( C^\infty \) map \( f_0: M \rightarrow P \) between surfaces and studied that of a \( C^\infty \) map \( S^2 \rightarrow S^2 \). Let \( f_0: M \rightarrow P \) be a \( C^\infty \) map and \( f: M \rightarrow P \) a \( C^\infty \) stable map which is homotopic to \( f_0 \) and satisfies \( i(f) = 1 \). Call \( \gamma(f) \) a minimal contour of \( f_0 \) if the number \( c(f) + n(f) \) is the smallest among \( C^\infty \) stable maps \( g: M \rightarrow P \) which are homotopic to \( f_0 \) and \( i(g) = 1 \). Then, Kamenosono and the author [7] studied minimal contours of \( C^\infty \) maps \( M \rightarrow S^2 \) of closed surfaces \( M \). Apparent contours of stable maps between surfaces were also studied in [15, 16, 3, 17]. Studying minimal contours of \( C^\infty \) maps makes the very first step toward classifying generic \( C^\infty \) maps of surfaces up to right-left equivalence.

In this paper, we study minimal contour of \( C^\infty \) maps of surfaces with boundary. More precisely, for a surface \( M \) with boundary and a surface \( P \) without boundary, we introduce the notion of admissibly homotopic which is an equivalence relation among admissible \( C^\infty \) maps \( M \rightarrow P \), and admissible minimal contour of an admissible \( C^\infty \) map \( M \rightarrow P \). Then, we study admissible minimal contours of admissible \( C^\infty \) maps \( M_1 \rightarrow \mathbb{R}^2 \).

This paper is organized as follows. In §2, we prepare some notions and introduce the main-theorems (Theorems 2.3 and 2.5). In §3, we prepare some notions concerning stable maps \( f: M_k \rightarrow \mathbb{R}^2 \) \((k \geq 1)\) and introduce the formula as an application of formulas obtained by Pignoni [9] and Imai [6]. In §4, we construct admissible stable maps \( \Sigma_{g,1} \rightarrow \mathbb{R}^2 \) \((g \geq 0)\) and \( N_{g,1} \rightarrow \mathbb{R}^2 \) \((g \geq 1)\) which are in the lists of Theorem 2.3 and 2.5 respectively. In §5, we show the contours of stable maps constructed in §4 are admissible minimal contours. In §6, we pose a problem which concerns the apparent contours of stable fold maps \( f: M_k \rightarrow \mathbb{R}^2 \), where a stable map \( f: M_k \rightarrow \mathbb{R}^2 \) of a surface with boundary is called fold map if it has no cups as its singularities.

Throughout this paper, all surfaces are connected and smooth of class \( C^\infty \), and all maps are smooth of class \( C^\infty \) unless stated otherwise. The symbols \( r \) and \( g \) \( \geq 0 \) denote integers. For a topological space \( X \), \( \text{id}_X \) denotes the identity map of \( X \).

2. Main-Theorem

In this section, we introduce some notions and introduce the main-theorems (Theorems 2.3 and 2.5).

Let \( M_k \) be a compact and connected surface with exactly \( k \) boundary components \( \partial_1 \cup \cdots \cup \partial_k \). Then, admissible \( C^\infty \) maps \( f_0, f_1: M_k \rightarrow \mathbb{R}^2 \) are said admissibly homotopic if there exists a \( C^\infty \) map \( H: M_k \times [0,1] \rightarrow \mathbb{R}^2 \) such that \( H_t = H(\cdot,t): M_k \rightarrow \mathbb{R}^2 \) is an admissible \( C^\infty \) map for each \( t \in [0,1] \), and \( H_0 = f_0 \) and \( H_1 = f_1 \).

Let \( f: M_k \rightarrow \mathbb{R}^2 \) be an admissible \( C^\infty \) map. Then, for each component \( \partial_j \), orient the regular curve \( f(\partial_j) \subset \mathbb{R}^2 \) so that at each point, the inner of \( f(M_k) \) is in the left hand side. Note that the definition of the orientation for \( f(\partial_j) \subset \mathbb{R}^2 \) is well-defined by virtue of the assumption that \( f \) is admissible. Then, call the rotation number of \( f(\partial_j) \subset \mathbb{R}^2 \) the boundary rotation number of \( \partial_j \) (or rotation number of \( \partial_j \) for short) with respect to \( f \) and denote it by \( W(f; \partial_j) \). If \( k = 1 \), then call the rotation number of \( f(\partial) \subset \mathbb{R}^2 \) the boundary rotation number of \( f \) and denote it by \( W(f) \). Furthermore, in the case that \( M = \Sigma_g \) and \( k = 1 \), define \( s(f) = +1 \) (or \(-1\)) if there exists a neighborhood of \( N(\partial) \) of \( \partial \) such that \( f|_{N(\partial)} \) preserves (resp. reverses) the orientation of \( N(\partial) \).

**Proposition 2.1.** (1) Admissible stable maps \( f_0, f_1: \Sigma_{g,1} \rightarrow \mathbb{R}^2 \) are admissibly homotopic if and only if \( W(f_0) = W(f_1) \) and \( s(f_0) = s(f_1) \).
Admissible stable maps \( f_0, f_1 : N_{g,1} \to \mathbb{R}^2 \) are admissibly homotopic if and only if \( W(f_0) = W(f_1) \).

**Proof.** (1) If \( f_0 \) and \( f_1 \) are admissibly homotopic, then \( s(f_0) = s(f_1) \) and regular curves \( f_0(\partial) \) and \( f_1(\partial) \) are regularly homotopic. It implies that \( W(f_0) = W(f_1) \).

We consider the opposite direction. If \( W(f_0) = W(f_1) \), then regular curves \( f_0(\partial) \) and \( f_1(\partial) \) with the canonical orientation are regularly homotopic. Thus, there exists a \( C^\infty \) map

\[
H' : \partial \times [0,1] \to \mathbb{R}^2
\]

so that \( H'(\cdot,0) = f_0|\partial \) and \( H'(\cdot,1) = f_1|\partial \). Then, we can extend \( H' \) to a \( C^\infty \) map

\[
H'' : N(\partial) \times [0,1] \to \mathbb{R}^2
\]
on a neighborhood of \( \partial \) so that \( H''|_{\partial \times [0,1]} = H' \) and \( H''|_{\partial \times [t,1]} : N(\partial) \to \mathbb{R}^2 \) is a submersion for any \( t \in [0,1] \). Note that if \( s(f_0) = s(f_1) = +1 \) (or \( s(f_0) = s(f_1) = -1 \)), then \( H''_{t} = H''(\cdot,t) \) is an immersion which preserves (resp. reverses) orientation of a neighborhood of \( \partial \) for each \( t \in [0,1] \).

On the other hand, we decompose \( \Sigma_{g,1} \) into a simplicial complex. We also decompose \( \Sigma_{g,1} \times [0,1] \) into a simplicial complex which is compatible with the simplicial decomposition of \( \Sigma_{g,1} \). We define a map \( H : \Sigma_{g,1} \times [0,1] \to \mathbb{R}^2 \) by the following manner:

**0-simplex:** If a 0-simplex \( \sigma = \{a_0\} \) is in \( N(\partial) \times [0,1] \) (or \( \Sigma_{g,1} \times \{0\}, \Sigma_{g,1} \times \{1\} \)), then we define \( H(a_0) = H''(a_0) \) (resp. \( H(a_0) = f_0(a_0), H(a_0) = f_1(a_0) \)). Otherwise, we define \( H(a_0) = 0 \in \mathbb{R}^2 \).

