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GENERALIZED BRIESKORN MODULES II: HIGHER BERNSTEIN POLYNOMIALS AND MULTIPLE POLES

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Si vous ne réussisez pas à tout intégrer
essayer donc d'intégrer par parties

Abstract. Our main result is to show that, if the p-th Bernstein polynomial of the (a, b)-module generated by a germ of a holomorphic volume form $\omega \in \Omega_0^{n+1}$ in the (convergent) Brieskorn (a, b)-module associated to f, has a root $-\alpha - \mathbb{N}$, there exists a pole of order at least p for the meromorphic extension of an analytic functional associated to ω at some point in $-\alpha - \mathbb{N}$, under the hypothesis that f has an isolated singularity at the origin relative to the corresponding eigenvalue $\exp(2i\pi\alpha)$ of the monodromy. This implies the existence of at least p roots in $-\alpha - \mathbb{N}$ (counting multiplicities) for the usual reduced Bernstein polynomial of the germ of f at 0.

We also obtain in the case of an isolated singularity for f that the largest root $-\alpha-m$ inside $\{-\alpha - \mathbb{N}\}$ of the reduced Bernstein polynomial of f produces a pole at the point $\lambda = -\alpha - m$ for the meromorphic extension of the distribution $|f|^{2\lambda}\bar{f}^{-h}$ for some $h \in \mathbb{N}$.

1. Introduction

1.1. **The aim of this article.** The roots of the reduced Bernstein polynomial $b_{f,0}$ of the germ of holomorphic function at the origin in $\mathbb{C}^{n+1}$ control the poles of the meromorphic extension of the distribution

$$\square \longrightarrow \frac{1}{\Gamma(\lambda)} \int_{\mathbb{C}^{n+1}} |f|^{2\lambda} \bar{f}^{-h} \square$$

defined in a neighborhood of $0 \in \mathbb{C}^{n+1}$ (see for instance [5] or [12]).

Contrary to the point of view in [4] and [3] which is to prove the existence of poles for the meromorphic extension of the distribution $|f|^{2\lambda}$ for a general germ of holomorphic function f at the origin of $\mathbb{C}^{n+1}$ under topological assumptions, we mainly consider here, in the case of an isolated singularity for the eigenvalue $\exp(2i\pi\alpha)$, the polar parts on $\{-\alpha-\mathbb{N}\}$ of the meromorphic extensions of the (conjugate) analytic functionals like

$$\omega' \in \Omega_0^{n+1} \mapsto \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-h} \rho\omega \wedge \bar{\omega}'$$

for a given germ $\omega \in \Omega_0^{n+1}$. We denote by h an integer and by ρ a function in $\mathscr{C}_c^\infty(X)$ which is identically 1 near 0.

Note that our hypothesis of an isolated singularity for the eigenvalue $\exp(2i\pi\alpha)$ of the monodromy is in fact interesting for the study of a general germ of a holomorphic function g at the origin of $\mathbb{C}^{n+1}$:

Let Σ be the biggest-dimensional stratum in $\{g = 0\}$ on which the vanishing cycle complex of g admits the eigenvalue $\exp(2i\pi\alpha)$ and let p be the codimension of Σ in $\mathbb{C}^{n+1}$. Let x_0 be a

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generic point in Σ and let Π be a transversal generic p-plane to Σ passing through x_0. Then the germ at x_0 of the restriction of g to Π has an isolated singularity for the eigenvalue $\exp(2i\pi\alpha)$ of its monodromy. So the results of the present article may be applied to such a situation and give tools for such a study.

The first goal of this article is to show that, assuming that 0 is an isolated singularity for the eigenvalue $\exp(2i\pi\alpha)$ of the monodromy (this corresponds to our hypothesis $H(\alpha, 1)$), the roots of the Bernstein polynomial of the (a, b)-module generated by the germ ω of holomorphic $(n+1)$-form at the origin in the Brieskorn (a, b)-module H_0^{n+1} of f at 0, control the poles of the (conjugate) analytic functional defined on Ω_0^{n+1} by polar parts of poles in $-\alpha - \mathbb{N}$ of the meromorphic functions

$$\omega' \mapsto \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-h} \rho\omega \wedge \bar{\omega}'$$

where $\rho \in \mathscr{C}_c^\infty(\mathbb{C}^{n+1})$ is identically 1 near 0 and has a sufficiently small support (note that the polar parts of these meromorphic extensions at points in $-\alpha - \mathbb{N}$ are independent of the choices of ρ thanks to our hypothesis $H(\alpha, 1)$).

Our goal is to give a sufficient condition, still on the (a, b)-module generated by the germ ω, to obtain higher order poles for such integrals.

The difficulty comes from the fact that it is not clear when, for instance, two roots, $-\alpha - m$ and $-\alpha - m'$ with $m, m' \in \mathbb{N}$, of the Bernstein polynomial give a simple pole or a double pole for such a meromorphic extension at points in $-\alpha - \mathbb{N}$, for some choices of ω' and $h \in \mathbb{Z}$.

So we attempt to understand when such a pair of roots are “linked”, thus producing a double pole for some choice of ω' and h, or is “independent”, thus producing at most a simple pole for any choices of ω' and h.

