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THE GAUSS MAP ON THETA DIVISORS WITH TRANSVERSAL A_1 SINGULARITIES

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Abstract. We use Lagrangian specialization to compute the degree of the Gauss map on Theta divisors with transversal A_1 singularities. This computes the Gauss degree for a general abelian variety in the loci $\mathcal{A}^{\delta}_{t,g-t}$ that form some of the irreducible components of the Andreotti-Mayer loci. We also prove that the first coefficient of the Lagrangian specialization is the Samuel multiplicity of the singular locus.

1. Introduction

The Gauss map relies on the linear nature of abelian varieties by attaching to a smooth point of a divisor, its tangent space translated at the origin. This map was already used by Andreotti [1] in his beautiful proof of the Torelli theorem, and its geometry is intimately connected with the singularities of the theta divisor. The degree of the Gauss map is also equal to the 0-th Chern-Mather class of the theta divisor. We will work over the complex numbers. Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties (ppav's) of dimension g, over the complex numbers. For a ppav $(A,\Theta)\in\mathcal{A}_g$, the Gauss map

$$\mathcal{G}_\Theta : \Theta \dashrightarrow \mathbb{P}(T_0^\vee A) \simeq \mathbb{P}^{g-1}$$

is the rational map defined by the complete linear system $|L|$ where $L=\mathcal{O}_A(\Theta)|_\Theta$ denotes the normal bundle to the hypersurface $\Theta\subset A$. The Gauss map is generically finite if and only if (A,Θ) is indecomposable as a ppav (see [5, Sec. 4.4] for generalities about $\mathcal{G}_\Theta$). The degree of $\mathcal{G}_\Theta$ is unknown beyond a few cases:

- For smooth Theta divisors, the degree is $[\Theta]^g = g!$ (Ex. 2.4).
- For non-hyperelliptic (resp. hyperelliptic) Jacobians, the degree is $\binom{2g-2}{g-1}$ (resp. 2^{g-1}) [3, 247].
- For a general Prym variety the degree is $D(g+1)+2^{g-2}$, where $D(g)$ is the degree of the variety of all quadrics of rank ≤ 3 in $\mathbb{P}^{g-1}$ [20].
- For the intermediate Jacobian of a cubic threefold the degree is 72 [7].

Another case where the degree of the Gauss map is straightforward to compute is when Θ has isolated singularities. In this case we have by [8, Rem. 2.8]

$$\deg \mathcal{G}_\Theta = g! - \sum_{z\in \mathrm{Sing}(\Theta)} \mathrm{mult}_z \Theta\,,$$

where $\mathrm{mult}_z\Theta$ is the Samuel multiplicity as defined for example in [13, Sec 4.3]. For a complex variety X of dimension n, we denote by

$$\chi(X) = \sum_{i=0}^{2n} (-1)^i \dim_{\mathbb{Q}} \mathrm{H}^i(X,\mathbb{Q})$$

the topological Euler characteristic. We say that X has *transversal* A_1 *singularities* if for all points $x\in\mathrm{Sing}(X)$, there is a local analytic isomorphism

$$(X,x) \simeq (V(x_1^2+\cdots+x_k^2),0) \subset (\mathbb{C}^{n+1},0)\,,$$

for some $k \geq 2$, where $x_1, \ldots, x_{n+1}$ are coordinates on $\mathbb{C}^{n+1}$. In a sense, this is the simplest kind of singularities to handle after isolated singularities. We have the following:

Theorem 1 (3.5)**.** *Let $(A, \Theta) \in \mathcal{A}_g$ such that Θ has transversal A_1 singularities, then*

$$\deg \mathcal{G}_\Theta = g! - 2(-1)^{\dim Z}\chi(Z) - (-1)^{\dim D}\chi(D)\,,$$

where $Z = \mathrm{Sing}(\Theta)$ and $D \in |L|_Z|$ is a general divisor in the linear system.

Recently, Codogni, Grushevsky and Sernesi [8] introduced the stratification of $\mathcal{A}_g$ by the *Gauss loci*

$$\mathcal{G}_d^{(g)} \coloneqq \{(A, \Theta) \in \mathcal{A}_g \mid \deg \mathcal{G}_\Theta \leq d\}\,.$$

These loci are closed by [9] and the Jacobian locus $\mathcal{J}_g$ is an irreducible component of $\mathcal{G}_d^{(g)}$ for

$$d = \binom{2g-2}{g-1}\,.$$

It is interesting to study how the Gauss loci interact with the stratification introduced by Andreotti and Mayer in [2], which consists of the loci

$$\mathcal{N}_k^{(g)} = \{(A, \Theta) \in \mathcal{A}_g \mid \dim \mathrm{Sing}(\Theta) \geq k\}\,.$$

Andreotti and Mayer prove that the Jacobian locus $\mathcal{J}_g$ is an irreducible component of $\mathcal{N}_{g-4}^{(g)}$. For $g \geq 5$, the known irreducible components of $\mathcal{N}_{g-4}^{(g)}$ away from the locus of decomposable ppav's are by [12] and [10]:

- the locus of Jacobians $\mathcal{J}_g$,
- two loci $\mathscr{E}_{g,0}$ and $\mathscr{E}_{g,1}$ arising from Prym varieties of certain étale double covers of bielliptic curves (for a definition see [10]),
- for $2 \leq t \leq g/2$, the loci $\mathcal{A}_{t,g-t}^2$ of ppav's containing two complementary abelian varieties of dimension t and $g-t$ respectively, such that the induced polarization is of type (2) (defined by Proposition 4.1).

It turns out that by Debarre, a general member of $\mathcal{A}_{t,g-t}^2$ for $2 \leq t \leq g/2$ satisfies the conditions of Theorem 1. As a consequence we have:

Theorem 2 (4.7)**.** *Let $2 \leq t \leq g/2$, then for a general $(A, \Theta) \in \mathcal{A}_{t,g-t}^2$, we have*

$$\deg \mathcal{G}_\Theta = t!(g-t)!g\,.$$

In particular, the degree of the Gauss map separates the components $\mathcal{A}_{t,g-t}^2$ from $\mathcal{J}_g$. The construction of $\mathcal{A}_{t,g-t}^2$ can be generalized for any polarization type $\delta = (a_1, \ldots, a_k)$. One has

$$\mathcal{A}_{t,g-t}^{\delta} \subset \mathcal{N}_{g-2d}^{(g)}\,, \quad \text{for } \deg \delta \leq t \leq g/2,$$

where $\deg \delta \coloneqq a_1 \cdots a_k$. Suppose $\delta \in \{(2), (3), (2,2)\}$, let $\deg \delta = d \leq t \leq g/2$, then $\mathcal{A}_{t,g-t}^{\delta}$ is an irreducible component of $\mathcal{N}_{g-2d}^{(g)}$ [10]. We compute the degree of the Gauss map for a general member of these loci as well, see Theorem 4.7. Using different techniques, it is also possible to compute the degree of the Gauss map on the loci $\mathscr{E}_{g,0}$ and $\mathscr{E}_{g,1}$, see the forthcoming paper [19]. This (with the analysis of Section 4.3) gives first partial results towards answering the following question, raised by Codogni, Grushevsky and Sernesi [8, Question 1.7]

Question 1.1. *For $g \geq 5$, what are the possible values of $\deg \mathcal{G}_\Theta$ for $(A, \Theta) \in \mathcal{A}_g$?*

The main tool in the proof of Theorem 1 is the notion of Lagrangian specialization, which was already employed by Codogni and Krämer to prove that the Gauss loci are closed [9]. Let us quickly recall the setup: Let W be a smooth variety. One defines the conormal variety to a closed subvariety $X \subset W$ as the Zariski closure

$$\Lambda_X \coloneqq \overline{\{(x, \xi) \in T^\vee(W) \mid x \in \mathrm{Sm}(X)\,, \xi \perp T_x(X)\}} \subset T^\vee W\,.$$

This can be done in a relative setting as well: Let S be a smooth curve, $q : \mathcal{W} \to S$ a smooth morphism and $\mathcal{X} \subset \mathcal{W}$ a subvariety that is flat over S. By replacing the tangent spaces in the above definition by the relative tangent spaces over S, one obtains the relative conormal variety

$$\Lambda_{\mathcal{X}/S} \subset T^\vee(\mathcal{W}/S)\,.$$

Let $0 \in S$ be a point and $W \coloneqq \mathcal{W}|_0$ and $X \coloneqq \mathcal{X}|_0$ be the fibers above 0. By [14], the specialization of $\Lambda_{\mathcal{X}/S}$ at 0 is a formal sum of conormal varieties to subvarieties $Z \subset W$, i.e.

$$\mathrm{sp}_0(\Lambda_{\mathcal{X}/S}) \coloneqq \Lambda_{\mathcal{X}/S}|_0 = \sum_{Z\subset W} m_Z \Lambda_Z\,,$$

for some positive integers m_Z.