**1-simplex:** If a 1-simplex \( \sigma = \{a_0, a_1\} \) is in \( N(\partial) \times [0,1] \) (or \( \Sigma_{g,1} \times \{0\}, \Sigma_{g,1} \times \{1\} \)), then \( H|_{\sigma} \) is defined by \( H|_{\sigma} = H''|_{\sigma} \) (resp. \( H|_{\sigma} = f_0|_{\sigma}, H|_{\sigma} = f_1|_{\sigma} \)). Otherwise, we define \( H|_{\sigma} \) by \( H(x) = \lambda_0 H(a_0) + \lambda_1 H(a_1) \), where \( x = \lambda_0 a_0 + \lambda_1 a_1 \in \sigma \) with the property that \( \lambda_i \in \mathbb{R}_{\geq 0} \) and \( \lambda_0 + \lambda_1 = 1 \).

**2-simplex:** If a 2-simplex \( \sigma = \{a_0, a_1, a_2\} \) is in \( N(\partial) \times [0,1] \) (or \( \Sigma_{g,1} \times \{0\}, \Sigma_{g,1} \times \{1\} \)), then \( H|_{\sigma} \) is defined by \( H|_{\sigma} = H''|_{\sigma} \) (resp. \( H|_{\sigma} = f_0|_{\sigma}, H|_{\sigma} = f_1|_{\sigma} \)). Otherwise, we define \( H|_{\sigma} \) by \( H(x) = \lambda_0 H(a_0) + \lambda_1 H(a_1) + \lambda_2 H(a_2) \), where \( x = \lambda_0 a_0 + \lambda_1 a_1 + \lambda_2 a_2 \in \sigma \) with the property that \( \lambda_i \in \mathbb{R}_{\geq 0} \) for \( i = 0, 1, 2 \), and \( \lambda_0 + \lambda_1 + \lambda_2 = 1 \).

**3-simplex:** If a 3-simplex \( \sigma = \{a_0, a_1, a_2, a_3\} \) is in \( N(\partial) \times [0,1] \), then \( H|_{\sigma} \) is defined by \( H|_{\sigma} = H''|_{\sigma} \). Otherwise, we define \( H|_{\sigma} \) by

\[
H(x) = \lambda_0 H(a_0) + \lambda_1 H(a_1) + \lambda_2 H(a_2) + \lambda_3 H(a_3),
\]

where \( x = \lambda_0 a_0 + \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 \in \sigma \) with the property that \( a_i \in \mathbb{R}, a_i > 0 \) for \( i = 0, 1, 2, 3 \), and \( a_0 + a_1 + a_2 + a_3 = 1 \).

Then, by perturbing \( H \) slightly, if necessary, we obtain a desired \( C^\infty \) map \( \Sigma_{g,1} \times [0,1] \to \mathbb{R}^2 \). Namely, \( f_0 \) and \( f_1 \) are admissibly homotopic.

(2) The case of \( C^\infty \) maps \( N_{g,1} \to \mathbb{R}^2 \) is also proved by similar way of (1). We omit the proof here.

\( C^\infty \) stable maps of compact and connected surfaces with boundary into surfaces without boundary are characterized by the following way.

**Proposition 2.2** (Bliuc and Giblin [1]). Let \( M \) be a compact and connected surface possibly with boundary \( \partial \) and \( P \) a surface without boundary. A \( C^\infty \) map \( f : M \to P \) is \( C^\infty \) stable if and only if it satisfies the following conditions.

1. **(Local conditions)** In the following, for \( p \in \partial \), we use local coordinates \( (x, y) \) around \( p \) such that \( \text{Int} M \) and \( \partial \) correspond to the sets \( \{y > 0\} \) and \( \{y = 0\} \) respectively.
(1a) For \( p \in \text{Int} M \), the germ of \( f \) at \( p \) is right-left equivalent to one of the following:

\[
(x, y) \mapsto \begin{cases} 
(x, y), & p: \text{regular point}, \\
(x, y^2), & p: \text{fold point}, \\
(x, y^3 + xy), & p: \text{cusp point}.
\end{cases}
\]

(1b) For \( p \in \partial \), the germ of \( f \) at \( p \) is right-left equivalent to one of the following:

\[
(x, y) \mapsto \begin{cases} 
(x, y), & p: \text{regular point of } f|_{N(\partial M)}, \\
(x, y^2 + xy), & p: B_2 \text{ point}.
\end{cases}
\]

(2) (Global conditions) For each \( q \in f(S(f) \cup \partial) \), the multi-germ

\[
(f|_{S(f) \cup \partial}, f^{-1}(q) \cap (S(f) \cup \partial))
\]

is right-left equivalent to one of the four multi-germs whose images are as depicted in Figure 1, where blue curves and gray curves represent \( f(S(f)) \) and \( f(\partial) \) respectively:

(1) represent immersion mono-germs \((\mathbb{R}, 0) \ni t \mapsto (t, 0) \in (\mathbb{R}^2, 0)\) which correspond to a single fold point or a single boundary point respectively, and (2) represents cusp mono-germ \((\mathbb{R}, 0) \ni t \mapsto (t^2, t^3) \in (\mathbb{R}^2, 0)\) which correspond to a cusp point, (3) represents \(B_2\) multi-germ which corresponds to a single point in \(\partial \cap S(f)\), (4) represent normal crossings of two immersion germs, each of which corresponds to a fold point or a boundary point.

![Figure 1](image-url)

**Figure 1.** The images of multi-germs of \( f|_{S(f) \cup \partial} \)

Let \( f_0 : M_1 \to P \) be an admissible \( C^\infty \) map and \( f : M_1 \to P \) an admissible \( C^\infty \) stable map which is admissibly homotopic to \( f_0 \). Call \( \gamma(f) \) an *admissible minimal contour* of \( f_0 \) if the number \( c(f) + n(f) \) is the smallest among stable maps \( g : M_1 \to \mathbb{R}^2 \) which are admissibly homotopic to \( f_0 \) and \( i(g) = 1 \). Note that the number of connected components of the set of singular points is allowed to vary during admissible homotopy.