Since it is known that the nilpotent part of the monodromy is related to this phenomenon (see [4] and [3]) we have defined, in our previous paper [6], the action of the monodromy on a simple pole geometric[1] (a, b)-modules and we have shown in the cited paper that the natural semi-simple filtration of a geometric (a, b)-module $\mathcal{E}$ is related to the filtration induced by the nilpotent part of the monodromy action on its saturation $\mathcal{E}^\sharp$ by $b^{-1}a$.

Section 2 of this article is devoted to some reminder of part I (see [6]) and to the definition and study of the **higher Bernstein polynomials** of a geometric (a, b)-module. The case of frescoes which is used in our main result, is examined in detail in Section 3.

1.2. **The main results.** We now describe our main results, which are obtained in Section 4, using the tools introduced in our previous paper [6] and the first sections.
The proofs will be given in Section 4 and the reader will find the notation and definitions used in the following presentation explained further in this article.

We consider a germ f of a holomorphic function at the origin of $\mathbb{C}^{n+1}$ with **an isolated singularity at** 0 **for the eigenvalue** $\exp(2i\pi\alpha)$ **of its monodromy**, that is to say, with the hypothesis $H(\alpha, 1)$ where α is in $]0, 1] \cap \mathbb{Q}$.

We denote $f : X \to D$ a Milnor's representative of the germ f near 0.

In this situation which will be called **the standard situation**, we consider the **generalized Brieskorn module** H_0^{n+1} which is the quotient of the $(n+1)$-cohomology group of the complex $(Ker^\bullet df_0, d)$ by its b-torsion. This complex is the sub-complex of the germ at 0 of the holomorphic de Rham complex at 0 where $Ker^p df_0$ is the kernel of $\wedge df : \Omega_0^p \to \Omega_0^{p+1}$ for $p \geq 2$ and $Ker^1 df$ is the quotient of the kernel of the map $\wedge df : \Omega_0^1 \to \Omega_0^2$ by $\mathbb{C}df$. Its cohomology groups, after quotienting by their torsion, are geometric (a, b)-modules (see [8] or [9]).

[1]The (a, b)-modules deduced from the Gauss-Manin connection which appear here are always geometric. See below.

For $\omega \in \Omega_0^{n+1}$ we denote by $\mathcal{F}_\omega$ the fresco generated by $[\omega]$ in H_0^{n+1}; it is defined as $\mathcal{F}_\omega = B[a]\omega \subset H_0^{n+1}$ where $B := \mathbb{C}\{\{b\}\}$ and $ab - ba = b^2$.

We refer the reader to Section 2 for the definitions of the Bernstein polynomial $B_{\mathcal{E}}$ and of the definition of **higher Bernstein polynomials** $B^p_{\mathcal{E}}$ of a geometric (a, b)-module $\mathcal{E}$. Remind that $B_{\mathcal{E}}$ always divides the product of the $B^j_{\mathcal{E}}$. Also any root of each $B^j_{\mathcal{E}}$ is a root of $B_{\mathcal{E}}$ and that $B_{\mathcal{E}}$ is the product of the $B^j_{\mathcal{E}}$ for all j when $\mathcal{E}$ is a **fresco**. The nilpotent order $d(\mathcal{E})$ of a geometric (a, b)-module $\mathcal{E}$ is the smallest integer $d \geq 0$ such that $B^{d+1}_{\mathcal{E}} \equiv 1$.

We remind the reader that for a geometric (a, b)-module $\mathcal{E}$ there exists a maximal quotient $\mathcal{E}^{[\alpha]}$ of $\mathcal{E}$ which is $[\alpha]$-primitive. This means that $B_{\mathcal{E}^{[\alpha]}}$ is the quotient of $B_{\mathcal{E}}$ by all its roots not in $-\alpha - \mathbb{N}$. Then, for each p, $B^p_{\mathcal{E}^{[\alpha]}}$ is also the quotient of $B^p_{\mathcal{E}}$ by all its roots not in $-\alpha - \mathbb{N}$.

Theorem 1.2.1. *Under the hypothesis $H(\alpha, 1)$ for f, let $\omega \in \Omega_0^{n+1}$ such that the nilpotent order of the fresco $\mathcal{F}^{[\alpha]}_\omega$ is equal to $p \in \mathbb{N}^*$, where $\mathcal{F}_\omega := B[a]\omega \subset H_0^{n+1}$. Then there exists $\omega' \in \Omega_0^{n+1}$ and $h \in \mathbb{N}$ such that the meromorphic extension of the function*

$$F_h^{\omega,\omega'}(\lambda) := \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-h} \rho\omega \wedge \bar{\omega}' \tag{1}$$

holomorphic for $\Re(\lambda) \gg 1$, has a pole of order at least p at a point in $-\alpha - \mathbb{N}$, where $\rho \in \mathscr{C}^\infty_c(X)$ is identically 1 near 0.

Conversely, if the meromorphic extension of $F_h^{\omega,\omega'}(\lambda)$, for some choices of ω' and h, has a pole of order p at a point in $-\alpha - \mathbb{N}$, the nilpotent order of the fresco $\mathcal{F}^{[\alpha]}_\omega$ (and the nilpotent order of the $[\alpha]$-primitive quotient of H_0^{n+1}) is at least equal to p.

The following lemma (see [6]) enlightens how the previous result gives a relation between the nilpotent order of $(H_0^{n+1})^{[\alpha]}$ and the order of poles in $-\alpha - \mathbb{N}$ of the meromorphic extensions of $\frac{1}{\Gamma(\lambda}|f|^{2\lambda}\bar{f}^{-h}$ under our hypothesis $H(\alpha, 1)$.