Remark. Although we will not use it in what follows, let us recall the sheaf-theoretic interpretation of Lagrangian specialization: Let ψ_q and ϕ_q be the nearby and vanishing cycle functors associated to $q : \mathcal{W} \to S$. Suppose that $\mathcal{X}$ is smooth away from $\mathcal{X}_0$ and $\dim \mathcal{X} = n+1$. In that case the Lagrangian specialization computes the characteristic cycle of the nearby cycle functor (see [17, Th. 3.55])

$$\mathrm{CC}(\psi_q(\underline{\mathbb{C}}_{\mathcal{X}}[n])) = \mathrm{sp}_0(\Lambda_{\mathcal{X}/S})\,.$$

Our next result is in a sense the leading term of the Lagrangian specialization in the codimension 1 case (2.6):

Proposition 1 (Leading term of the Lagrangian specialization)**.** *In the above setting assume that $\mathcal{X} \subset \mathcal{W}$ is of codimension* 1*. Then*

$$\mathrm{sp}_0(\Lambda_{\mathcal{X}/S}) = \sum_i (\mathrm{mult}_{X_i} X)\cdot \Lambda_{X_{i,\mathrm{red}}} + \sum_{Z\subsetneq X_{i,\mathrm{red}}} m_Z \Lambda_Z\,,$$

where X_i are the reduced irreducible components of X and $\mathrm{mult}_{X_i} X = \mathrm{len}(\mathcal{O}_{X,X_i})$ *is the geometric multiplicity.*

When X is reduced, we can go one step further (2.8):

Theorem 3 (Second term of the Lagrangian specialization)**.** *In the above setting assume moreover that X is reduced and $\mathcal{X}|_s$ is smooth for $s \neq 0$. Let* $\mathrm{Sing}(X) = \cup_i Z_i$ *be the decomposition of the singular locus into its scheme-theoretic irreducible components. Then*

$$\mathrm{sp}_0\Lambda_{\mathcal{X}/S} = \Lambda_X + \sum_i (\mathrm{mult}_{Z_i} X)\cdot \Lambda_{Z_{i,red}} + \sum_{Y\subsetneq Z_{i,\mathrm{red}}} m_Y \Lambda_Y$$

where $\mathrm{mult}_Z X$ *is the Samuel multiplicity of Z in X as defined in* [13, Sec. 4.3].

Let us consider the setting of plane curve singularities to illustrate the above theorem.

Example 1. *Let $f : \mathbb{A}^2 \to \mathbb{A}$ be a map with an isolated critical point at* 0*. Let $C_t \coloneqq \{f = t\} \subset \mathbb{A}^2$. Let μ, κ, m be the Milnor number, Samuel multiplicity (of* $\mathrm{Sing}(C_0)$ *in C_0), and the order of vanishing of f at* 0 *respectively. We then have*

$$\begin{aligned}\mathrm{CC}(\phi_f(\underline{\mathbb{Q}}_{\mathbb{A}^2}[1])) &= \mu\Lambda_x\,,\\ \mathrm{CC}(\underline{\mathbb{Q}}_{C_0}[1]) &= \Lambda_{C_0} + (m-1)\Lambda_x\,.\\ \mathrm{CC}(\psi_f(\underline{\mathbb{Q}}_{\mathbb{A}^2}[1]) &= \Lambda_{C_0} + \kappa\Lambda_x\,,\end{aligned}$$

where the two first equalities can be deduced for example from [17, Ex. 3.58] *and the last equality is a consequence of the above theorem. We thus obtain the well-known relation (see for instance* [18]*)*

$$\kappa = \mu + m - 1\,.$$

As a corollary to Theorem 3 we obtain an upper bound on the degree of the Gauss map in terms of the degree of the conormal variety to the singular locus (2.9):

Corollary 1. *Let $(A,\Theta)\in\mathcal{A}_g$ and let $\operatorname{Sing}(\Theta)=\cup_i Z_i$ be the decomposition of the singular locus of Θ into its scheme-theoretic irreducible components. We have*

$$\deg\mathcal{G}_\Theta \le g! - \sum_i (\operatorname{mult}_{Z_i}\Theta)\deg(\Lambda_{Z_{i,\mathrm{red}}}),$$

where

$$\deg\Lambda_Z := [\Lambda_Z]\cdot[W]\in \mathrm{H}_0(T^\vee W,\mathbb{Z})$$

is the degree of the intersection with the zero section $W\hookrightarrow T^\vee W$.

It is interesting to compare this with the formula obtained by Codogni, Grushevski and Sernesi in [8, Cor. 2.6]:

$$\deg\mathcal{G}_\Theta \le g! - \sum_i (\operatorname{mult}_{Z_i}\Theta)\deg(L|_{Z_{i,\mathrm{red}}}),$$

where $\deg(L|_Z)=c_1(L)^{\dim Z}\cap[Z]$ is the degree of the polarization $L=\mathcal{O}_A(\Theta)$ restricted to Z. Although the similarity is striking, there is no obvious relation between both bounds: Indeed, let $2\le t\le g/2$ and let $(A,\Theta)\in\mathcal{A}^2_{t,g-t}$ be general. Then $\operatorname{Sing}(\Theta)$ is smooth and by 4.6 we have

$$\deg\Lambda_{\operatorname{Sing}(\Theta)} = t!(g-t)!(t-1)(g-t-1).$$

A direct computation using 4.4 shows

$$\deg L|_{\operatorname{Sing}(\Theta)} = t!(g-t)!\binom{g-4}{t-2},$$

which is less than $\deg\Lambda_{\operatorname{Sing}(\Theta)}$ for small values of t and greater than $\deg\Lambda_{\operatorname{Sing}(\Theta)}$ for big values of t. The proof of Codogni, Grushevski and Sernesi relies on Vogel cycles and it is not clear how both techniques relate. It would also be interesting to know how the next coefficients in the Lagrangian specialization relate to known invariants of the singularity.

The text is organized as follows: In Section 2, we recall some well-known facts about the Lagrangian specialization, and prove Theorem 3. In Section 3 we then prove Theorem 1. Finally, in Section 4 we prove Theorem 2, and analyse the result numerically.

2. Lagrangian Specialization

2.1. **Generalities on Lagrangian Specialization.** We recall some facts about Lagrangian specialization, see [9] for an introduction. Let W be a smooth variety of dimension n. To a closed subvariety $X\subset W$ we define its *conormal variety* by

$$\Lambda_X := \overline{\{(x,\xi)\in T^\vee W \mid x\in\operatorname{Sm}(X),\, \xi\perp T_x(X)\}}\subset T^\vee W.$$

The degree of a conormal variety is defined by

$$\deg(\Lambda_X) := \deg([\Lambda_{X/S}]\cdot[W]),$$

where $W\hookrightarrow T^\vee(W)$ is embedded as the zero section and the product is in the Chow ring of $T^\vee W$. Let $\mathbb{P}\Lambda_X\subset\mathbb{P}T^\vee W$ denote the projectivization of the conormal variety. Let

$$h := c_1(\mathcal{O}_{\mathbb{P}T^\vee W}(1))\in \mathrm{CH}^1(\mathbb{P}T^\vee W)$$

be the hyperplane class of the projective bundle. The *Chern-Mather* class of X (or Λ_X) is defined as

$$c_M(X)=c_M(\Lambda_X) := c(T^\vee W)\cap p_*\left((1-h)^{-1}\cap[\mathbb{P}\Lambda_X]\right)\in \mathrm{CH}_*(W),$$

where $p:\mathbb{P}T^\vee W\to W$ is the projection and $c(-)$ is the usual Chern class. We thus have $\deg(\Lambda_X)=\deg(c_M(X))$.

Example 2.1. *Let $X \subset W$ be a subvariety. MacPherson [16] introduced a constructible function called the local Euler obstruction* $\mathrm{Eu_X} : W \to \mathbb{Z}$*, such that*

$$\mathrm{Eu_X}(x) = \begin{cases} 0 & \text{if } x \in W \setminus X, \\ 1 & \text{if } x \in X_{\mathrm{sm}}. \end{cases}$$

By Kashiwara's index formula (see [11, 123]) we then have

$$\deg \Lambda_X = (-1)^{\dim X} \chi(X, \mathrm{Eu}_X) \coloneqq (-1)^{\dim X} \sum_{n \in \mathbb{Z}} n \cdot \chi(\mathrm{Eu}_X^{-1}(n)) .$$

In particular, if $X \subset W$ is a smooth subvariety, then $\deg \Lambda_X = (-1)^{\dim X} \chi(X)$*, where*

$$\chi(X) = \sum_{n \in \mathbb{Z}} (-1)^n \mathrm{h}^n(X, \mathbb{Q})$$

denotes the usual topological Euler characteristic.

Conormal varieties can be defined in families: Let S be a smooth (quasi-projective) curve, and

$$q : \mathcal{W} \to S$$

is a smooth dominant morphism of varieties. Let $\mathcal{X} \subset \mathcal{W}$ be a closed subvariety, flat over S. One defines the *relative conormal variety* to $\mathcal{X}$ as the closure

$$\Lambda_{\mathcal{X}/S} \coloneqq \overline{\{(x, \xi) \in T^\vee(\mathcal{W}/S) \,|\, x \in \mathrm{Sm}(\mathcal{X}/S),\ \xi \perp T_x \mathcal{X}_{f(x)}\}} \subset T^\vee(\mathcal{W}/S) ,$$

where

$$T^\vee(\mathcal{W}/S) \coloneqq T^\vee \mathcal{W} / f^{-1} T^\vee(S)$$

is the relative cotangent bundle. Let

$$\mathscr{L}(\mathcal{W}/S) = \bigoplus_{\mathcal{X} \subset \mathcal{W}} \mathbb{Z} \cdot \Lambda_{\mathcal{X}/S}$$

denote the free abelian group generated by relative conormal varieties to closed subvarieties $\mathcal{X} \subset \mathcal{W}$ that are flat over S. The Lagrangian specialization of $\Lambda_{\mathcal{X}/\mathcal{W}} \in \mathscr{L}(\mathcal{W}/S)$ is the intersection with the fiber above $s \in S$. This is again a Lagrangian cycle on $\mathcal{W}_s$ [14] [15]

$$\mathrm{sp}_s(\Lambda_{\mathcal{X}/S}) \coloneqq \Lambda_{\mathcal{X}/S} \cap T^\vee \mathcal{W}_s = m_{\mathcal{X}_s} \Lambda_{\mathcal{X}_s} + \sum_{Z \subset \mathrm{Sing}(\mathcal{X}_s)} m_Z \Lambda_Z ,$$

where $m_{\mathcal{X}_s}, m_Z > 0$ and the sum runs over finitely many subvarieties $Z \subset \mathrm{Sing}(\mathcal{X}_s)$. Moreover, for a general $s \in S$,

$$\mathrm{sp}_s(\Lambda_{\mathcal{X}/S}) = \Lambda_{\mathcal{X}_s} .$$

Remark. Note that the definition of the conormal variety and of the Lagrangian specialization are local. Thus, we can compute the coefficients m_Z above locally.