**Theorem 2.3.** Let \( g \geq 0 \) be an integer and \( f : \Sigma_g, 1 \to \mathbb{R}^2 \) be a rotation number \( r \) admissible stable map. The contour \( \gamma(f) \) is an admissible minimal contour if and only if the pair \((c(f), n(f))\) is one of the pairs below:

\[
g = 0: \\
(c(f), n(f)) = \begin{cases} 
(r + 1, 0) & \text{if } r \geq 0, \\
(-r - 1, -r - 1) & \text{if } r \leq -1.
\end{cases}
\]
$$g = 1: \quad (c(f), n(f)) = \begin{cases} (r + 3, 0) \text{ or } (r - 1, 4) & \text{if } r \geq 1, \\ (r + 3, 0) & \text{if } -2 \leq r \leq 0, \\ (-r - 3, -r - 3) & \text{if } r \leq -3, \end{cases}$$

$$g = 2: \quad (c(f), n(f)) = \begin{cases} (r - 3, 6) & \text{if } r \geq 3, \\ (1, 5) & \text{if } r = 2, \\ (r + 1, 4) \text{ or } (r + 5, 0) & \text{if } -1 \leq r \leq 1, \\ (r + 5, 0) & \text{if } -4 \leq r \leq -2, \\ (-r - 5, -r - 5) & \text{if } r \leq -5, \end{cases}$$

$$g \geq 3: \quad (c(f), n(f)) = \begin{cases} (r - 2g + 1, 2g + 2) & \text{if } r \geq 2g - 1, \\ (2, 6 + 2k) & \text{if } r = 9 - 2g + 4k, k = 0, \ldots, g - 3, \\ (1, 6 + 2k) & \text{if } r = 8 - 2g + 4k, k = 0, \ldots, g - 3, \\ (0, 6 + 2k) & \text{if } r = 7 - 2g + 4k, k = 0, \ldots, g - 3, \\ (1, 5 + 2k) & \text{if } r = 6 - 2g + 4k, k = 0, \ldots, g - 2, \\ (r + 2g - 3, 4) \text{ or } (r + 2g + 1, 0) & \text{if } 3 - 2g \leq r \leq 5 - 2g, \\ (r + 2g + 1, 0) & \text{if } -2g \leq r \leq 2 - 2g, \\ (-r - 2g - 1, -r - 2g - 1) & \text{if } r \leq -1 - 2g. \end{cases}$$

Remark that the number \(c + n\) of an admissible minimal contour of a \(C^\infty\) map \(f_0: \Sigma_{g, 1} \to \mathbb{R}^2\) depend only on the boundary rotation number \(W(f_0)\). It does not depend on the sign \(s(f_0)\).

**Corollary 2.4.** The number \(c + n\) of an admissible minimal contour of a rotation number \(r\) admissible stable map \(\Sigma_{g, 1} \to \mathbb{R}^2\) is one of the items below:

$$c + n = \begin{cases} r + 3 & \text{if } r \geq 2g - 1, \\ (r + 2g + 5)/2 & \text{if } 3 - 2g \leq r < 2g - 1 \text{ and } r \equiv 3 - 2g \mod 4, \\ (r + 2g + 6)/2 & \text{if } 2 - 2g \leq r < 2g - 1 \text{ and } r \equiv 2 - 2g \text{ or } -2g \mod 4, \\ (r + 2g + 7)/2 & \text{if } 1 - 2g \leq r < 2g - 1 \text{ and } r \equiv 1 - 2g \mod 4, \\ r + 2g + 1 & \text{if } -2g \leq r \leq 2 - 2g, \\ -2(r + 1 + 2g) & \text{if } r \leq -1 - 2g. \end{cases}$$

**Theorem 2.5.** Let \(g \geq 1\) be an integer and \(h: N_{g, 1} \to \mathbb{R}^2\) be a rotation number \(r\) admissible stable map. The contour \(\gamma(h)\) is an admissible minimal contour if and only if the pair \((c(h), n(h))\) is one of the items below:

$$(c(h), n(h)) = \begin{cases} (1, |g + r - 4|/2) & \text{if } r \geq 2 - g \text{ and } r \equiv g \mod 2, \\ (0, |g + r - 3|/2) & \text{if } r \geq 1 - g \text{ and } r \not\equiv g \mod 2, \\ (1, -(g + r)/2) & \text{if } r \leq -g \text{ and } r \equiv g \mod 2, \\ (0, -(g + r + 1)/2) & \text{if } r \leq -1 - g \text{ and } r \not\equiv g \mod 2. \end{cases}$$
3. Topological formula of apparent contour

In this section, we introduce topological formula of apparent contours of admissible stable maps \( M \rightarrow \mathbb{R}^2 \) of surfaces with boundary.

Let us recall some notions introduced by Pignoni [9]. Let \( M_k \) be a compact and connected surface with exactly \( k \) boundary components \( \partial = \partial_1 \cup \cdots \cup \partial_k \) and \( f: M_k \rightarrow \mathbb{R}^2 \) an admissible stable map whose contour is non-empty. Then, for each component \( \partial_j \), orient the regular curve \( f(\partial_j) \subset \mathbb{R}^2 \) so that at each point, the inner of \( f(M_k) \) is in the left hand side. Note that the definition of the orientation for \( f(\partial_j) \) is well-defined by virtue of the assumption that \( f \) is admissible. Let \( S \) be a compact and connected surface with exactly \( k \) boundary components \( \partial = \partial_1 \cup \cdots \cup \partial_k \) and \( f: \mathbb{R}^2 \rightarrow M_k \) an admissible stable map whose contour is non-empty. Then, for each component \( \partial_j \), orient the regular curve \( f(\partial_j) \subset \mathbb{R}^2 \) so that at each point, the inner of \( f(M_k) \) is in the left hand side. Note that the definition of the orientation for \( f(\partial_j) \) is well-defined by virtue of the assumption that \( f \) is admissible.

Let \( S(f) = S_1 \cup \cdots \cup S_\ell \) be the decomposition of \( S(f) \) into the connected components and set \( \gamma_i = f(S_i) \) \((i = 1, \ldots, \ell)\). Note that \( \gamma(f) = \gamma_1 \cup \cdots \cup \gamma_\ell \). For each \( \gamma_i \), denote by \( U_i \) the unbounded component of \( \mathbb{R}^2 \setminus \gamma_i \). Note that \( \partial U_i \subset \gamma_i \).

Orient \( \gamma_i \) so that at each fold point image, the surface is “folded to the left hand side”. More precisely, for a point \( y \in \gamma_i \) which is not a cusp or a node, choose a normal vector \( v \) of \( \gamma_i \) at \( y \) such that \( f^{-1}(y') \) contains more elements than \( f^{-1}(y) \), where \( y' \) is a regular value of \( f \) close to \( y \) in the direction of \( v \). Let \( \tau \) be a tangent vector of \( \gamma_i \) at \( y \) such that the ordered pair \((\tau, v)\) is compatible with the given orientation of \( \mathbb{R}^2 \). It is easy to see that \( \tau \) gives a well-defined orientation for \( \gamma_i \).

**Definition 3.1.** A point \( y \in \partial U_i \setminus \{\text{cusps, nodes}\} \) is said to be **positive** if the normal orientation \( v \) at \( y \) points toward \( U_i \). Otherwise, it is said to be **negative**.

A component \( \gamma_i \) is said to be **positive** if all points of \( \partial U_i \setminus \{\text{cusps, nodes}\} \) are positive; otherwise, \( \gamma_i \) is said to be **negative**. The numbers of positive and negative components are denoted by \( i^+ \) and \( i^- \) respectively.

By the geometrical condition of the surface \( \Sigma_{g,1} \), we obtain the following lemma.

**Lemma 3.2.** Let \( f: \Sigma_{g,1} \rightarrow \mathbb{R}^2 \) be an admissible stable map whose singular points set consists of one component. Then, the contour is a negative component.

**Definition 3.3.** A point \( y \in \partial U_i \setminus \{\text{cusps, nodes}\} \) is called an **admissible starting point** if \( y \) is a positive (or negative) point of a positive (resp. negative) component \( \gamma_i \). Note that for each \( i \), there always exists an admissible starting point on \( \gamma_i \).