Lemma 1.2.2. *For any geometric (a, b)-module $\mathcal{E}$ such that $-\beta$ is a root of $B^p_{\mathcal{E}}$ there exists $\omega \in \mathcal{E}$ and an integer $q \geq p$ such that $B^q_{\mathcal{F}_\omega}$ has a root $-\beta + m$ with $m \in \mathbb{N}$, where $\mathcal{F}_\omega := B[a]\omega \subset \mathcal{E}$ is the fresco generated by ω in $\mathcal{E}$.*

Moreover, if $p = d(\mathcal{E}^{[\alpha]})$ where $[\alpha]$ is the class of β in $\mathbb{Q}/\mathbb{Z}$ we may choose ω such that $m = 0$.

Corollary 1.2.3. *In the previous situation, if p is the nilpotent order of $(H_0^{n+1})^{[\alpha]}$ there exists $\omega \in \Omega_0^{n+1}$ which satisfies the hypothesis of Theorem 1.2.1.*
Moreover, for each root $-\beta$ of $B^p_{(H_0^{n+1})^{[\alpha]}}$ we may find a form $\omega \in \Omega_0^{n+1}$ such that $-\beta$ is a root of $B^p_{\mathcal{F}_\omega}$.

The next result makes precise that the biggest root in $-\alpha - \mathbb{N}$ of $B^p_{\mathcal{F}_\omega}$ is a pole for the function $F_h^{\omega,\omega'}(\lambda)$ for a convenient choice of ω' and h.

Theorem 1.2.4. *Under the hypothesis $H(\alpha, 1)$ for f, let $\omega \in \Omega_0^{n+1}$ be such that the nilpotent order of $\mathcal{F}^{[\alpha]}_\omega$, where $\mathcal{F}_\omega := B[a]\omega \subset H_0^{n+1}$, is equal to $p \in \mathbb{N}^*$. Let $-\alpha - m$ be the biggest root of $B^p_{\mathcal{F}_\omega}$ inside $-\alpha - \mathbb{N}$. Then there exists ω' and h such that $F_h^{\omega,\omega'}(\lambda)$ has a pole of order p at $\lambda = -\alpha - m$.*

The last result (see Corollary 4.5.3) greatly strengthens the main theorem in [1].

Theorem 1.2.5. *Under the hypothesis $H(\alpha, 1)$ for f, let $\omega \in \Omega_0^{n+1}$ be such that the nilpotent order of $\mathcal{F}^{[\alpha]}_\omega$ is equal to p. For each $s \in [1, p]$ let ξ_s be the biggest number in $-\alpha - \mathbb{N}$ for which there exists ω' and h such that $F_h^{\omega,\omega'}(\lambda)$ has a pole of order $\geq s$ at ξ_s. Then there exists $j \in \mathbb{N}$ such that ξ_s is a root of $B^{s+j}_{\mathcal{F}_\omega}$.*

Moreover, if $\xi_s = \xi_{s+1} = \cdots = \xi_{s+h}$, then there exist at least h distinct integers $q_1, \dots, q_h$ such that ξ_s is a root of $B^{s+q_j}_{\mathcal{F}_\omega}$ for $j \in [1, h]$.

The proofs and some more precise results are given in Sections 4.4 and 4.5.

I thank the referee whose questions and comments allowed me to improve and clarify many points.

2. Asymptotics and geometric (a, b)-modules

2.1. **Some facts from** [6]**.** Let $B := \mathbb{C}\{\{b\}\}$ be the algebra of constant coefficients micro-differential operators of degree ≤ 0. So $b := \partial_s^{-1}$ and a series $S(b) := \sum_{p=0}^{\infty} c_p b^p$ is in B when there exists $R > 1$ and an a constant C_R such that $\forall p \geq 0 \quad |c_p| \leq C_R R^p p!$.

Then B acts on $\mathbb{C}\{s\}$, the algebra of germs of holomorphic functions at the origin in $\mathbb{C}$, by the rule

$$S(b)[f] = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\sum_{p=0}^{n}(n-p)!c_p\gamma_{n-p}\Big)s^n \quad \text{where } f(s) := \sum_{q=0}^{\infty}\gamma_q s^q.$$

Note that the derivation in the variable b acts continuously[2] on B and that the multiplication by s on $\mathbb{C}\{s\}$, denoted by a, satisfies the commutation relation

$$aS(b) = S(b)a + b^2 S'(b) \qquad \forall S \in B$$

in the algebra $End_c(\mathbb{C}\{s\})$ of continuous endomorphisms of $\mathbb{C}\{s\}$.

We denote by $B[a]$ the sub-algebra of $End_c(\mathbb{C}\{s\})$ generated by B and $a := \times s$.

We denote by $A := \mathbb{C}\{a\}$ the sub-algebra of $End_c(\mathbb{C}\{s\})$ of elements given by multiplication by some $f \in \mathbb{C}\{s\}$.