We define the *projectivised* conormal variety $\mathbb{P}\Lambda_{Z/S}$ by taking the image in the projectivised cotangent space $\mathbb{P}T^\vee(\mathcal{W}/S)$. From now on we will assume $\mathcal{X} \subset \mathcal{W}$ to be of codimension 1. Recall that the *relative singular locus* is the scheme defined locally by

$$\mathrm{Sing}(\mathcal{X}/S) = V(F, \partial_1 F, \ldots, \partial_n F) \subset \mathcal{W} ,$$

where F is a holomorphic function defining $\mathcal{X}$ and ∂_i generate the relative tangent space $T(\mathcal{W}/S)$ (recall that $\mathcal{W}$ is smooth, thus $\mathcal{X}$ is a Cartier divisor). We have the following:

Proposition 2.2. *Let S be a curve or a point, and suppose $\mathcal{X} \subset \mathcal{W}$ is of codimension* 1*. There is a canonical identification*

$$\mathbb{P}\Lambda_{\mathcal{X}/S} = \mathrm{Bl}_{\mathrm{Sing}(\mathcal{X}/S)} \mathcal{X} .$$

Proof. In this setting we define the (relative) Gauss map by

$$\begin{aligned} \gamma : \mathcal{X} &\dashrightarrow \mathbb{P}T^\vee(\mathcal{W}/S) \\ x &\mapsto (T_x \mathcal{X}_{f(x)})^* \,. \end{aligned}$$

The statement of the proposition is local, thus after restricting to an open set of $\mathcal{W}$ we can assume that there are coordinates $x_1, \ldots, x_n$ (and s in the relative case) on $\mathcal{W}$, and a function F defining $\mathcal{X} \subset \mathcal{W}$. Then $\mathbb{P}T^\vee(\mathcal{W}/S) = \mathcal{W} \times \mathbb{P}^{n-1}$ and

$$\gamma(x) = \left(x, \frac{\partial F}{\partial x_1}(x) : \cdots : \frac{\partial F}{\partial x_n}(x) \right) .$$

Thus $Z \coloneqq \mathrm{Sing}(\mathcal{X}/S)$ is exactly the base locus of γ. We claim that there exist an embedding $\tilde{\gamma} : \mathrm{Bl}_Z \mathcal{X} \hookrightarrow \mathbb{P}T^\vee(\mathcal{W}/S)$ such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Bl}_Z \mathcal{X} & & \\ {\scriptstyle p}\downarrow & \overset{\tilde{\gamma}}{\searrow} & \\ \mathcal{X} & \overset{\gamma}{\dashrightarrow} & \mathbb{P}T^\vee(\mathcal{W}/S) \end{array} .$$

This completes the proof as

$$\mathbb{P}\Lambda_{\mathcal{X}/S} = \overline{\gamma(\mathcal{X})} = \tilde{\gamma}(\mathrm{Bl}_Z \mathcal{X}) \simeq \mathrm{Bl}_Z \mathcal{X} \,.$$

We prove the claim following [13, Sec. 4.4]. Let $\mathcal{I}$ be the ideal sheaf of $Z \subset \mathcal{X}$, and $L \coloneqq \gamma^* \mathcal{O}_{\mathbb{P}T^\vee(\mathcal{W}/S)}(1)$. Let $V \coloneqq \langle \partial_{x_1} F, \ldots, \partial_{x_n} F \rangle \subset \mathrm{H}^0(\mathcal{X}, L)$ and $V_{\mathcal{X}} = V \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{X}}$. There is a canonical surjection $V_{\mathcal{X}} \to \mathcal{I} \otimes L$, inducing an embedding

$$\begin{aligned} \mathrm{Bl}_Z \mathcal{X} = \mathrm{Proj}\,(\oplus_n \mathcal{I}^n) \hookrightarrow \mathrm{Proj}\left(\mathrm{Sym}(V_{\mathcal{X}} \otimes L^{-1}) \right) &= \mathrm{Proj}\,(\mathrm{Sym}\, V_{\mathcal{X}}) \\ &= \mathbb{P}T^\vee(\mathcal{W}/S)|_{\mathcal{X}} \\ &\subset \mathbb{P}T^\vee(\mathcal{W}/S) \,. \end{aligned}$$

Denoting by $\tilde{\gamma}$ the above composition, we are done. □

In the case of ppav's we have the following:

Proposition 2.3. *Let (A, Θ) be a polarized abelian variety, with Θ reduced and let*

$$\mathcal{G} : \Theta \dashrightarrow \mathbb{P}T_0^\vee A$$

be the Gauss map. Then

$$\mathbb{P}\Lambda_\Theta = \Gamma_{\mathcal{G}} \subset A \times \mathbb{P}T_0^\vee A \simeq \mathbb{P}T^\vee A \,,$$

where $\Gamma_{\mathcal{G}}$ is the closure of the graph of $\mathcal{G}$. In particular:

$$\deg \Lambda_\Theta = \deg \mathcal{G} \,.$$

Proof. The first part of the proposition follows immediately from the proof of Proposition 2.2. Let $v \in T_0^\vee A$ be a general point and $\overline{v} \in \mathbb{P}T_0^\vee A$ the projectivization. Then in $\mathrm{CH}^*(T^\vee W)$ (resp. $\mathrm{CH}^*(\mathbb{P}T^\vee W)$) we have

$$\begin{aligned} \deg \Lambda_\Theta &\coloneqq [\Lambda_\Theta] \cdot [A \times \{0\}] \\ &= [\Lambda_\Theta] \cdot [A \times \{v\}] \\ &= [\mathbb{P}\Lambda_\Theta] \cdot [A \times \{\overline{v}\}] \\ &\eqqcolon \deg \mathcal{G} \,. \end{aligned}$$

□

Example 2.4. *Let $(A,\Theta)\in\mathcal{A}_g$ and suppose that Θ is smooth, then*

$$\deg\Lambda_\Theta = \deg\mathcal{G} = g!\,.$$

Recall that the Gauss map corresponds to the complete linear system $|L|_\Theta|$ with $L=\mathcal{O}_A(\Theta)$. If Θ is smooth, the Gauss map is defined everywhere thus

$$\deg\mathcal{G} = [\Theta]^{\cdot g} = g!$$

by Riemann Roch.

We end the section with the following computation of Lagrangian specialization:

Example 2.5. *Let $n\geq 3$, $x_1,\dots,x_n$ be coordinates on $\mathbb{A}^n$ and s be a coordinate on $\mathbb{A}$. Consider the following deformation*

$$\begin{array}{ccc} \mathcal{X}=\{x_1^2+\cdots+x_{n-1}^2+x_ns=0\} & \subset & \mathbb{A}^n\times\mathbb{A} \\ & \searrow & \downarrow q \\ & & \mathbb{A} \end{array},$$

where q is the projection onto the second factor. Then

$$\mathrm{sp}_0\Lambda_{\mathcal{X}/S} = \Lambda_X + 2\Lambda_B + \Lambda_C\,,$$

where

$$\begin{aligned} B &= \{s=x_1=\cdots=x_{n-1}=0\} \quad \textit{is the singular locus of } X=\mathcal{X}_0,\\ C &= \{s=x_1=\cdots=x_n=0\} \quad \textit{is the singular locus of } \mathcal{X}. \end{aligned}$$

As this example is central in our proof, we will do the computation: By 2.2 we have

$$\begin{aligned} \Lambda_{\mathcal{X}} &= \mathrm{Blow}_B\mathcal{X} \\ &= V(u_1x_1+\cdots+u_nx_n,\, x_iu_j-x_ju_i,\, x_iu_n-su_i)_{1\leq i,j\leq n-1} \\ &\subset \mathcal{X}\times\mathbb{P}^{n-1}\,, \end{aligned}$$

where u_i are homogeneous coordinates on $\mathbb{P}^{n-1}$. Specializing to $s=0$ and restricting to the open set $U=\{u_1\neq 0\}$ we have (with $a_i=u_i/u_1$ for $2\leq i\leq n$)

$$\begin{aligned} \mathrm{sp}_0\Lambda_{\mathcal{X}}|_U = V\Big(&x_1^2(1+a_2^2+\cdots+a_{n-1}^2), \\ &x_1(1+a_2^2+\cdots+a_{n-1}^2)+a_nx_n,\ x_1a_n\Big) \\ \subset\ &\mathbb{A}^2\times\mathbb{A}^{n-1}\,. \end{aligned}$$

Thus $\mathrm{sp}_0\Lambda_{\mathcal{X}}$ has 3 irreducible components:

$$\begin{aligned} \Lambda_X &= V(1+a_2^2+\cdots+a_{n-1}^2, a_n)\subset\mathbb{A}^2\times\mathbb{A}^{n-1} && \text{with multiplicity 1,}\\ \Lambda_B &= V(x_1,a_n) && \text{with multiplicity 2,}\\ \Lambda_C &= V(x_1,x_n) && \text{with multiplicity 1.} \end{aligned}$$

2.2. **First Coefficients in the Lagrangian Specialization.** Let $q:\mathcal{W}\to S$ be a smooth morphism to a quasi-projective curve S. Let $\mathcal{X}\subset\mathcal{W}$ be a variety of codimension 1, flat over S. Let $0\in S$ be a point and $X=\mathcal{X}_0$, $W=\mathcal{W}_0$ be the special fibers

$$\mathcal{X}\subset\mathcal{W}\xrightarrow{q}S\,.$$

We have the following:

Proposition 2.6 (Zeroth-Order Approximation of the Lagrangian Specialization)**.** *Let $X = \cup_i X_i$ be the scheme-theoretic irreducible components of X. We have*

$$\mathrm{sp}_0(\Lambda_{\mathcal{X}/S}) = \sum_i (\mathrm{mult}_{X_i} X) \cdot \Lambda_{X_{i,\mathrm{red}}} + \sum_{Z \subsetneq X_{i,\mathrm{red}}} m_Z \Lambda_Z \,,$$

where $\mathrm{mult}_{X_i} X = \mathrm{len}(\mathcal{O}_{X,X_i})$ *is the geometric multiplicity and the last sum runs over subvarieties $Z \subset \mathrm{Sing}(X_{\mathrm{red}}) \cup (\mathrm{Sing}(\mathcal{X}) \cap X)$.*

Proof. By the principle of Lagrangian specialization [9, Lem 2.3], we have

$$\mathrm{sp}_0(\Lambda_{\mathcal{X}/S}) = \sum_i m_{X_i} \Lambda_{X_{i,\mathrm{red}}} + \sum_{Z \subset \mathrm{Sing}(X)} m_Z \Lambda_Z \,, \tag{2.7}$$

for some coefficients m_{X_i}, m_Z. The definition of the coefficients m_{X_i} is local, thus we can assume that we are working on an affine neighborhood where X_{red} is smooth and irreducible. Let $x_1, \ldots, x_n, s$ be coordinates on $\mathcal{W}$ such that q is the projection onto s. $\mathcal{X}$ is defined locally by a function $F(x_1, \ldots, x_n, s)$. We will show that the ideal of the relative singular locus $\mathrm{Sing}(\mathcal{X}/S)$

$$I := \left\langle F, \frac{\partial F}{\partial x_i} \right\rangle_{1 \le i \le n}$$

is locally principal in the affine coordinate ring of $\mathcal{X}$ away from a strict subset $Z \subseteq X \cap \mathrm{Sing}(\mathcal{X})$. If X is reduced, then it is smooth and $I = \langle 1 \rangle$ so there is nothing to prove. We assume now that X is non-reduced. X is a Cartier divisor in W, thus defined by the vanishing of f^k for some $k \ge 2$, where X_{red} is defined by the vanishing of $f(x_1, \ldots, x_n)$. We have

$$F = f^k + s^l \cdot g \,,$$

for some function g defined on $\mathcal{W}$ not divisible by s, and $l \ge 1$. g does not vanish identically on X_{red}, else g would be divisible by f and $\mathcal{X}$ would not be integral. Notice that $V(g|_X) \subset \mathrm{Sing}(\mathcal{X}) \cap X$ thus we can restrict to $\{g \ne 0\}$ and assume g is a unit. We have

$$I = \left\langle f^k + s^l \cdot g, \frac{\partial f}{\partial x_i} f^{k-1} + s^l \frac{\partial g}{\partial x_i} \right\rangle_{1 \le i \le n} \,.$$

As X_{red} is smooth, we have $\langle f, \partial_i f \rangle = \langle 1 \rangle$, thus $(f^{k-1} + s^l \cdot h) \in I$ for some function h. Thus $s^l(g - fh) \in I$. As $g - fh$ is a unit near X after restricting to a smaller open set we can assume

$$s^l \in I \,, \quad \text{thus} \quad I = \langle f^{k-1}, F \rangle \,.$$

In particular, $\mathrm{Sing}(\mathcal{X}/S)$ is principal in $\mathcal{X}$ (defined by f^{k-1}), thus by Proposition 2.2 we have

$$\begin{aligned} \mathrm{sp}_0(\Lambda_{\mathcal{X}/S}) &= (\mathrm{Bl}_{\mathrm{Sing}\mathcal{X}/S}\, \mathcal{X})|_0 \\ &\simeq \mathcal{X}|_0 \\ &= X \\ &= \mathrm{len}(\mathcal{O}_{X,X_{\mathrm{red}}}) \cdot X_{\mathrm{red}} \\ &\simeq \mathrm{len}(\mathcal{O}_{X,X_{\mathrm{red}}}) \cdot \Lambda_{X_{\mathrm{red}}} \,. \end{aligned}$$

This proves the claim on the coefficients of the $\Lambda_{X_{i,\mathrm{red}}}$. As we only needed to restrict to complements of closed sets in $\mathrm{Sing}(X_{\mathrm{red}}) \cup \mathrm{Sing}(\mathcal{X})$ during the proof, we have that every other cycle Λ_Z in the specialization must verify $Z \subset \mathrm{Sing}(X_{\mathrm{red}}) \cup (\mathrm{Sing}(\mathcal{X}) \cap X)$. □

When the special fiber is reduced, we can go one step further:

Theorem 2.8 (First-Order Approximation of the Lagrangian Specialization)**.** *Assume that X is reduced and $\mathcal{X}|_s$ is smooth for $s \neq 0$. Let $\operatorname{Sing}(X) = \cup_i Z_i$ be the decomposition of the singular locus into its scheme-theoretic irreducible components. Then*

$$\mathrm{sp}_0 \Lambda_{\mathcal{X}/S} = \Lambda_X + \sum_i (\mathrm{mult}_{Z_i} X) \cdot \Lambda_{Z_{i,red}} + \sum_{Y \subset Z_{i,\mathrm{red}}} m_Y \Lambda_Y$$

where $\mathrm{mult}_Z X$ *is the Samuel multiplicity of Z in X as defined for example in* [13, Sec. 4.3]*, and the last sum runs over* proper *subvarieties $Y \subsetneq Z_{i,\mathrm{red}}$.*

Remark. If the singular locus is 0-dimensional, this computes the full Lagrangian specialization.

Proof. By Proposition 2.6, we have

$$\mathrm{sp}_0 \Lambda_{\mathcal{X}/S} = \Lambda_X + \sum_i \left(m_{Z_i} \Lambda_{Z_{i,\mathrm{red}}} + \sum_{Y \subset Z_{i,\mathrm{red}}} m_Y \Lambda_Y \right)$$

for some coefficients m_{Z_i}, m_Y. The coefficients m_{Z_i} are defined locally, thus after restricting to an open set we can assume that $Z = \operatorname{Sing}(X)$ is irreducible and

$$\mathrm{sp}_0 \Lambda_{\mathcal{X}/S} = \Lambda_X + m_Z \Lambda_{Z_{\mathrm{red}}} \,.$$

Let $\mathcal{Z} := \operatorname{Sing}(\mathcal{X}/S)$. Note that by assumption $\operatorname{Supp}(\mathcal{Z}) = \operatorname{Supp}(Z)$ and $Z = \mathcal{Z} \cap X$. By Proposition 2.2 we have $\mathbb{P}\Lambda_{\mathcal{X}/S} = \mathrm{Bl}_{\mathcal{Z}} \mathcal{X}$. We have the following two fiber squares,

$$\begin{array}{ccccc} \mathrm{sp}_0(\mathbb{P}\Lambda_{\mathcal{X}/S}) & \hookrightarrow & \mathbb{P}\Lambda_{\mathcal{X}/S} & \xleftarrow{\ j\ } & E \\ \downarrow & & \downarrow f & & \downarrow g \\ X & \hookrightarrow & \mathcal{X} & \xleftarrow{\ i\ } & \mathcal{Z} \end{array}$$

where the right square is the diagram associated to the blowup f and E is the exceptional divisor. By definition, we have

$$f^*[X] = [\mathbb{P}\Lambda_{\mathcal{X}/S}|_0] =: \mathrm{sp}_0(\mathbb{P}\Lambda_{\mathcal{X}/S}) = [\mathbb{P}\Lambda_X] + m_Z [\mathbb{P}\Lambda_{Z_{\mathrm{red}}}] \in \mathrm{CH}^1(\mathbb{P}\Lambda_{\mathcal{X}/S}) \,.$$

Let $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}\Lambda_{\mathcal{X}/S}}(-E)|_E$ denote the canonical bundle on E associated to the blowup. By 2.2 and [13, B.6.9] there is a canonical embedding

$$\mathbb{P}\Lambda_X = Bl_Z X \hookrightarrow Bl_{\mathcal{Z}} \mathcal{X} = \mathbb{P}\Lambda_{\mathcal{X}/S}$$

and the restriction of the exceptional divisor $E' := E \cap Bl_Z X$ is the exceptional divisor of $Bl_Z X$. Thus

$$\begin{aligned} g_* \left(j^*[\mathbb{P}\Lambda_X] \cap c_1(\mathcal{O}(1))^{d-2} \right) &= g_* \left(j^*[\mathrm{Bl}_{Z'} X] \cap c_1(\mathcal{O}(1))^{d-2} \right) \\ &= g_* \left([E'] \cap c_1(\mathcal{O}(1))^{d-2} \right) \\ &= \mathrm{mult}_Z X \cdot [Z_{\mathrm{red}}] \in \mathrm{CH}^0(Z) \,, \end{aligned}$$

by [13, Sec. 4.3]. Notice that $\mathbb{P}\Lambda_{Z_{\mathrm{red}}} \subset E$. We make the abuse of notation to write $[\mathbb{P}\Lambda_{Z_{\mathrm{red}}}]$ for the cycle in $\mathrm{CH}^\bullet(\mathbb{P}\Lambda_{\mathcal{X}/S})$ as well as $\mathrm{CH}^\bullet(E)$, when it is clear from the context which Chow ring we mean. Recall $\mathcal{O}_{\mathbb{P}\Lambda_{\mathcal{X}/S}}(E)|_E = \mathcal{O}(-1)$, thus

$$\begin{aligned} g_* \left(j^*[\mathbb{P}\Lambda_{Z_{\mathrm{red}}}] \cap c_1(\mathcal{O}(1))^{d-2} \right) &= g_* \left(-[\mathbb{P}\Lambda_{Z_{\mathrm{red}}}] \cap c_1(\mathcal{O}(1))^{d-1} \right) \\ &= -[Z_{\mathrm{red}}] \in \mathrm{CH}^0(Z) \,, \end{aligned}$$

as a generic fiber of $\mathbb{P}\Lambda_{Z_{\mathrm{red}}} \to Z_{\mathrm{red}}$ is a $(d-1)$-plane. By definition $[X] = q^*[0] \in \mathrm{CH}^1(\mathcal{X})$. Since $\mathcal{Z}$ is supported on a fiber of q we have

$$0 = i^* q^*[0] = i^*[X] \,.$$

Putting all of this together we have

$$\begin{aligned} 0 &= g_* \left(g^* i^* [X] \cap c_1(\mathcal{O}(1))^{d-2} \right) \\ &= g_* \left(j^* f^* [X] \cap c_1(\mathcal{O}(1))^{d-2} \right) \\ &= g_* \left(j^* (\Lambda_X + m_Z \Lambda_{Z_{\mathrm{red}}}) \cap c_1(\mathcal{O}(1))^{d-2} \right) \\ &= (\mathrm{mult}_Z X - m_Z)[Z_{\mathrm{red}}] \in \mathrm{CH}^0(Z)\,. \end{aligned}$$