**Definition 3.4.** Let \( y \in \gamma_i \) be an admissible starting point and \( Q \in \gamma_i \) a node. Let \( \alpha: [0,1] \rightarrow \gamma_i \) be a parameterization consistent with the orientation which is singular only when the image is a cusp such that \( \alpha^{-1}(y) = \{0,1\} \). Then, there are two numbers \( 0 < t_1 < t_2 < 1 \) satisfying \( \alpha(t_1) = \alpha(t_2) = Q \).

We say that \( Q \) is **positive** if the orientation of \( \mathbb{R}^2 \) at \( Q \) defined by the ordered pair \((\alpha'(t_1), \alpha'(t_2))\) coincides with that of \( \mathbb{R}^2 \) at \( Q \); **negative**, otherwise.

The number of positive (or negative) nodes on \( \gamma_i \) is denoted by \( N_i^+ \) (resp. \( N_i^- \)). The definition of a positive (or negative) node on \( \gamma_i \) depends on the choice of an admissible starting point \( y \). However, it is known that the algebraic number \( N_i^+ - N_i^- \) does not depend on the choice of \( y \), see [12] for details. Thus, the algebraic number \( N_i^+ - N_i^- = \sum_{j=1}^{\ell} (N_i^+ - N_i^-) \) is well defined. Note that nodes arising from \( \gamma_i \cap \gamma_j \) \((i \neq j)\) play no role in the computation.

Then, we have the following formula as an application of the formula of Pignoni [9] and Imai [6].

**Proposition 3.5.** For an admissible stable map \( f: M_k \rightarrow \mathbb{R}^2 \), we have

\[
g = \varepsilon(M_k) \left( (N^+ - N^-) + \frac{\varepsilon(f)}{2} + (1 + i^+ - i^-) - \frac{1}{2} \sum_{j=1}^{k} (r_j + 1) \right)
\]
where $\varepsilon(M_k)$ is equal to 1 if $M_k$ is orientable or 2 if $M_k$ is non-orientable, and $r_j$ denotes the rotation number of $f|\partial_i$.

Proof. To compute the Euler characteristic $\chi(M_k)$, apply a result of Levine [8]: For an admissible stable map $f: M_k \to \mathbb{R}^2$, we have

$$\chi(M_k) = \sum_{i=1}^{\ell} \tau(\gamma_i) + \frac{1}{2} \sum_{j=1}^{k} \tau(e_j),$$

where $\gamma_i$ and $e_j$ denote $f(S_i)$ and $f(\partial_j)$ respectively, and $\tau(\gamma_i)$ and $\tau(e_j)$ denote the double tangent turning number of $\gamma_i$ and $e_j$ with respect to the canonical orientation respectively. For an oriented closed curve $\alpha$, the double tangent turning number $\tau(\alpha)$ is defined as the degree of the map $\alpha \to \mathbb{R}P^1$ assigning to each point on the curve its tangent line. This map is also defined at cusp points. If $\alpha$ has no cusps, then $\tau(\alpha) = 2r(\alpha)$ where $r(\alpha)$ denotes the normal degree of $\alpha$. To compute $\tau(\alpha)$, apply a result of Quine [10]: For a closed plane curve $\alpha$, we have

$$\tau(\alpha) = 2\eta(\alpha) + 2n^+ - 2n^- + c^+ - c^-,$$

where $\eta(\alpha) = \pm 1$ is defined according to the orientation of the curve $\alpha$, $c^+$ (or $c^-$) denotes the number of positive (resp. negative) cusps of $\alpha$, and $n^+$ (or $n^-$) the number of positive (resp. negative) nodes of $\alpha$, see [10] for details. Comparing the definitions of the items in Quine’s formula with the ones introduced in this paper, we see: (a) the sign of the double points is the opposite of that defined by Quine; (b) when the contour is endowed with its canonical orientation, each cusp is negative. Thus,

$$\tau(\gamma_i) = 2\eta(\gamma_i) + 2N_i^- - 2N_i^+ - c_i,$$

where $c_i$ denotes the number of cusps of $\gamma_i$. $\eta(\gamma_i) = +1$ if and only if $\gamma_i$ is negative.

$$\sum_{i=1}^{k} \tau(\gamma_i) = 2i^- - 2i^+ + 2N^- - 2N^+ - c(f).$$

Each $f(\partial_j)$ is a closed curve with no cusp: $\tau(f(\partial_j)) = 2r_j$. Hence, by applying the formula of Levine to $f$, we obtain

$$\chi(M_k) = 2i^- - 2i^+ + 2N^- - 2N^+ - c(f) + \sum_{j=1}^{k} r_j.$$

Then, the result follows immediately. \qed

Corollary 3.6. Let $f: \Sigma_{g,1} \to \mathbb{R}^2$ be an admissible stable map of rotation number $r$. Then, the number of cusps of $f$ and the rotation number $r$ never have the same parity.

Lemma 3.7. Let $f: \Sigma_{g,1} \to \mathbb{R}^2$ be an admissible stable map. If $\gamma(f)$ has a node, then it has at least one negative node.

4. Admissible stable maps $M_1 \to \mathbb{R}^2$

In this section, we construct boundary rotation number $r \in \mathbb{Z}$ stable maps $f_{r,g}: \Sigma_{g,1} \to \mathbb{R}^2$ ($g \geq 0$) and $h_{r,g}: N_{g,1} \to \mathbb{R}^2$ ($g \geq 1$) whose singular points set consist of one component and whose pairs $(c, n)$ are in the lists of Theorems 2.3 and 2.5 respectively. Note that constructing such stable maps is a part of a proof of Theorem 2.3 (or Theorem 2.5).

Note that in Figures, boundary curves are drawn in gray and the image of boundary curves are also drawn in gray.
4.1. **Admissible stable maps** $\Sigma_{1,1} \to \mathbb{R}^2$. For a boundary rotation number $r'$ admissible stable map $f': \Sigma_{g',1} \to \mathbb{R}^2$ whose singular points set consists of $i'$ components and have $c'$ cusps and $n'$ nodes, by applying modifications I (or II, III) defined by Figure 2 (resp. Figures 3, 4), we obtain a boundary rotation number $r$ admissible stable map $f: \Sigma_{g,1} \to \mathbb{R}^2$ whose singular points set consists of $i$ components and has $c$ cusps and $n$ nodes. Note that a $C^\infty$ map $\Sigma_{g,1} \to \mathbb{R}^2$ is locally defined by the projection $\mathbb{R}^3 \to \mathbb{R}^2$ into the $xz$-plane composed with a $C^\infty$ map $\iota: D^2 \to \mathbb{R}^3$ of the 2-dimensional disc. Figures 2, 3 and 4 represent modifications for a $C^\infty$ map $\iota': D^2 \to \mathbb{R}^3$.