Define

$$e_{\alpha,j} := s^{\alpha-1}\frac{(Log\, s)^j}{j!}$$

for $\alpha \in]0, 1] \cap \mathbb{Q}$ and for $j \in \mathbb{N}$ and let $\Xi_\alpha^{(N)}$ be the free $\mathbb{C}\{s\}$-module with basis $e_{\alpha,j}$ for $j \in [0, N]$. Then define the action of $\mathbb{C}[b]$ on $\Xi_\alpha^{(N)}$ inductively by the formulas

$$be_{\alpha,j} = \frac{1}{\alpha}\big(se_{\alpha,j} - be_{\alpha,j-1}\big) \qquad \text{for} \quad j \geq 1 \qquad \text{and} \quad be_{\alpha,0} = \frac{1}{\alpha}se_{\alpha,0}$$

with the commutation relations

$$S(b)a = aS(b) - b^2 S'(b) \qquad \forall S \in \mathbb{C}[b]$$

where S' is the derivative (in the variable b) in the algebra $\mathbb{C}[b]$.

Lemma 2.1.1. *For each $\alpha \in]0, 1] \cap \mathbb{Q}$ and each $N \in \mathbb{N}$ the action of $\mathbb{C}[b]$ on $\Xi_\alpha^{(N)}$ extends to an action of B and makes it isomorphic to the free B-module with basis $e_{\alpha,j}$ for $j \in [0, N]$, on which the action of a is continuous*[3] *and satisfies $ab - ba = b^2$.*

So it is a left $B[a]$-module for which the action of $\mathbb{C}[a]$ extends continuously to an action of $\mathbb{C}\{a\}$.

For a proof, see Section 2.3 in [6]. □

Definition 2.1.2. *For $\alpha \in]0, 1[\cap\mathbb{Q}$ we define*

$$S\Xi_\alpha^{(N)} := \Xi_\alpha^{(N)}$$

and for $\alpha = 1$

$$S\Xi_1^{(N)} := \Xi_1^{(N+1)}\big/\Xi_1^{(0)}.$$

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[2]For its natural dual Fréchet topology associated to the "pseudo-norms" :
$|| \quad ||_R := Sup\{|c_p|R^{-p}/p!,\ p \geq 0\}$.

[3]for the topology deduced from this isomorphism of free finite type B-modules.

Then for $\mathscr{A}$ a finite subset in $]0,1]\cap\mathbb{Q}$, $N \in \mathbb{N}$ and V a finite-dimensional complex vector space, define $S\Xi^{(N)}_{\mathscr{A}} \otimes V$ as before, replacing $\Xi^{(N)}_\alpha$ by $S\Xi^{(N)}_\alpha$ for each $\alpha \in \mathscr{A}$.

Note that $\Xi^{(0)}_1 = \mathbb{C}\{s\}$; so it is the non-singular part (i.e. the uni-valued holomorphic part) of the expansion at the origin.

From now on a **geometric** (a,b)**-module**, we also use the terminology **Generalized Brieskorn Modules**, will be, by definition, a sub $B[a]$-module of some $S\Xi^{(N)}_R \otimes V$ where $\mathscr{A}$ is a finite subset in $]0,1]\cap\mathbb{Q}$, N a non-negative integer and V a finite-dimensional complex vector space.

Remark 2.1.3.

1. *Of course, thanks to the results in [6] Section 5, this is equivalent to the "standard definition" of a convergent (a,b)-module, given in [6] Section 2, plus the condition of regularity (see Section 3 of [6]) and the condition that the Bernstein polynomial has its roots in $-\mathbb{Q}^*$ (see Theorem 5.1.4 of [6]).*
2. *Since B is a local noetherian algebra and $S\Xi^{(N)}_{\mathscr{A}} \otimes V$ is a free and finite type B-module, any geometric (a,b)-module is a free finite type module over B with a continuous $\mathbb{C}$-linear action of a.*
3. *Also any $B[a]$-sub-module of a geometric (a,b)-module is again a geometric (a,b)-module.*
4. *Fix $\alpha \in]0,1]$. For each integer n we have in $\Xi^{(0)}_\alpha$:*

$$b^n s^{\alpha-1} = \frac{s^{\alpha+n-1}}{\prod_{p=1}^n(\alpha+p-1)}$$

and so

$$\Big(\sum_{n=0}^{\infty} c_n b^n\Big)s^{\alpha-1} = \Big(\sum_{n=0}^{\infty}\frac{c_n}{\prod_{p=1}^n(\alpha+n-1)}a^n\Big)s^{\alpha-1}$$

and the estimates

$$\frac{|c_n|}{\prod_{p=2}^n(\alpha+p-1)}R^n(n-1)! \leq |c_n|R^n \leq \frac{|c_n|}{\prod_{p=1}^n(\alpha+p-1)}R^n n!$$

show that the topology of $\Xi^{(0)}_\alpha$ as a free rank 1 A-module or as free rank 1 B-module are the same.

This extends easily to any $S\Xi^{(N)}_{\mathscr{A}} \otimes V$ and shows that, in any sub-$B[a]$-module of some $S\Xi^{(N)}_{\mathscr{A}} \otimes V$, the action of a extends continuously to an action of $A = \mathbb{C}\{a\}$. So any geometric convergent (a,b)-module is a free finite type A-module.

We shall use the following definitions which are equivalent to the definitions given and used in [6], where E is a geometric (a,b)-module. The reader may find proofs there.