Thus $m_Z = \mathrm{mult}_Z X$, as $\mathrm{CH}^0(Z) = [Z_{\mathrm{red}}] \cdot \mathbb{Z}$. □

2.3. **Application to Theta Divisors.** We have the following corollary to Theorem 2.8:

Corollary 2.9. *Let $(A, \Theta) \in \mathcal{A}_g$ and $\cup_i Z_i = \mathrm{Sing}(\Theta)$ the decomposition of the singular locus of Θ into its scheme-theoretic irreducible components. Then*

$$\deg \mathcal{G} \leq g! - \sum_i (\mathrm{mult}_{Z_i} \Theta) \deg(\Lambda_{Z_{i,\mathrm{red}}})\,,$$

where $\mathcal{G} : \Theta \dashrightarrow \mathbb{P}^{g-1}$ is the Gauss map.

Proof. Let (A_S, Θ_S) be a 1-dimensional deformation of (A, Θ), i.e. an abelian scheme over a smooth curve S with special fiber (A, Θ), such that Θ_s is smooth for general s. The degree is invariant in flat families [9, Prop. 2.4], thus

$$\begin{aligned} g! &= \deg \Lambda_{\Theta_s} && \text{(Ex. 2.4)} \\ &= \deg(\mathrm{sp}_0 \Lambda_{\Theta_S/S}) \\ &= \deg(\Lambda_\Theta) + \sum_i (\mathrm{mult}_{Z_i} \Theta) \deg(\Lambda_{Z_{i,\mathrm{red}}}) + \sum_{Z \subset Z_i} m_Z \deg(\Lambda_Z) && \text{(Thm. 2.8)} \\ &\geq \deg(\mathcal{G}) + \sum_i (\mathrm{mult}_{Z_i} \Theta) \deg(\Lambda_{Z_{i,\mathrm{red}}})\,. \end{aligned}$$

The last assertion follows from the fact that $\deg \Lambda_Z \geq 0$ for a subvariety Z of an abelian variety [9, Lem. 5.1]. □

Remark. If the singular locus of Θ is 0-dimensional, there are no other terms in the Lagrangian specialization and one recovers a result of [8, Rem 2.8]

$$\deg \mathcal{G} = g! - \sum_{z \in \mathrm{Sing}(\Theta)} \mathrm{mult}_z \Theta\,.$$

3. Theta Divisors with Transversal A$_1$ Singularities

The idea of the proof of Theorem 3.5 is to deform a given ppav to a ppav with a smooth theta divisor. Using the heat equation verified by theta functions, it is then possible to compute the Lagrangian specialization explicitly. Finally, we use the fact that the degree of Lagrangian cycles is invariant in flat families.

3.1. **Deformation of PPAV's.** Let $(A, \Theta) \in \mathcal{A}_g$ and denote by T_A the tangent bundle on A. It is well-known that there is a canonical identification between $\mathrm{H}^0(A, \mathrm{Sym}_2(T_A))$ and infinitesimal deformations of (A, Θ) [21] and [6, Sec. 3]. Specifically, let

$$\mathscr{D} = \sum_{i,j} \lambda_{ij} \frac{\partial^2}{\partial z_i \partial z_j} \in \mathrm{H}^0(A, \mathrm{Sym}_2(T_A))\,,$$

where $\lambda_{ij} \in \mathbb{C}$ and $\partial_{z_1}, \ldots, \partial_{z_g}$ is a basis of $\mathrm{H}^0(A, T_A)$. Then there exists a deformation of (A, Θ), i.e. an abelian scheme over a smooth quasi-projective curve S

$$\Theta_S \subset A_S \to S\,,$$

such that the fiber above $0 \in S$ is (A, Θ). Moreover, locally there are coordinates $(z_1, \ldots, z_g, s)$ on A_S such that the map to S is given by the projection onto the last coordinate, and if θ is the theta-function defining Θ_S we have

$$\mathscr{D}(\theta) = \sum_{i,j} \lambda_{ij} \frac{\partial^2 \theta}{\partial z_i \partial z_j} = \frac{\partial \theta}{\partial s}\,. \tag{3.1}$$

We call this a deformation in the $\mathscr{D}$ direction.

3.2. **Computation of the Gauss Degree.** We will need the following technical lemma [22, Lem. 2.26]:

Lemma 3.2 (Morse Lemma with parameters)**.** *Let $f(x;s) : (\mathbb{C}^d \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ be a holomorphic function such that the hessian matrix in the first d coordinates*

$$\mathrm{H}(f)_0 = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0) \right)_{1 \le i,j \le d}$$

is non-degenerate. Then there is a local holomorphic change of coordinates $h_s : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ such that

$$f(h_s(y); s) = \sum_{i=1}^{d} y_i^2 + f(0; s)\,.$$

Let $(X, 0) = V(f) \subset (\mathbb{C}^n, 0)$ be a hypersurface singularity germ. The *scheme-theoretic* singular locus of X is defined as

$$\mathrm{Sing}(X) \coloneqq V\left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) \subset (\mathbb{C}^n, 0)\,, \tag{3.3}$$

where $x_1, \ldots, x_n$ are some coordinates on $\mathbb{C}^n$. We have the following:

Proposition 3.4. *Let $(X, 0) = V(f) \subset (\mathbb{C}^n, 0)$ be a hypersurface singularity germ, and $d = \mathrm{codim}_{\mathbb{C}^n} \mathrm{Sing}(X)$. The Hessian of f*

$$H(f) \coloneqq \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{1 \le i,j \le n}$$

is of rank at most d. The following conditions are equivalent:

i) $\mathrm{Sing}(X)$ *is smooth at* 0.
ii) $H(f)$ *is of rank* d *at* 0.
iii) *There is a holomorphic change of coordinates* $x = h(y)$ *such that*

$$f(h(y)) = y_1^2 + \cdots + y_d^2\,.$$

In this case, we say that X has a transversal A_1 *singularity at* 0.

Proof. ($i \iff ii$) and ($iii \implies i$) are trivial. Let us show (i and ii) $\implies iii$.

After a first change of coordinates we can assume $\mathrm{Sing}(X) = V(x_1, \ldots, x_d)$. We have $T_0\mathrm{Sing}(X) = \mathrm{Ker} H(f)_0$, thus

$$H(f)_0 = \begin{pmatrix} \tilde{H}(f)_0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\tilde{H}(f)$ is the Hessian in the first d coordinates. Thus $\tilde{H}(f)$ is non-degenerate at 0. By the Morse Lemma with parameter $x' = (x_{d+1}, \dots, x_n)$, there is a change of coordinates

$$(x_1, \dots, x_d) = h_{x'}(y_1, \dots, y_d)$$

such that

$$f(h_{x'}(y); x') = y_1^2 + \cdots + y_d^2 + f(0; x') = y_1^2 + \cdots + y_d^2 \,,$$

as $(0, x') \in \mathrm{Sing}(X)$. □

We say that a variety X has transversal A_1 singularities if the equivalent conditions of Proposition 3.4 hold at every singular point of X. We now compute the degree of the Gauss map using the results of the previous section:

Theorem 3.5. *Let $(A, \Theta) \in \mathcal{A}_g$ such that Θ has transversal A_1 singularities. Let $Z := \mathrm{Sing}(\Theta)$ and*

$$D \in |\mathcal{O}_A(\Theta)|_Z|$$

be any smooth divisor of the linear system. Then the degree of the Gauss map $\mathcal{G} : \Theta \dashrightarrow \mathbb{P}^{g-1}$ is

$$\deg \mathcal{G} = g! - 2(-1)^{\dim Z} \chi(Z) - (-1)^{\dim D} \chi(D) \,,$$

where χ denotes the usual topological Euler characteristic.