Note that the modified maps $\iota: D^2 \to \mathbb{R}^3$ in Figure 2 and 3, 4 have one cross-cap:

1. **Modification I** (Figure 2):
   $$(r, g, i, c, n) = (r' + 1, g', i', c' + 1, n')$$

2. **Modification II** (Figures 3):
   $$(r, g, i, c, n) = (r' - 1, g', i', c' + 1, n' + 1)$$

3. **Modification III** (Figure 4):
   $$(r, g, i, c, n) = (r' - 2, g' + 1, i', c', n')$$

Figure 5 define a rotation number $-1$ admissible stable map $f_{-1,0}: \Sigma_{0,1} \to \mathbb{R}^2$ whose triple $(i, c, n)$ is equal to $(1, 0, 0)$. More precisely, $f_{-1,0}$ is defined by $f_{-1,0} = \pi_{xz} \circ \iota$.

By applying modification I inductively to $f_{-1,0}$, we obtain an admissible stable map

$$f_{r,0}: \Sigma_{0,1} \to \mathbb{R}^2$$

whose triple $(i, c, n)$ is equal to $(1, r + 1, 0)$ for each integer $r \geq -1$.

By applying modification II inductively to $f_{-1,0}$, we obtain an admissible stable map

$$f_{r,0}: \Sigma_{0,1} \to \mathbb{R}^2$$

whose triple $(i, c, n)$ is equal to $(1, -r - 1, -r - 1)$ for each integer $r \leq -1$.

4.2. **Admissible stable maps** $\Sigma_{1,1} \to \mathbb{R}^2$. For each integer $r' \leq 2$, by applying modification III to $f_{r',0}$, we obtain boundary rotation number $r \leq 0$ admissible stable maps $f_{r,1}$ whose triples $(i, c, n)$ are one of the items below:

$$(i, c, n) = \begin{cases} 
(1, r + 3, 0) & \text{if } -2 \leq r \leq 0, \\
(1, -r - 3, -r - 3) & \text{if } r \leq -3.
\end{cases}$$
More precisely

Figure 3. Modification II: By applying this modification, the rotation number decrease by one.

\[ D^2 \xrightarrow{\iota'} \Sigma_{1,1}, \]

Figure 4. Modification III: By applying this modification, the rotation number decrease by two and the genus of the source surface increase by one.

\[ D^2 \xrightarrow{\iota} \Sigma_{1,1}. \]

Figure 5. Admissible stable map \( D^2 \to \mathbb{R}^2 \) of rotation number \(-1\).

Let us construct stable maps \( f_{r,1} \) \((r \geq 1)\). Figures 6 and 7 show degree one stable maps \( f'_1, f'_2 : \Sigma_1 \to S^2 \) obtained by Kamenosono and the author [7]. Note that the contours of these
Proposition 4.1. For each \( r = f \) modification III to \( r \) number \( f \) maps are minimal contours. Stable maps

\[
\text{Proposition 4.1. For each } r = f \text{ modification III to } r \text{ number } f \text{ maps are minimal contours. Stable maps}
\]

maps are minimal contours. Stable maps \( f_1', \Sigma_1 \to S^2 \) induce rotation number one admissible stable maps \( f^{1,1}_1, f^{1,1}_2 : \Sigma_{1,1} \to \mathbb{R}^2 \) whose contours are as depicted in right-hand side of Figures 8 and 9 respectively. By applying modification I inductively to \( f^{1,1}_1 \) and \( f^{1,1}_2 \), we obtain rotation number \( r \geq 1 \) admissible stable maps \( f^{1,1}_1, f^{1,1}_2 : \Sigma_{1,1} \to \mathbb{R}^2 \) whose triples \((i,c,n)\) are equal to \((1, r - 1, 4)\), \((1, r + 3, 0)\) respectively.

4.3. Admissible stable maps \( \Sigma_{2,1} \to \mathbb{R}^2 \). For each \( r' \leq 0 \) (or \( r' = 1, 2, 3 \), by applying modification III to \( f_{r',1} \) (resp. \( f^{r',1}_1, f^{r',1}_2 \)), we obtain boundary rotation number \( r \leq -2 \) (resp. \( r = -1, 0, 1 \)) admissible stable maps \( f_{r,2} \) (resp. \( f^{r,2}_1, f^{r,2}_2, f^{r,2}_3, f^{r,2}_4, f^{r,2}_5 \)) whose triples \((i,c,n)\) are one of the items below:

\[
(i,c,n) = \begin{cases} 
(1, r + 1, 4) \text{ or } (r + 5, 0) & \text{if } -1 \leq r \leq 1, \\
(1, r + 5, 0) & \text{if } -4 \leq r \leq -2, \\
(1, -r - 5, -r - 5) & \text{if } r \leq -5.
\end{cases}
\]

Let us construct rotation number \( r \geq 2 \) admissible stable maps \( \Sigma_{2,1} \to \mathbb{R}^2 \).

**Proposition 4.1.** For each \( g \geq 2 \), there are rotation numbers \( 2g - 2 \) and \( 2g - 1 \) admissible stable maps \( f_{2g-2,g} \) and \( f_{2g-1,g} : \Sigma_{g,1} \to \mathbb{R}^2 \) whose triples \((i,c,n)\) are equal to \((1, 1, 2g + 1)\) and \((1, 0, 2g + 2)\) respectively.

![Figure 6. A degree one stable map \( f_1 : \Sigma_g \to S^2 \); \( f_1' \) is obtained by the following manner: (1) Define \( S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid z^2 + y^2 + z^2 = r^2\} \) and put \( M = S^2_{1/2} \cup S^2_2 \). Define \( t_1 : M \to S^2_1 \) by \( x \mapsto x / |x| \). (2) By attaching two handles vertically between \( S^2_{1/2} \) and \( S^2_2 \), we obtain a degree one stable map \( t'_1 : S^2 \to S^2 \) whose triple is equal to \((2,0,0)\). (3) By attaching a handle horizontally as the Figure, we obtain a degree one stable map \( f'_1 : \Sigma_g \to S^2 \) whose triple \((i,c,n)\) is equal to \((1,0,4)\).](image-url)
Attach a handle horizontally

\[ \Sigma_1 \]

\[ \downarrow \text{id}^2_S \]

\[ \Sigma_1 \]

\[ \downarrow f_2' \]

**Figure 7.** A degree one stable map \( f'_2: \Sigma \rightarrow S^2 \): \( f_2 \) is obtained by attaching a handle horizontally to the source sphere of the identity map on \( S^2 \).

\[ \Sigma_1 \]

\[ \downarrow f_1 \]

\[ \Sigma_1,1 \]

\[ \downarrow f_{1,1} \]

**Figure 8.** Admissible stable map \( f^1_{1,1}: \Sigma_{1,1} \rightarrow \mathbb{R}^2 \).

**Proof.** Figures 10 and 11 define boundary rotation number two and three admissible stable maps \( f_{2,2} \) and \( f_{3,2}: \Sigma_{2,1} \rightarrow \mathbb{R}^2 \) whose triples \((i, c, n)\) are equal to \((1, 1, 5)\) and \((1, 0, 6)\) respectively. More precisely, to define \( f_{2,2} \) (or \( f_{3,2} \)), we decompose \( \Sigma_{2,1} \) into three pieces. Then, define inclusions of each pieces into \( \mathbb{R}^3 \) as depicted in Figure 10 (resp. Figure 11). Note that \( \Sigma_{2,1} \) is restored by attaching the three pieces along bold curves and dotted lines which are labeled in Figure 10 (resp. Figure 11). An admissible stable map \( f_{2,2} \) (resp. \( f_{3,2} \)) is defined by the projection \( \pi_{xz} \) composed with the inclusion.