1. A sub-module $F \subset E$ is **normal** when $F \cap bE = bF$. In this case the quotient E/F is again without B-torsion so free and finite type as a B-module and it is again a geometric (a,b)-module[4].
2. For an arbitrary sub-module F in E we denote by

$$N_E(F) = \{x \in E \ / \ \exists n \in \mathbb{N} \ \textit{such that} \ b^n x \in F\}.$$

We call this sub-module the **normalisation** of F in E. It is the smallest normal sub-module in E which contains F.

[4]This point is not obvious with the definition adopted here but is rather easy with the standard definition; see [6].

(3) We denote by $E^\sharp$ the **saturation** of E in $b^{-1}a$ that is to say the sub-module

$$E^\sharp := \sum_{m\in\mathbb{N}} (b^{-1}a)^m E \subset E[b^{-1}] := E \otimes_B B[b^{-1}].$$

Note that any $S\Xi_R^{(N)} \otimes V$ is stable by $b^{-1}a$. So $E^\sharp$ is again a finite type free B-module and there exists an integer $m \geq 0$ such that $b^m E^\sharp \subset E$. So $E^\sharp\big/E$ is a finite-dimensional $\mathbb{C}$-vector space.

We shall say that E has a **simple pole** when $E = E^\sharp$. This is equivalent to the fact that $aE \subset bE$ (since a is injective).

The reader may consult the first section in [10] to see how to construct a natural bijective correspondance between free, finite rank $\mathbb{C}[[z]]$-modules with a simple pole connection and formal simple pole (a,b)-modules. Note that in both cases we dispose of an action of $z\partial_z = b^{-1}a - 1$.

(4) The rank 1 geometric (a,b)-modules are classified by positive rational numbers, so elements in $\mathbb{Q}^{*+}$. To $\alpha \in \mathbb{Q}^{*+}$ corresponds $E_\alpha := B[a]\big/B[a](a - \alpha b)$. Then E_α is the free rank 1 B-module Be_α where the action of a is defined by $aSe_\alpha = \alpha bSe_\alpha + b^2 S' e_\alpha$ for $S \in B$, where S' is the derivative (in b) of S.

(5) For a given $\alpha \in]0,1] \cap \mathbb{Q}$ and any geometric (a,b)-module E we denote by $E^{[\alpha]}$ the quotient of E by elements in E that present no non-zero term like $s^{\alpha+m-1}(Log\, s)^j \otimes v$ for any integers j and m and any $v \in V$. Since we make the quotient of E by $E \cap (S\Xi^{(N)}_{\mathscr{A}\setminus\alpha} \otimes V)$ and since we have

$$S\Xi^{(N)}_{\mathscr{A}} \otimes V \big/ S\Xi^{(N)}_{\mathscr{A}\setminus\alpha} \otimes V \simeq S\Xi^{(N)}_\alpha \otimes V$$

the quotient $E^{[\alpha]}$ is again a geometric (a,b)-module. A geometric (a,b)-module E such that $E = E^{[\alpha]}$ will be called **$[\alpha]$-primitive** and, for a general E, $E^{[\alpha]}$ will be called **the $[\alpha]$-primitive quotient** of E (see [6] Proposition 3.3.3).

When E has a simple pole, E is the direct sum of the $E^{[\alpha]}$ when α describe the image in $\mathbb{Q}/\mathbb{Z} \simeq]0,1] \cap \mathbb{Q}$ of the roots of B_E (see below).

(6) The (usual) **Bernstein polynomial** B_E of the geometric (a,b)-module E is, by definition the minimal polynomial of the endomorphism $-b^{-1}a$ acting on the finite-dimensional complex vector space $E^\sharp\big/bE^\sharp$ whose dimension is equal to the B-rank of E.

Of course, when f is a germ of holomorphic function at the origin of $\mathbb{C}^{n+1}$, the reduced Bernstein polynomial of f and the Bernstein polynomial of the Brieskorn module of f (which is a geomeric (a, b)-module) coincide (see [15]).

Remark 2.1.4. *If we have an exact sequence of* **simple poles** (a,b)*-modules*

$$0 \to E_1 \to E \to E_2 \to 0$$

then B_{E_1} and B_{E_2} divide B_E since the maps in the exact sequence of finite-dimensional vector spaces

$$0 \to E_1/bE_1 \to E/bE \to E_2/bE_2 \to 0$$

commute with the respective actions $-b^{-1}a$ on these vector spaces.
Remark that if we have such an exact of (a,b)-modules with a simple pole for E, then E_1 (which must be normal in E !) and E_2 have simple poles too.

Warning. Without the assumption that E has a simple pole it is still true that B_{E_2} divides B_E since the induced map $E^\sharp \to E_2^\sharp$ is still surjective.

But the kernel of this induced map is not equal to $E_1^\sharp$ in general (it is equal to the normalisation of $E_1^\sharp$ in $E^\sharp$), so it is not true in general that B_{E_1} divides B_E. See, for instance, the case where E is a fresco described below.

(7) A **fresco**[5] is a geometric (a, b)-module F such that the (finite-dimensional) complex vector space $F/(aF + bF)$ has dimension 1 (note that this vector space is a quotient of F/bF). This is equivalent to ask that F is a geometric (a, b)-module which is isomorphic to the quotient of $B[a]$ by a left ideal.
For a fresco, the Bernstein polynomial is equal to the **characteristic polynomial** of $-b^{-1}a$ acting on $F^\sharp/bF^\sharp$. So its degree is the B-rank of F.