Proof. Let $d = \dim(Z)$. Let $\theta \in \mathrm{H}^0(A, \mathcal{O}_A(\Theta))$ be a non-zero section. Let $\partial z_1, \dots, \partial z_g$ be a basis of $\mathrm{H}^0(A, T_A)$. Consider the linear subspace

$$V := \left\langle \frac{\partial^2 \theta}{\partial z_i \partial z_j}|_Z \right\rangle_{1 \le i,j \le g} \subset \mathrm{H}^0(Z, \mathcal{O}_A(\Theta)|_Z) \,,$$

and let $\mathscr{V} = \mathbb{P}V \subset |\mathcal{O}_A(\Theta)|_Z|$ be the associated linear series. Notice that by Proposition 3.4, the Hessian of θ is of rank d. In particular, at every point $x \in Z$ there is a section in V that is non-zero at x. Thus $\mathscr{V}$ is base-point free and by Bertini's theorem, a general divisor $D \in \mathscr{V}$ is smooth. Fix such a smooth divisor $D \in \mathscr{V}$. We have

$$D = \mathrm{div}\left(\sum_{i,j} \lambda_{ij} \frac{\partial^2 \theta}{\partial z_i \partial z_j}|_Z \right)$$

for some $\lambda_{ij} \in \mathbb{C}$. By the previous section, there is a deformation $q : (A_S, \Theta_S) \to S$ in the $\mathscr{D} = \sum_{i,j} \lambda_{ij} \partial_i \partial_j \theta$ direction. By 3.1, there are locally coordinates $z_1, \dots, z_g, s$ on A_S such that q is the projection onto the last coordinate and

$$D = V\left(\frac{\partial \theta}{\partial s}|_Z \right) .$$

Let $\Lambda_{\Theta_S/S}$ be the relative conormal variety. By Lemma 3.6 below and Example 2.5 we have

$$\mathrm{sp}_0 \Lambda_{\Theta_S/S} = \Lambda_{\Theta_0} + 2\Lambda_Z + \Lambda_D \,.$$

By the local normal form in the lemma below (3.7), Θ_s is smooth for $s \neq 0$. Thus

$$\begin{aligned}
g! &= \deg \Lambda_{\Theta_s} && \text{(Ex. 2.4)} \\
&= \deg(\mathrm{sp}_s \Lambda_{\Theta_S/S}) \\
&= \deg(\mathrm{sp}_0 \Lambda_{\Theta_S/S}) && \text{([9, Prop. 2.4])} \\
&= \deg\left(\Lambda_{\Theta_0} + 2\Lambda_Z + \Lambda_D\right) \\
&= \deg(\mathcal{G}) + 2(-1)^{\dim Z}\chi(Z) + (-1)^{\dim D}\chi(D) && \text{(Prop. 2.3 and Ex. 2.1)} .
\end{aligned}$$

□

We have the following lemma:

Lemma 3.6. *Let $f(z;s) : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function such that the singular locus and the critical locus of $f|_{s=0}$*

$$Z \coloneqq V(f, \partial_1 f, \dots, \partial_n f, s)\,, \quad D \coloneqq V(\partial_s f|_Z) \subset Z\,,$$

are smooth. Then there is a local holomorphic change of coordinates $z = h_s(\tilde{z})$ such that

$$f(h_s(\tilde{z}); s) = \tilde{z}_1^2 + \cdots + \tilde{z}_d^2 + \tilde{z}_{d+1} s\,. \tag{3.7}$$

Proof. After a change of coordinates in $\mathbb{C}^n$ we can assume $Z = \{z_1 = \cdots = z_d = 0\}$. We have $f|_Z = 0$ thus

$$\frac{\partial^2 f}{\partial z_i \partial z_j}|_0 = 0 \quad \text{for } 1 \leq i \leq n \text{ and } d+1 \leq j \leq n.$$

Thus the Hessian of F in the first n coordinates is

$$H(F)_0 = \begin{pmatrix} H(F|_{\mathbb{C}^d})_0 & 0 \\ 0 & 0 \end{pmatrix}.$$

By Proposition 3.4, $H(F)$ and thus $H(F|_{\mathbb{C}^d})$ is of rank d at 0. By the Morse Lemma with parameters $(z_{d+1}, \dots, z_n, s)$, there is a holomorphic change of coordinates

$$(z_1, \dots, z_d) = h_{(z',s)}(\tilde{z}_1, \dots, \tilde{z}_d)$$

where $z' = (z_{d+1}, \dots, z_n)$, such that

$$f(h_{(z',s)}(\tilde{z}), z', s) = \sum_{i=1}^{d} \tilde{z}_i^2 + f(0, z', s)\,.$$

We have

$$f(0, z', 0) = 0\,,$$

and $\frac{\partial f}{\partial s}|_Z$ has a simple 0 in zero, thus after a change of coordinates (in z'), we can assume

$$\frac{\partial f}{\partial s}|_Z = z_{d+1}\,.$$

Thus

$$f(0, z', s) = s(z_{d+1} + sg(z', s))\,,$$

for some holomorphic g. Making the coordinate change

$$\tilde{z}_{d+1} = z_{d+1} + sg(z', s)\,,$$

the lemma follows. □

4. The Family $\mathcal{A}^{\delta}_{t,g-t}$

We apply Theorem 3.5 to the families $\mathcal{A}^{\delta}_{t,g-t}$ studied by Debarre in [10]. First we recall the definition and known results about $\mathcal{A}^{\delta}_{t,g-t}$. Then we compute the Gauss degree for a general member of these families. Finally we analyse the degree numerically and show that it separates the corresponding components of the Andreotti-Mayer locus.

4.1. **Definition of the Family.** Let A be an abelian variety and L an ample line bundle on A. Recall that the type $\delta = (a_1, \dots, a_k)$ of L is defined by

$$\mathrm{Ker}(\Phi_L) \simeq \bigoplus_{i=1}^{k} (\mathbb{Z}/a_i\mathbb{Z})^2 , \quad \text{and} \quad a_i | a_{i+1} \quad \text{for } 1 \leq i < k ,$$

where $\Phi_L : A \to \hat{A}$ is the polarization induced by L. We have the following [10], [5, Th. 5.3.5]:

Proposition 4.1 (Complementary Abelian Varieties)**.** *Let $(A, \Theta) \in \mathcal{A}_g$ and δ be a polarization type. Suppose there is an abelian subvariety $X \subset A$ of dimension t and the induced polarization $L_X = L|_X$ is of type δ. Then there is a unique abelian subvariety $Y \subset A$ (of dimension $g - t$) such that:*

a) The morphism $\pi : X \times Y \overset{i_X + i_Y}{\longrightarrow} A$ is an isogeny.
b) We have

$$\pi^\star L = L_X \boxtimes L_Y , \quad \textit{where} \quad L_Y = L|_Y .$$

Moreover L_Y is also of type δ. We define $\mathcal{A}^\delta_{t,g-t} \subset \mathcal{A}_g$ to be the set of ppav's verifying the above conditions.

Reciprocally, if (X, L_X), (Y, L_Y) are two abelian varieties of the same type δ, of dimension t and $g - t$ respectively, and $\psi : \mathrm{Ker}(\Phi_{L_X}) \to \mathrm{Ker}(\Phi_{L_Y})$ is an antisymplectic isomorphism, then

$$A \coloneqq X \times Y / K \in \mathcal{A}^\delta_{t,g-t} \quad \text{where} \quad K \coloneqq \{(x, \psi x) \,|\, x \in K(L_X)\} .$$

Thus $\mathcal{A}^\delta_{t,g-t}$ is irreducible loci of codimension $t(g-t)$ in $\mathcal{A}_g$ [10, Sec. 9.3]. Clearly $\mathcal{A}^\delta_{t,g-t} = \mathcal{A}^\delta_{g-t,t}$, so from now on we will assume $t \leq g/2$. The loci $\mathcal{A}^\delta_{t,g-t}$ are all distinct. Let B_X (resp. B_Y) be the base locus of L_X (resp. L_Y). Recall that by the Riemann-Roch theorem,

$$\mathrm{h}^0(X, L_X) = \mathrm{h}^0(Y, L_Y) = \deg \delta =: d .$$

Thus, for $t \geq d$, the base loci B_X and B_Y are non-empty of codimension at most d in X and Y respectively. Let $s^X_1, \dots, s^X_d$ and $s^Y_1, \dots, s^Y_d$ denote a basis of $\mathrm{H}^0(X, L_X)$ and $\mathrm{H}^0(Y, L_Y)$ respectively. Let s be a generator of $\mathrm{H}^0(A, L)$. Then

$$\pi^* s = \sum_{i,j} \lambda_{ij} s^X_i \boxtimes s^Y_j$$

for some λ_{ij}. Taking derivatives, we have

$$\mathrm{d}(\pi^* s) = \sum_{i,j} \lambda_{ij} ((\mathrm{d}s^X_i) \boxtimes s^Y_j + s^X_i \boxtimes (\mathrm{d}s^Y_j)) ,$$

which vanishes on $B_X \times B_Y$. Thus

$$\pi(B_X \times B_Y) \subset \mathrm{Sing}(\Theta) , \tag{4.2}$$

$$\text{and} \quad \mathcal{A}^\delta_{t,g-t} \subset \mathcal{N}^g_{g-2d} . \tag{4.3}$$

The main result of Debarre concerning the families $\mathcal{A}^\delta_{t,g-t}$ is the following:

Theorem 4.4 ([10, Thm. 10.4 and 12.1])**.** *Let $\delta \in \{(2), (3), (2, 2)\}$, and $d = \deg \delta$.*

i) If $t \geq d$, then $\mathcal{A}^\delta_{t,g-t}$ is an irreducible component of $\mathcal{N}^{(g)}_{g-2d}$. Moreover, for a general $(A, \Theta) \in \mathcal{A}^\delta_{t,g-t}$, there is equality in 4.2 and Θ has transversal A_1 singularities, and $\dim \mathrm{Sing}(\Theta) = g - 2d$.
ii) If $t = \lfloor d/2 \rfloor$, a general $(A, \Theta) \in \mathcal{A}^\delta_{t,g-t}$ has smooth theta divisor. In this case,

$$\deg \mathcal{G}\left(\mathcal{A}^\delta_{t,g-t}\right) = g!.$$

We end this section with a result on the dimension of the fibers of the Gauss map. This is a slight slight improvement on [4, Thm. 1.1] (the bound on the dimension is stronger):

Proposition 4.5. *Let $(A,\Theta)\in\mathcal{A}^{\delta}_{t,g-t}$, $d:=\deg\delta$, and suppose $2\le d\le t\le g/2$. Suppose there is a divisor $D\in|L_X|$ such that D is smooth at some point $x\in B_X$. Then some fibers of the Gauss map $\mathcal{G}:\Theta\dashrightarrow\mathbb{P}^{g-1}$ are of dimension at least $g-t-d+1$.*

Proof. Let $\pi: X\times Y\to A$ denote the isogeny of 4.1. Let $\tilde{\Theta}:=\pi^*\Theta\subset X\times Y$. By [10, Prop. 9.1], there is a basis $s_1^X,\dots,s_d^X$ (resp. $s_1^Y,\dots,s_d^Y$) of $\mathrm{H}^0(X,L_X)$ (resp. $\mathrm{H}^0(Y,L_Y)$), such that

$$\tilde{\Theta}=\operatorname{div}s\,,\quad\text{where}\quad s=\sum_{i=1}^{d}s_i^X\otimes s_i^Y\,.$$