We can construct such admissible stable maps \( f_{2g-2,g} \) and \( f_{2g-1,g} \) as well as the cases \( f_{2,2} \) and \( f_{3,2} \).

By applying modification I inductively to \( f_{3,2} \), we obtain a rotation number \( r \) admissible stable map \( f_{r,2}: \Sigma_{2,1} \rightarrow \mathbb{R}^2 \) whose triple \((i, c, n)\) is equal to \((1, r - 3, 6)\) for each \( r \geq 3 \).
4.4. Admissible stable maps $\Sigma_{g,1} \to \mathbb{R}^2$ $(g \geq 3)$. Let us consider the case $g = 3$. In this case, we already have admissible stable maps $f_{4,3}$ and $f_{5,3}$ whose triples $(i, c, n)$ are equal to $(1, 1, 7)$ and $(1, 0, 8)$ respectively by Proposition 4.1.

By applying modification III to $f_{r',2}$ where $2 \leq r' \leq 5$ or $r' \leq -2$ (or $f_{r',2}^1, f_{r',2}^2$ where $-1 \leq r' \leq 1$), we obtain boundary rotation number $0 \leq r \leq 3$ or $r \leq -4$ (resp. $-3 \leq r \leq -1$) admissible stable maps $f_{r,3}$ (resp. $f_{r,3}^1, f_{r,3}^2$) whose triples $(i, c, n)$ are one of the items below:

$$(i, c, n) = \begin{cases} 
(1, r - 1, 6) & \text{if } 1 \leq r \leq 3, \\
(1, 1, 5) & \text{if } r = 0, \\
(1, r + 3, 4) & \text{if } -3 \leq r \leq -1, \\
(1, r + 7, 0) & \text{if } -6 \leq r \leq -4, \\
(1, -r - 7, -r - 7) & \text{if } r \leq -7.
\end{cases}$$

Then, by applying modification I inductively to $f_{5,3}$, we obtain a boundary rotation number $r$ admissible stable map $f_{r,3} : \Sigma_{3,1} \to \mathbb{R}^2$ whose triple $(i, c, n)$ is equal to $(1, r - 5, 8)$ for each $r \geq 5$. 
4.5. **Admissible stable maps** $N_{g,1} \to \mathbb{R}^2$. By applying modification IV (or V, VI) defined by Figure 12 (resp. Figures 13, 14) for a boundary rotation number $r'$ admissible stable map $h: N_{g',1} \to \mathbb{R}^2$ whose singular points set consists of $i'$ components and has $c'$ cusps and $n'$ nodes, we obtain a boundary rotation number $r$ admissible stable map $h: N_{g,1} \to \mathbb{R}^2$ whose singular points set consists of $i$ components and has $c$ cusps and $n$ nodes:

1. **Modification IV**

   $$(r, g, i, c, n) = (r' + 2, g', i', c', n' + 1)$$

2. **Modification V**

   $$(r, g, i, c, n) = (r' - 2, g', i', c', n' + 1)$$
Figure 13. Modification V: By applying this modification, the rotation number increases by two.

Figure 14. Modification VI: By applying this modification, the rotation number decreases by one.

(6) Modification VI

\[(r, g, i, c, n) = (r' - 1, g + 1, i', c', n')\]

Note that the modified map \(\iota': D^2 \rightarrow \mathbb{R}^3\) have one cross-cap.

Furthermore, by applying modification III to a boundary rotation number \(r'\) admissible stable map \(h': N_{g',1} \rightarrow \mathbb{R}^2\), we obtain a boundary rotation number \(r' - 2\) admissible stable map \(h': N_{g'+2,1} \rightarrow \mathbb{R}^2\).

Figure 15 defines \(C^\infty\) maps \(\iota_i: N_{1,1} \rightarrow \mathbb{R}^3\) \((i = -2, -1, 2\) and \(3)\). Then, the projection \(\pi_{xz}\) composed with \(\iota_{-2}, \iota_{-1}, \iota_2\) and \(\iota_3\) define boundary rotation number \(-2, -1, 2\) and \(3\) admissible stable maps \(h_{-2,1}, h_{-1,1}, h_{2,1}\) and \(h_{3,1}: N_{1,1} \rightarrow \mathbb{R}^2\) whose triples \((i, c, n)\) are equal to \((1, 0, 0), (1, 1, 0), (1, 0, 0)\) and \((1, 1, 0)\) respectively.

By applying modification IV to \(h_{-2,1}\) and \(h_{-1,1}\), we obtain boundary rotation number zero and one admissible stable maps \(h_{0,1}\) and \(h_{1,1}: N_{1,1} \rightarrow \mathbb{R}^2\) whose triples \((i, c, n)\) are equal to \((1, 0, 1)\) and \((1, 1, 1)\) respectively.

By applying modification IV inductively to \(h_{2,1}\) and \(h_{3,1}\), we obtain a boundary rotation number \(r \geq 2\) admissible stable map \(h_{r,1}: N_{1,1} \rightarrow \mathbb{R}^2\) whose triple \((i, c, n)\) is equal to \((1, 0, (r - 2)/2)\) if \(r \geq 2\) is even, \((1, 1, (r - 3)/2)\) otherwise.

Similarly, by applying modification V inductively to \(h_{-2,1}\) and \(h_{-1,1}\), we obtain a boundary rotation number \(r \leq -1\) admissible stable map \(h_{r,1}: N_{1,1} \rightarrow \mathbb{R}^2\) whose triple \((i, c, n)\) is equal to \((1, 0, (-r - 2)/2)\) if \(r \leq -1\) is even, \((1, 1, (-r - 1)/2)\) otherwise.

Thus, we see that for each triple \((i, c, n)\) in the list of Theorem 2.5 \((g = 1)\), there exists an admissible stable map \(N_{1,1} \rightarrow \mathbb{R}^2\) whose triple \((i, c, n)\) is the triple.

Then, by applying modification III inductively to \(h_{r,1}: N_{1,1} \rightarrow \mathbb{R}^2\), we obtain a boundary rotation number \(r\) admissible stable map \(h_{r,g}: N_{g,1} \rightarrow \mathbb{R}^2\) whose triples \((i, c, n)\) are in the list of Theorem 2.5 for each odd number \(g \geq 1\) and each \(r \in \mathbb{Z}\). Furthermore, by applying modification
VI inductively to \(h_{r',g'}: N_{g',1} \to \mathbb{R}^2\) with odd \(g' \geq 1\), we obtain \(h_{r,g}: N_{g,1} \to \mathbb{R}^2\) whose triples \((i,c,n)\) are in the list of Theorem 2.5 for each even \(g \geq 2\) and \(r \in \mathbb{Z}\).

Thus, we see that for each \((i,c,n)\) in the list of Theorem 2.5, there is a boundary rotation number \(r\) admissible stable map \(h: N_{g,1} \to \mathbb{R}^2\) whose triple \((i,c,n)\) is equal to the triple.

5. Proof of minimum of \(c + n\) in Theorem 2.3

Let \(g \in \mathbb{Z}_{\geq 0}\) and \(r \in \mathbb{Z}\). To prove Theorem 2.3 we need the following Lemmas.