(8) A normal sub-module G of a fresco F is a fresco and the quotient F/G is again a fresco.

(9) For each element x in a geometric (a, b)-module $E \subset S\Xi_{\mathscr{A}}^{(N)} \otimes V$ we define $d(x)$ as the maximal integer $d \in \mathbb{N}$ such that a non-zero term like $s^{\alpha+m-1}(Log\, s)^d \otimes v$ appears in x for some $\alpha \neq 1$ or like $s^m(Log\, s)^{d+1} \otimes v$ for $\alpha = 1$, for some integer $m \in \mathbb{N}$ and some $v \in V$.
The integer $d(x)$ is called the **nilpotent order** of x and does not depend on the realization of E as a sub-module of some $S\Xi_R^{(N)} \otimes V$.
Then we define $d(E) := \sup_{x \in E}\{d(x)\}$ and call this integer **the nilpotent order** of E.

(10) The following properties for a geometric (a, b)-module E are equivalent:
   (i) $d(E) \leq 1$.
   (ii) E is a sub-module of a finite direct sum $\oplus_{j=1}^p E_{\alpha_j}$.
   (iii) $E^\sharp$ is isomorphic to a finite direct sum $\oplus_{j=1}^p E_{\alpha_j}$.

A geometric (a, b)-module is **semi-simple** when it satisfies one of the properties above.
Note that the Bernstein polynomial of a semi-simple geometric (a, b)-module has only simple roots (obvious from property (iii) above).

(11) We define the **semi-simple filtration**[6] $\big(S_j(E), j \in \mathbb{N}\big)$ by
$$S_j(E) := \{x \in E \ / \ d(x) \leq j\}.$$
The sub-module $S_1(E)$ is called "the **semi-simple part** of E".
Then $S_0(E) = \{0\}, S_{d(E)}(E) = E$ and for $j \in [0, d(E) - 1]$ we have a strict inclusion $S_j(E) \subsetneq S_{j+1}(E)$. Moreover, for each integer j, $S_j(E)$ is a normal sub-module of E.
A geometric (a, b)-module is semi-simple when $d(E) \leq 1$. For instance, each quotient $S_j(E)/S_{j-1}(E)$ is semi-simple because another way to define the semi-simple filtration is to see that $S_j(E)/S_{j-1}(E)$ is the semi-simple part of $E/S_{j-1}(E)$, that is to say the subset of x in this quotient with $d(x) \leq 1$.

(12) For any sub-module G of a geometric (a, b)-module E and any integer j we have $S_j(G) = S_j(E) \cap G$.
When G is normal, the induced map $S_j(E) \to S_j(E/G)$ is not surjective in general when $j < d(E)$. For instance, it is proved below that for $G = S_1(E)$ the image by the map induced by the quotient map $S_j(E) \to S_j(E/G)$ is equal to $S_{j-1}(E/G)$ (see Lemma 2.1.5 below.)

(13) It is proved in [6] Proposition 4.2.8 that the rank (as B-module) of $S_{j+1}(E)/S_j(E)$ is non-increasing in j.

The following easy lemma will be used later on.

[5]See [7] for a detailed study of frescos.
[6]See [6] Section 4 for more details.

Lemma 2.1.5. *Let E be a geometric (a,b)-module and put $G := E/S_1(E)$. Then for each positive integer there exists a natural isomorphism of $B[a]$-modules*

$$S_{h+1}(E)/S_1(E) \to S_h(G)$$

for each $h \geq 0$. This implies the isomorphisms of $B[a]$-modules

$$S_{h+1}(E)/S_h(E) \to S_h(G)/S_{h-1}(G).$$

for each $h \geq 1$.

Proof. It is clear with the definition of a geometric (a,b)-module adopted here that if x belongs to $S_{h+1}(E)$ then its image in G is in $S_h(G)$. This gives an injective morphism

$$S_{h+1}(E)/S_1(E) \to S_h(G)$$

which is clearly surjective. □

2.2. **Higher Bernstein Polynomials.**

Definition 2.2.1. *Let E be a geometric (a,b)-module. For any $j \in [1, d(E)]$ we denote by B_E^j the Bernstein polynomial of the semi-simple (a,b)-module $S_j(E^\sharp)/S_{j-1}(E^\sharp)$. We call it* **the j-th Bernstein polynomial of** *E.*

Remark 2.2.2.

(i) Like B_E the (usual) Bernstein polynomial of E, the polynomials B_E^j only depend on $E^\sharp$ the saturation of E by $b^{-1}a$.

(ii) Since for each $j \in [1, d(E)]$ the geometric (a,b)-module $S_j(E^\sharp)/S_{j-1}(E^\sharp)$ is semi-simple and has a simple pole, it is a direct sum of rank 1 geometric (a,b)-modules. So each B_E^j has only simple roots.

The first interesting point of this definition is given by the following proposition.

Proposition 2.2.3. *Let E be a geometric (a,b)-module. Then each root of B_E is a root of B_E^j for at least one $j \in [1, d(E)]$. Moreover for each integer j in $[1, d(E)]$ the polynomial B_E^j divides B_E.*

Proof. We have an exact sequence of simple poles (a,b)-modules

$$0 \to S_j(E^\sharp)/S_{j-1}(E^\sharp) \to E^\sharp/S_{j-1}(E^\sharp) \to E^\sharp/S_j(E^\sharp) \to 0$$

which gives the fact that a root of B_E^j is a root of the quotient $E^\sharp/S_{j-1}(E^\sharp)$ and then a root of $B_E = B_{E^\sharp}$.