By assumption, there is a divisor $D\in|L_X|$ that is smooth at some point $x\in B_X$. Thus, after relabeling the s_i^X, we can assume $\operatorname{div}(s_d^X)$ is smooth at $x\in B_X$. Let

$$F=V(s_1^Y,\dots,s_{d-1}^Y)\setminus V(s_d^Y)\subset Y.$$

For a line bundle L on A we denote by $d: L\to L\otimes T^\vee A$ the usual differential on A. For $y\in F$, we have

$$\mathrm{d}_{x,y}s=\sum_i \mathrm{d}_x\left(s_i^X\right)\otimes s_i^Y(y)+s_i^X(x)\otimes\mathrm{d}_y\left(s_i^Y\right)=\mathrm{d}_x\left(s_d^X\right)\otimes s_d^Y(y)\neq 0\,.$$

Moreover for a fixed $x\in B_X$, varying $y\in F$ only multiplies the conormal vector $\mathrm{d}_x s_d^X$ by a constant. Thus the image $v=[\mathrm{d}_x s_d^X]\in\mathbb{P}T_0^\vee(X\times Y)$ is the same. Thus the preimage of v by the Gauss map contains $\{x\}\times F$ which is of dimension at least

$$\dim Y-(d-1)=g-t-d+1\,.$$

□

4.2. **Gauss Degree on $\mathcal{A}^{\delta}_{t,g-t}$.** Knowing 3.5 and 4.4, the computation of the Gauss Degree on a general $(A,\Theta)\in\mathcal{A}^{\delta}_{t,g-t}$ boils down to a relatively simple Euler characteristic computation:

Lemma 4.6. *Let $(A,\Theta)\in\mathcal{A}^{\delta}_{t,g-t}$, let $d:=\deg\delta$, assume that $\mathrm{Sing}(\Theta)$ is smooth, that $\mathrm{codim}_A\mathrm{Sing}(\Theta)=2d$ and equality holds in 4.2. Then*

$$\chi(\mathrm{Sing}(\Theta))=(-1)^{g-2d}t!(g-t)!\binom{t-1}{d-1}\binom{g-t-1}{d-1}.$$

If $C\in|\mathcal{O}_A(\Theta)|_{\mathrm{Sing}(\Theta)}|$ is smooth, then

$$\chi(C)=(-1)^{g-2d-1}t!(g-t)!c_{t-d,g-t-d}\,,$$

where $c_{m,n}$ is defined by the generating series

$$\frac{x+y}{(1-x)^d(1-y)^d(1-x-y)}=\sum_{m,n}c_{m,n}x^my^n\,.$$

Proof. We keep the notation of the previous section. By assumption $\pi: B_X\times B_Y\to\mathrm{Sing}(\Theta)$ is an isogeny of degree d^2, thus

$$\chi(\mathrm{Sing}(\Theta))=\chi(B_X\times B_Y)/d^2\,.$$

By Riemann-Roch we have $\mathrm{h}^0(X,L_X)=\mathrm{h}^0(Y,L_Y)=d$. Thus, the base locus B_X (resp. B_Y) is the intersection of d divisors in $|L_X|$ (resp. $|L_Y|$). By assumption, we have

$$\mathrm{codim}_{X\times Y}B_X\times B_Y=\mathrm{codim}_A\mathrm{Sing}(\Theta)=2d.$$

Thus B_X (resp. B_Y) is of codimension d in X (resp. Y), and is a complete intersection of d divisors in $|L_X|$ (resp. $|L_Y|$). We thus have $N_{B_X/X} = L_X^{\oplus d}|_{B_X}$ and by [13, Ex. 3.2.12] and Riemann-Roch we have

$$\begin{aligned}\chi(B_X) &= \deg(c(T_{B_X}))\\ &= \deg\left(c(T_X)|_{B_X} \cdot (c(L_X|_{B_X}))^{-d}\right)\\ &= \deg\left(1 \cdot (1+c_1(L_X))^{-d} \cap [B_X]\right)\\ &= \deg \sum_{k\geq 0} \binom{d+k-1}{d-1} (-1)^k c_1(L_X)^{k+d} \cap [X]\\ &= (-1)^{t-d} d \binom{t-1}{d-1} t!\,.\end{aligned}$$

The same computation applies to B_Y, thus

$$\chi(B_X \times B_Y) = (-1)^{g-2d} d^2 \binom{t-1}{d-1} \binom{g-t-1}{d-1} t!(g-t)!\,.$$

We now compute $\chi(C)$. Let $C' = \pi^* C \subset X \times Y$. Let $x = c_1(p_X^* L_X) \in \mathrm{CH}^1(X\times Y)$ and $y = c_1(p_Y^* L_Y) \in \mathrm{CH}^1(X \times Y)$. We have $C' \in |(L_X \boxtimes L_Y)|_{B_X\times B_Y}|$,

$$\left[C'\right] = x^d y^d (x+y) \in \mathrm{CH}^{2d+1}(X\times Y)\,,$$

and $N_{C'/X\times Y} = \left((p_X^* L_X)^{\oplus d} \oplus (p_Y^* L_Y)^{\oplus d} \oplus (L_X \boxtimes L_Y)\right)|_{C'}$. By [13, Ex. 3.2.12] we have

$$\begin{aligned}c(T_{C'}) &= c(T_{X\times Y}|_{C'}) \cdot c(N_{C'/X\times Y})^{-1}\\ &= (1+x)^{-d}(1+y)^{-d}(1+x+y)^{-1} \cap [C']\\ &= \frac{x^d y^d (x+y)}{(1+x)^d(1+y)^d(1+x+y)} \cap [X\times Y]\\ &= x^d y^d \sum_{m,n} (-1)^{m+n+1} c_{m,n} x^m y^n\,.\end{aligned}$$

The only term of degree g in the above series which does not vanish is $x^t y^{g-t}$ and

$$\deg(x^t y^{g-t}) = d^2 t!(g-t)!$$

by Riemann-Roch. Thus

$$\chi(C) = \chi(C')/d^2 = (-1)^{g-2d-1} t!(g-t)! c_{t-d,g-t-d}\,.$$

□

We have the following:

Theorem 4.7. *Let $\delta \in \{(2),(3),(2,2)\}$, let $t \geq d \coloneqq \deg\delta$, let $(A,\Theta) \in \mathcal{A}^{\delta}_{t,g-t}$ be general and $\mathcal{G} : \Theta \dashrightarrow \mathbb{P}^{g-1}$ be the Gauss map. Then*

$$\deg \mathcal{G} = g! - t!(g-t)! a_{t-d,g-t-d}\,.$$

where $a_{m,n}$ is defined by the generating series

$$\frac{1}{(1-x)^d(1-y)^d} + \frac{1}{(1-x)^d(1-y)^d(1-x-y)} = \sum_{m,n} a_{m,n} x^m y^n\,.$$

More explicitly,

$$\deg \mathcal{G} = t!(g-t)!\left(\binom{t-1}{d-1}\binom{g-t-1}{d-2} + \sum_{k=2}^{d}\binom{t-k}{d-k}\binom{g-t-1+k}{d-1}\right).$$

Remark. The theorem above holds more generally when $(A,\Theta) \in \mathcal{A}^{\delta}_{t,g-t}$, $\mathrm{Sing}(\Theta)$ is smooth of dimension $g-2d$ and equality holds in 4.2, but we do not know for which values of δ, t and g this happens in general.

Proof. Let $(A,\Theta) \in \mathcal{A}^{\delta}_{t,g-t}$ be general. By Theorem 4.4 and Theorem 3.5, there is a smooth divisor $C \in |L_A|_{\Theta_{\mathrm{sing}}}|$ such that

$$\deg \mathcal{G} = g! - 2(-1)^{g-2d}\chi(\mathrm{Sing}(\Theta)) - (-1)^{g-2d-1}\chi(C)).$$

Moreover, by Theorem 4.4 we have $\mathrm{Sing}(\Theta)$ smooth, $\mathrm{codim}_A \mathrm{Sing}(\Theta) = 2d$ and equality holds in 4.2. Thus, by Lemma 4.6 we have

$$\begin{aligned}(-1)^{g-2d}\chi(\mathrm{Sing}(\Theta)) &= t!(g-t)!\binom{t-1}{d-1}\binom{g-t-1}{d-1} \\ &= t!(g-t)!\left\{\frac{1}{(1-x)^d(1-y)^d}\right\}_{x^{t-d}y^{g-t-d}},\end{aligned}$$

Thus

$$2\chi(\mathrm{Sing}(\Theta)) + \chi(C) = t!(g-t)!a_{m,n},$$

where

$$\begin{aligned}\sum_{m,n\geq 0} a_{m,n}x^m y^n &= \frac{2}{(1-x)^d(1-y)^d} + \frac{x+y}{(1-x)^d(1-y)^d(1-x-y)} \\ &= \frac{1}{(1-x)^d(1-y)^d} + \frac{1}{(1-x)^d(1-y)^d(1-x-y)}.\end{aligned}$$

We use the combinatorial Lemma 4.8 below to conclude

$$\begin{aligned}\deg \mathcal{G} &= g! - t!(g-t)!a_{t-d,g-t-d} \\ &= g! - t!(g-t)!\left(\binom{t-1}{d-1}\binom{g-t-1}{d-1} + \binom{t+g-t}{t}\right. \\ &\quad \left. - \sum_{k=1}^{d}\binom{t-k}{t-d}\binom{g-t-1+k}{d-1}\right) \\ &= t!(g-t)!\left(\binom{t-1}{d-1}\binom{g-t-1}{d-2} + \sum_{k=2}^{d}\binom{t-k}{d-k}\binom{g-t-1+k}{d-1}\right).\end{aligned}$$