**Lemma 5.1** (M. Yamamoto [14]). Let \(f: \Sigma_{g,1} \to \mathbb{R}^2\) be a rotation number \(r\) admissible stable map whose singular points set consists of one component. Then, \(c(f) \geq |r + 1| - 2g\) and \(c(f) \not\equiv r \mod 2\).

**Lemma 5.2.** Let \(f: \Sigma_{g,1} \to \mathbb{R}^2\) be a rotation number \(r\) admissible stable map whose singular points set consists of one component.

1. If \(f\) has no cusps, then \(r \equiv 2g - 1 \mod 4\).
2. If \(r \equiv 2g + 1 \mod 4\), then \(\gamma(f)\) has at least two cusps.

**Proof.** (1) For such stable map \(f: \Sigma_{g,1} \to \mathbb{R}^2\), \(\Sigma_{g,1}\) is decomposed into three pieces as

\[
\Sigma_{g,1} = \Sigma_{g-t,1} \cup N(S(f)) \cup \Sigma_{t,2}, \quad 0 \leq t \leq g,
\]

where \(N(S(f))\) denote a tubular neighborhood of \(S(f)\). Note that \(f_1 := f|\Sigma_{g-t,1}\) and \(f_2 := f|\Sigma_{t,2}\) are immersions. Then, by applying a result of Haefliger:

*For an immersed surface \(M_k \subset \mathbb{R}^2\), the Euler-Poincare characteristic \(\chi(M_k)\) is equal to the normal degree of \(\partial M_k\).*

If \(W(f_1) = k\), then we have \(\chi(\Sigma_{g-t,1}) = k\) and \(\chi(\Sigma_{t,2}) = k + r\). This shows that \(2g = 1 + r + 4t\).

(2) Put \(r = 2g + 1 + 4k\). Then, formula (3.1) implies the conclusion.

Let us divide a proof into two cases \(g = 0\) and \(g \geq 1\).
5.1. $g = 0$. Lemma 5.1 shows that the contour $\gamma_{(f_r,0)}$ is an admissible minimal contour for each $r \geq 0$.

Let us consider the case $r \leq -1$. Let $f : \Sigma_{g,0} \to \mathbb{R}^2$ be an admissible stable map of rotation number $r$ whose singular points set consists of one component. Then, Lemma 5.1 implies that $c(f) \geq -(r + 1)$. In this case, (3.1) and Lemma 3.2 show that

$$\frac{r + 1}{2} = (N^+ - N^-) + \frac{c(f)}{2}.$$  
Then, we have

$$\frac{r + 1}{2} = (N^+ - N^-) + \frac{c(f)}{2} \geq (N^+ - N^-) - \frac{r + 1}{2}.$$  
This implies that $(r + 1) \geq (N^+ - N^-)$. Note that $(r + 1)$ is negative. Thus, we have $N^- \geq -(r + 1)$. Then,

$$c(f) + n(f) \geq \frac{c(f)}{2} + \frac{r + 1}{2} + 2N^- \geq -2(r + 1).$$
Thus, for such admissible stable maps, we have $c(f) + n(f) \geq -2(r + 1)$. This shows that the contour $\gamma_{(f_r,0)}$ ($r \leq -1$) is an admissible minimal contour.

5.2. $g \geq 1$. At first, let us consider the case $r \geq 2g - 1$. Let $f : \Sigma_{g,1} \to \mathbb{R}^2$ be an admissible stable map of rotation number $r$ whose singular points set consists of one component. Then the formula (3.1) and Lemma 3.2 show that

$$\frac{r + 1}{2} = (N^+ - N^-) + \frac{c(f)}{2}.$$  
If $\gamma(f)$ has no node, then $c(f) = 2g + r + 1$. If $\gamma(f)$ has a node, then Lemma 3.7 and Lemma 5.1 yield that

$$c(f) + n(f) \geq \frac{c(f)}{2} + g + \frac{r + 1}{2} + 2N^- \geq r + 3.$$  
This shows that the contour $\gamma_{(f_r,g)}$ ($r \geq 2g - 1$) is an admissible minimal contour.

The case $-2g \leq r \leq 2g$ is also proved by using Lemmas 5.1, 5.2 and the similarly argument as the above case.

Then, let us consider the case $r \leq -2g - 1$. Let $f : \Sigma_{g,1} \to \mathbb{R}^2$ be a rotation number $r$ admissible stable map whose singular points set consists of one component. The formula (5.1) and Lemma 5.1 imply

$$g + \frac{r + 1}{2} \geq (N^+ - N^-) + \frac{-r - 2g}{2}.$$  
Thus, we have

$$2g + r + 1 \geq (N^+ - N^-).$$  
Note that $2g + r + 1$ is negative. Thus, $N^- \geq -(2g + r + 1)$. Then,

$$c(f) + n(f) \geq \frac{c(f)}{2} + g + \frac{r + 1}{2} + 2N^- \geq -2(r + 2g + 1).$$  
Therefore, the contour $\gamma_{(f_r,g)}$ ($r \leq -2g - 1$) is admissible minimal contour.

It completes the proof of Theorem 2.3.
6. Proof of minimum of $c + n$ in Theorem 2.5

Let $g \in \mathbb{Z}_{\geq 1}$ and $r \in \mathbb{Z}$. Proposition 3.5 yields the following lemma.

**Lemma 6.1.** Let $h : N_{g,1} \to \mathbb{R}^2$ be a boundary rotation number $r$ admissible stable map whose singular points set consists of one component. Then, the numbers $g + r$ and $c(h)$ never have the same parity. In particular, if $g + r$ is an even number, then $h$ has at least one cusp.

**Proof.** Let $h : N_{g,1} \to \mathbb{R}$ be a such stable map. Then, formula (3.1) induces the following modulo two equation

$$g \equiv c(h) - (r + 1).$$

It implies the conclusion. 

We divide a proof into two cases $g = 1$ and $g \geq 2$.

6.1. $g = 1$. Lemma 6.1 shows that the contours $\gamma(h_{r,1})$ ($r = -2, -1, 2, 3$) are admissible minimal contours.

At first, let us consider the case $r \geq 4$. Let $h : N_{1,1} \to \mathbb{R}^2$ be a boundary rotation number $r$ admissible stable map whose singular points set consists of one component.

(i) $i^+ = 1$. Then, the formula (3.1) implies $2(N^+ - N^-) + c(h) = r - 2$. If $\gamma(h)$ has no nodes, then $c(h) = r - 2$. If $\gamma(h)$ has a node, then

$$c(h) + n(h) = \frac{r - 2 + c(h)}{2} + 2N^- \geq \frac{r - 2 + c(h)}{2}.$$ 

This yields that if $r \geq 4$ is odd (or even), then

$$c(h) + n(h) \geq (r - 1)/2$$

(resp. $c(h) + n(h) \geq (r - 2)/2$).

(ii) $i^- = 1$. Then, the formula (3.1) implies $2(N^+ - N^-) + c(h) = r + 2$. If $\gamma(h)$ has no nodes, then $c(h) = r + 2$. If $\gamma(h)$ has a node, then

$$c(h) + n(h) = \frac{r + c(h) + 2}{2} + 2N^- \geq \frac{r + c(h) + 2}{2}.$$ 

This yields that if $r \geq 4$ is odd (or even), then

$$c(h) + n(h) \geq (r + 3)/2$$

(resp. $c(h) + n(h) \geq (r + 2)/2$).