We have also an exact sequence of simple pole (a,b)-modules

$$0 \to S_{j-1}(E^\sharp) \to S_j(E^\sharp) \to S_j(E^\sharp)/S_{j-1}(E^\sharp) \to 0$$

so it is clear that B_E^j divides $B_{S_j(E^\sharp)}$. This already proves that for $j = d(E) = d(E^\sharp)$ the polynomial $B_E^{d(E)}$ divides B_E. Applying this to $F := S_j(E^\sharp)$ gives, since $d(F) = j$ and

$$S_{j-1}(F) = S_{j-1}(E^\sharp) \cap F = S_{j-1}(E^\sharp)$$

that $B_F^j = B_E^j$ divides B_F. But F is a normal sub-module of $E^\sharp$ so, thanks to point 6 above (F has a simple pole) B_F divides B_E. □

Proposition 2.2.4. *Let E be a geometric (a,b)-module and put $d := d(E) \geq 2$. If $-\beta$ is a root of B_E^d there exists at least a root of B_E^{d-1} in $-\beta + \mathbb{N}$.*

Proof. It is of course enough to consider the case where E has a simple pole by definition of the standard and higher Bernstein polynomials.

We are only concerned by the $[\beta]$-primitive part of E so we may assume that E is $[\beta]$-primitive, and we may assume that $-\beta$ is the biggest root of B^d_E in $-\beta + \mathbb{Z}$.

Since $E/S_{d-1}(E)$ is a direct sum of E_{α_j}, we may always solve in $E/S_{d-1}(E)$ the equation $ax - (\beta - p)bx = by$ for any $p \in \mathbb{N}^*$ and any $y \in E/S_{d-1}(E)$ because $b^{-1}a - (\beta - p)$ is invertible in $E/S_{d-1}(E)$ for each integer $p \geq 1$[7].

But our hypothesis implies the existence of $x_0 \in E \setminus (bE + S_{d-1}(E))$ such that

$$ax_0 - \beta bx_0 = b^2y_0 + z_0$$

for some $y_0 \in E$ and $z_0 \in S_{d-1}(E)$. This is equivalent to

$$(b^{-1}a - \beta)(x_0) \in bE + S_{d-1}(E)$$

since the simple pole of E implies that we may write $z_0 = bz_1$ with $z_1 \in S_{d-1}(E)$, since z_0 is in $bE \cap S_{d-1}(E) = bS_{d-1}(E)$, as $S_{d-1}(E)$ is normal in E.

Now there exists $y_1 \in E$ such that $ay_1 - (\beta - 1)by_1 = by_0 + \xi_0$ with $\xi \in S_{d-1}(E)$ (invertibility of $b^{-1}a - (\beta - 1)$ in $E\big/S_{d-1}(E)$). And again we may write $\xi_0 = b\xi_1$ with $\xi_1 \in S_{d-1}(E)$. Applying b to this equality gives

$$aby_1 - \beta b^2y_1 = b^2y_0 + b^2\xi_1$$

and then

$$a(x_0 - by_1) - \beta b(x_0 - by_1) = bz_1 - b^2\xi_1 \in bS_{d-1}(E).$$

From now on, assume that $d = 2$.

If we assume that there is no root of the Bernstein polynomial of $S_{d-1}(E) = S_1(E)$ in $-\beta + \mathbb{N}$ we may solve in $S_1(E)$, which is semi-simple and has a simple pole (so it is a direct sum of E_{α_j}), the equation

$$az_2 - \beta bz_2 = b(z_1 - b\xi_1)$$

with $z_2 \in S_1(E)$. Then we find that $x = x_0 - by_1 - z_2$ satisfies $ax - \beta bx = 0$. Since x is not in bE, Bx is a normal rank 1 sub-module of E isomorphic to E_β, so it is contained in $S_1(E)$. Contradiction!

So there exists a root in $-\beta + \mathbb{N}$ for $S_{d-1}(E)$. Since we assume $d = 2$ the proof is complete for this case.

Now for $d(E) \geq 3$, define $F := E/S_{d-2}(E)$. We have the equalities :

$$S_1(F) = S_{d-1}(E)/S_{d-2}(E), \quad d(F) = 2 \quad \text{and} \quad B^d_E = B^2_F, \quad B^{d-1}_E = B^1_F$$

and applying to F the previous result gives that B^{d-1}_E has a root in $-\beta + \mathbb{N}$. □

Corollary 2.2.5. *Let E be a geometric (a, b)-module and let j be an integer in $[1, d(E)]$. Then if $-\beta$ is a root of B^j_E, for each $h \in [1, j]$ there exists a root of B^h_E in $-\beta + \mathbb{N}$.*

Proof. It is enough to apply the previous proposition successively to each simple pole (a, b)-module $S_j(E^\sharp)$ since the equality $S_{j-h}(S_j(E^\sharp)) = S_{j-h}(E^\sharp)$ for each $h \in [0, j-1]$ shows the equality $B^{j-h}_{S_j(E^\sharp)} = B^{j-h}_E$. □

Note that if the j roots find in $-\beta + \mathbb{N}$ are two by two distinct, we have found j roots of B_E, since each B^h_E divides B_E.