□

The generating series of the theorem has the following combinatorial interpretation:

Lemma 4.8. *Consider the generating series*

$$\frac{1}{(1-x)^d(1-y)^d(1-x-y)} = \sum_{m,n\geq 0} A_{m,n}x^m y^n.$$

then the coefficient $A_{m,n}$ is equal to the number of (weak) $m+d+1$ compositions of $n+d$

$$0 \leq a_1 \leq \cdots \leq a_{m+d} \leq a_{m+d+1} = n+d,$$

such that $a_{m+1} \geq d$*. This number is equal to*

$$A_{m,n} = \binom{m+n+2d}{m+d} - \sum_{k=1}^{d} \binom{m+d-k}{m} \binom{n+d-1+k}{d-1}.$$

Proof. Recall that by a (here we mean weak) m composition of n we mean an m-tuple $(a_1, \dots, a_m)$ such that

$$0 \leq a_1 \leq \dots \leq a_{m-1} \leq a_m = n\,.$$

The number of m compositions of n is equal to

$$\binom{n+m-1}{m-1}.$$

We know that

$$\frac{1}{(1-y)^d} = \sum_{n \geq 0} \binom{n+d-1}{d-1} y^n$$

is the generating series for the d-compositions of n. Moreover,

$$\frac{1}{1-x-y} = \sum_{n \geq 0} (x+y)^n = \sum_{m,n \geq 0} \binom{m+n}{m} x^m y^n$$

is the generating series for the $m+1$-compositions of n. Thus,

$$\frac{1}{(1-y)^d(1-x-y)}$$

is the generating series for the $(m+d+1)$-compositions of n. We can interpret $1/(1-x)^d$ as the generating series of the $m+1$ compositions of $d-1$. Thus the coefficient $A_{m,n}$ is in bijection with the set

$$\bigsqcup_{k=0}^{m} \{k+1 \text{ composition of } d-1\} \times \{m-k+1+d \text{ composition of } n\}\,.$$

Now to a $k+1$-composition of $d-1$ $(a_1, \dots, a_{k+1})$ and a $m-k+1+d$-composition of n $(b_1, \dots, b_{m-k+1+d})$, we associate a $m+d+1$-composition of $n+d$ in the following way:

$$\begin{aligned} \tilde{a}_i &= a_i \quad \text{for} \quad 1 \leq i \leq k \\ \tilde{a}_i &= b_{i-k} + a_{k+1} + 1 \quad \text{for} \quad k+1 \leq i \leq m+d+1\,. \end{aligned}$$

Clearly this gives a bijection to all the $m+d+1$-compositions of $n+d$ such that $a_{m+1} \geq d$. The inverse map is given by choosing $k+1$ to be the first coefficient of the composition above d. Thus

$$\begin{aligned} A_{m,n} &= \#\{m+d+1 \text{ compositions of } n+d\} \\ &\quad - \sum_{k=0}^{d-1} \#\{m+d+1 \text{ compositions of } n+d \text{ such that } a_{m+1} = k\} \\ &= \#\{m+d+1 \text{ compositions of } n+d\} \\ &\quad - \sum_{k=0}^{d-1} \#\{m+1 \text{ compositions of } k\} \times \{d \text{ compositions of } n+d-k\} \\ &= \binom{m+n+2d}{m+d} - \sum_{k=0}^{d-1} \binom{m+k}{m} \binom{n+2d-1-k}{d-1} \end{aligned}$$

□

4.3. **Numerical Analysis of the Degree.** By [9, Th. 1.1] the degree of the Gauss map $\deg(\mathcal{G}) : \mathcal{A}_g \to \mathbb{N}$ is a lower-semicontinuous function. For an irreducible locus $Z \subset \mathcal{A}_g$, the degree of the Gauss map thus has to be constant on a dense open subset, and we denote this degree by $\deg \mathcal{G}(Z)$. We close this section with a numerical analysis of the degree $\deg \mathcal{G}\left(\mathcal{A}^{\delta}_{t,g-t}\right)$ for $t \in \{d, d+1, \dots, \lfloor g/2 \rfloor\}$. We have the following:

Proposition 4.9. *For $\delta \in \{(2),(3),(2,2)\}$ and $g \geq 2d \coloneqq 2\deg\delta$, the degree of the Gauss map on the loci $\mathcal{A}^{\delta}_{t,g-t}$,*

$$\{d, d+1, \dots, \lfloor g/2 \rfloor\} \to \mathbb{N}$$
$$t \mapsto \deg \mathcal{G}\left(\mathcal{A}^{\delta}_{t,g-t}\right)$$

is a strictly decreasing function of t. In particular, the degree of the Gauss map separates these loci.

Remark. The proposition states that the degree separates the loci $\mathcal{A}^{\delta}_{t,g-t}$ for fixed δ, fixed g, and varying t. One could ask if this still hold when δ varies, i.e. that the degree of the Gauss map is different on all loci $\mathcal{A}^{\delta}_{t,g-t}$ for a fixed dimension g. We verified this with a computer up to $g = 1000$. This is not true anymore when g varies, as one can check that the lowest pair of genera where we have an equality of degrees is $g_1 = 28$ and $g_2 = 30$, with

$$\deg \mathcal{G}\left(\mathcal{A}^{3}_{5,28-5}\right) = \deg \mathcal{G}\left(\mathcal{A}^{2}_{7,30-7}\right) = 390882493091940846796800000 0.$$

Proof. We will prove this by looking at the explicit description of the degree. Recall that by 4.7 the degree is given by

$$F_g(t) = t!(g-t)!\left(\binom{t-1}{d-1}\binom{g-t-1}{d-2} + \sum_{k=2}^{d}\binom{t-k}{t-d}\binom{g-t-1+k}{d-1}\right).$$

We will now prove the proposition by doing each possible value of δ separately.

Case $\delta = (2)$. In this case the formula becomes

$$F_g(t) = t!(g-t)!g\,,$$

and this is obviously a decreasing function of t in the range $2 \leq t \leq \lfloor g/2 \rfloor$.

Case $\delta = (3)$. In this case,

$$F_g(t) = t!(g-t)!(-t^2 + gt + 3 - g)\,.$$

Let $f(x) = -x^2 + gx + 3 - g$. We have

$$\Delta F_g(t) \coloneqq F_g(t+1) - F_g(t) = t!(g-t-1)!(g-2t-1)h_g(t)\,,$$

with $h_g(t) = t^2 - (g-1)t + g - 2$. Evaluating we have

$$h_g(3) = 10 - 2g < 0 \quad \text{for } g \geq 6$$
$$h_g\left(\frac{g-1}{2}\right) = (-g^2 + 6g - 9)/4 < 0 \quad \text{for } g \geq 6\,.$$

h_g is convex, thus strictly negative on $[3, (g-1)/2]$, and so F_g is strictly decreasing.

Case $\delta = (2,2)$. Now

$$F_g(t) = t!(g-t)!\frac{g}{12}(2t + 1 - g)h_g(t)\,,$$

where

$$h_g(x) = x^4 + x^3(-2g+2) + x^2(g^2+g-7) + x(-3g^2+11g-8) + 2g^2 - 10g + 12\,.$$

We compute
$$\frac{\partial h_g}{\partial x} = (2x+1-g)(2x^2+2x(1-g)+3g-8)\,,$$
which is positive for $4 \le x \le (g-1)/2$. Evaluating at $x=4$ we have
$$h_g(4) = 6(g^2-13g+42) > 0 \quad \text{for } g \ge 8\,.$$
Thus
$$\Delta F_g(t) < 0 \quad \text{for } 4 \le t \le \lfloor g/2 \rfloor - 1 \quad \text{and } g \ge 8\,,$$
and thus the degree of the Gauss map is strictly decreasing on this range. □

Finally, we study how this degree compares with the degree of the Gauss map on Jacobians.

Proposition 4.10. *The degree of the Gauss map on Jacobians is always different than on a general member of the loci $\mathcal{A}^{\delta}_{t,g-t}$. Namely, for $g \ge 7$, $\delta \in \{(2),(3),(2,2)\}$ and $d \le t \le g/2$, we have*
$$\deg \mathcal{G}\left(\mathcal{A}^{\delta}_{t,g-t}\right) > \det \mathcal{G}\left(\mathcal{J}_g\right)\,.$$
For $g=5$ or $g=6$ the above inequality fails, but the degrees are still different.

Proof. By Proposition 4.9, the lowest term on the left hand side of the inequality is achieved when $\delta = (2)$ and $t = \lfloor g/2 \rfloor$. Thus we have to study
$$\deg \mathcal{G}\left(\mathcal{A}^{2}_{\lfloor g/2 \rfloor, g-\lfloor g/2 \rfloor}\right) - \deg \mathcal{G}(\mathcal{J}_g) \ge g(g/2)!^2 - \binom{2g-2}{g-1}\,.$$
Using Stirlings lower bound for the factorial we have, for $g \ge 22 > 8e$
$$g(g/2)!^2 > (g/2)!^2 > \binom{2g-2}{g-1}\,.$$
The remaining values can be checked by hand. For instance for $g=7$ we obtain
$$\deg \mathcal{G}\left(\mathcal{A}^{2}_{3,4}\right) = 1008 > \det \mathcal{G}\left(\mathcal{J}_7\right) = 924\,.$$
For $g=6$
$$\deg \mathcal{G}\left(\mathcal{A}^{2}_{3,3}\right) = 216\,, \quad \deg \mathcal{G}\left(\mathcal{A}^{2}_{2,4}\right) = 288\,, \quad \deg \mathcal{G}\left(\mathcal{J}_6\right) = 252\,.$$
For $g=5$
$$\deg \mathcal{G}\left(\mathcal{A}^{2}_{2,3}\right) = 60\,, \quad \deg \mathcal{G}\left(\mathcal{J}_5\right) = 70\,.$$
□

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