(i) and (ii) show that if $r \geq 4$ is odd (or even), then $c(h) + n(h) \geq (r - 1)/2$ (resp. $c(h) + n(h) \geq (r - 2)/2$). This implies that the contour $\gamma(h_{r,1})$ ($r \geq 4$) is an admissible minimal contour.

Then, let us consider the case $r \leq -3$. Let $h : N_{1,1} \to \mathbb{R}^2$ be a boundary rotation number $r$ admissible stable map whose singular points set consists of one component.

(i) $i^+ = 1$. Then, the formula (3.1) induces $2(N^+ - N^-) = r - c(h) - 2$. Note that $r - c(h) - 2 \leq 0$. Thus, we have $N^- \geq -(r - c(h) - 2)/2$. Then,

$$c(h) + n(h) = \frac{r + c(h) - 2}{2} + 2N^- \geq \frac{3c(h) - r + 2}{2}.$$ 


Lemma 6.1 yields that if \( r \leq -3 \) is odd (or even), then \( c(h) + n(h) \geq (r + 5)/2 \) (resp. \( c(h) + n(h) \geq (r + 2)/2 \).

(ii) \( \iota^- = 1 \). Then, the formula (3.1) induces \( 2(N^+ - N^-) = r - c(h) + 2 \). If \( \gamma(h) \) has no nodes, then \( c(h) = r + 2 \). If \( \gamma(h) \) has a node, then \( (N^+ - N^-) = (r - c(h) + 2)/2 \leq 0 \). Thus, we have \( N^- \geq -(r - c(h) + 2)/2 \). Then,
\[
c(h) + n(h) = \frac{r - c(h) + 2}{2} + 2N^- \geq \frac{3c(h) - r - 2}{2}.
\]

Lemma 6.1 shows that if \( r \leq -3 \) is odd (or even), then \( c(h) + n(h) \geq (r + 1)/2 \) (resp. \( (r - 2)/2 \)).

(i) and (ii) show that \( \gamma(h_{r,1}) \) (\( r \leq -3 \)) is an admissible minimal contour.

Formula (3.1) implies the following.

**Lemma 6.2.** Let \( h: N_{1,1} \to \mathbb{R}^2 \) be a boundary rotation number \( 0 \) admissible stable map whose singular points set consists of one component. Then, \( c(h) + n(h) \geq 1 \).

Therefore, \( \gamma(h_{0,1}) \) is an admissible minimal contour.

We can show that \( \gamma(h_{1,1}) \) is minimal as the above case.

Thus, we complete the proof of the Theorem 2.5 for \( g = 1 \).

6.2. \( g \geq 2 \). Lemma 6.1 shows that the contours \( \gamma(h_{-g,g}) \) and \( \gamma(h_{-g-1,g}) \) are admissible minimal contours.

At first, let us consider \( r \geq -g + 1 \). Let \( h: N_{g,1} \to \mathbb{R}^2 \) be a boundary rotation number \( r \) admissible stable map whose singular points set consists of one component.

(i) \( \iota^+ = 1 \). Then, formula (3.1) shows that \( 2(N^+ - N^-) = c = g + r - 3 \). If \( \gamma(h) \) has no nodes, then \( c(h) = g + r - 3 \). If \( \gamma(h) \) has a node, then
\[
c(h) + n(h) = c(h) + \frac{g + r - c(h) - 3}{2} + 2N^- \geq \frac{g + r + c(h) - 3}{2}.
\]

Lemma 6.1 shows that if \( g + r \) is even (or odd), then \( c(h) + n(h) \geq (g + r - 2)/2 \) (resp. \( c(h) + n(h) \geq (g + r - 3)/2 \)).

(ii) \( \iota^- = 1 \). Then, formula (3.1) shows that \( 2(N^+ - N^-) + c(h) = g + r + 1 \). If \( \gamma(h) \) has no nodes, then \( c(h) = g + r + 1 \). If \( \gamma(h) \) has a node, then
\[
c(h) + n(h) = c(h) + \frac{g + r - c(h) + 1}{2} + 2N^- \geq \frac{g + r + c(h) + 1}{2}.
\]

Lemma 6.1 shows that if \( g + r \) is even (or odd), then \( c(h) + n(h) \geq (g + r + 2)/2 \) (resp. \( c(h) + n(h) \geq (g + r + 1)/2 \)).

(i) and (ii) implies that the contours \( \gamma(h_{r,g}) \) (\( r \geq -g + 1 \)) are an admissible minimal contours.

Then, let \( r \leq -g - 2 \). Let \( h: N_{g,1} \to \mathbb{R}^2 \) be a boundary rotation number \( r \) admissible stable map whose singular points set consists of one component.

(i) \( \iota^+ = 1 \). Formula (3.1) shows that \( 2(N^+ - N^-) = g + r - c(h) - 3 \leq 0 \). Thus, we have \( N^- \geq -(g + r - c(h) - 3)/2 \). Then,
\[
c(h) + n(h) = c(h) + \frac{g + r - c(h) - 3}{2} + 2N^- \geq \frac{-g - r + c(h) + 3}{2}.
\]

Lemma 6.1 shows that if \( g + r \) is even (or odd), then \( c(h) + n(h) \geq (-g - r + 2)/2 \) (resp. \( c(h) + n(h) \geq (-g - r + 3)/2 \)).
(i2) \( i^- = 1 \). Formula (3.1) shows that \( 2(N^+ - N^-) = g + r - c(h) + 1 \leq 0 \). Thus, we have \( N^- \geq -(g + r - c(h) + 1)/2 \). Then,
\[
c(h) + n(h) = c(h) + \frac{g + r - c(h) + 1}{2} + 2N^- \geq \frac{-g - r + 3c(h) - 1}{2}.
\]
Lemma 6.1 shows that \( g + r \) is even (or odd), then
\[
c(h) + n(h) \geq \frac{(-g - r + 2)}{2}
\]
(resp. \( c(h) + n(h) \geq \frac{(-g - r - 1)}{2} \)).

(i1) and (i2) implies that \( \gamma(h_{r,g}) \) \( r \geq -g - 2 \) is an admissible minimal contour. It completes the proof of Theorem 2.5.

7. Problem

Let \( M \) be a compact connected surface with boundary and \( P \) a surface without boundary. A \( C^\infty \) map \( f : M \to P \) is called a fold map if \( f \) has only fold points as its singularities.

Let \( f : M \to \mathbb{R}^2 \) be a boundary rotation number \( r \) admissible stable fold map. Then, call the contour \( \gamma(f) \) an \( \mathcal{F}-(i,n)-\text{minimal contour} \) of boundary rotation number \( r \) maps \( M \to \mathbb{R}^2 \) if the pair \( (i(f), n(f)) \) is the smallest among rotation number \( r \) admissible stable fold maps \( M \to \mathbb{R}^2 \) with respect to the lexicographic order.

**Problem 7.1.** Let \( M = \Sigma_{g,1} \) or \( N_{g,1} \). Study an \( \mathcal{F}-(i,n) \)-minimal contour of boundary rotation number \( r \) maps \( M \to \mathbb{R}^2 \).

References


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