[7] To show that in $E_\alpha := Be_\alpha$ we can always solve the equation $(a - (\alpha - p)b)x = by$ for each given $y \in E_\alpha$ reduces to show that for each $S \in B$ we may find $T \in B$ such that $bT'(b) + pT(b) = S(b)$ for each $p \in \mathbb{N}^*$. This is an easy exercise.

Corollary 2.2.6. *Let E be a geometric (a, b)-module and assume that $-\beta$ is the greatest root of B_E in $-\beta + \mathbb{Z}$. Then $-\beta$ is a root of B^1_E.*

Moreover, if $-\beta$ is a root of B^j_E then $-\beta$ is a root of B^h_E for each $h \in [1, j]$.

Proof. This is an obvious consequence of Corollary 2.2.5. □

Proposition 2.2.7. *Let E be a geometric (a, b)-module with Bernstein polynomial B_E. Assume that B_E has the root $-\alpha$ with multiplicity $p \geq 1$. Then there exists at least p distinct values of the integer j such that $-\alpha$ is a root of B^j_E.*

Proof. It is enough to prove the result when E has a simple pole, by definition of the standard and higher Bernstein polynomials.

We shall make an induction on the rank $r \geq 1$ of E. Since the case $r = 1$ is trivial, assume that the result is proved (for any $p \geq 1$) for each integer $r \leq r_0$ and assume that the rank of E is $r_0 + 1$. Then $G = E/S_1(E)$ has rank at most r_0 and we may apply the induction hypothesis to G if $-\alpha$ is a root of order $p \geq 1$ of B_G.

Consider the exact sequence of geometric (a, b)-modules

$$0 \to S_1(E) \to E \to G \to 0.$$

First assume that $B_{S_1(E)}(-\alpha) \neq 0$. Then the Bernstein polynomial of G is divisible by $(x + \alpha)^p$ since we have an exact sequence of monodromic vector spaces

$$0 \to S_1(E)/bS_1(E) \to E/bE \to G/bG \to 0$$

corresponding to the previous exact sequence ($S_1(E)$ is normal in E).

In this case we conclude by the induction hypothesis applied to G which has rank at most r_0 (since $E \neq \{0\}$ the rank of $S_1(E)$ is al least 1). The conclusion follows from the remark following point 6 (on B_E) recalled at the beginning of section 2.1.

Assume now that $B_{S_1(E)}(-\alpha) = 0$. Then, since $S_1(E)$ is semi-simple, $-\alpha$ is a simple root of $B_{S_1(E)}$ and the exact sequence above implies that $-\alpha$ is a root of order at least $p-1$ of B_G. Then the induction hypothesis and Lemma 2.1.5 give the existence of $(p - 1)$ distinct values of $j \geq 2$ such that $B^j_E(-\alpha) = 0$. Since $B_{S_1(E)} = B^1_E$ we have found p values of j with $B^j_E(-\alpha) = 0$. □

Warning. It happens that if $-\alpha$ is a root of multiplicity $p \geq 1$ of B_E the p values of j for which $B^j_E(-\alpha) = 0$ do not contain $j = 1$ as is it shown by the following example.

Example. Define now the fresco $E := B[a]\varphi_p \subset \Xi^{(N)}_\alpha$ with

$$\varphi_p := s^{\alpha+m-1}\frac{(Log\, s)^p}{p!} + s^{\alpha-1}$$

where $m \geq 1, p \geq 2$ and $\alpha \in]0, 1[\cap\mathbb{Q}$. Then the semi-simple part of $E^\sharp$ is $E_\alpha = Bs^{\alpha-1}$ and the Bernstein polynomial of E is equal to $(x + \alpha + m)^p(x + \alpha)$ since we have

$$(b^{-1}a - (\alpha + m))[\varphi_p] = \varphi_{p-1} - (m + 1)s^{\alpha-1}.$$

So $-\alpha - m$ is a root of B^j_E for each $j \in [2, p + 1]$ but not of $B^1_E(x) = x + \alpha$. Note that $S_1(E) \simeq E_{\alpha+p}$. □

Corollary 2.2.8. *Let E be a geometric (a, b)-module with Bernstein polynomial B_E. Then B_E divides $\prod_{j=1}^{d(E)} B^j_E(x)$, the product of the higher Bernstein polynomials B^j_E of E.*

Proof. It is enough to consider the case of a simple pole E. We already know that each root of a polynomial B^j_E is a root of B_E thanks to Proposition 2.2.3 and the fact that a root of multiplicity p in B_E is a root of at least p polynomials B^j_E, thanks to Proposition 2.2.7. So B_E divides the product $\prod_{j=1}^{d(E)} B^j_E$. □

2.3. **Complements.** We give now some more properties of the higher Bernstein polynomials of a geometric (a,b)-module which are useful in the sequel.

Lemma 2.3.1. *Let $G \subset H$ be two geometric (a,b)-modules such that $H\big/G$ is a finite-dimensional complex vector space. If $-\beta$ is a root of B_G then there exists a root of B_E in $-\beta+\mathbb{N}$.*

Proof. We may assume that G and H has simple poles.

We prove the lemma by induction on the rank of H (equal to the rank of G).

In rank 1 we have $G = E_\beta$ and $H = E_{\beta-\nu}$ with $\nu \in \mathbb{N}$, so the result is clear.

Assume that the result is proved for $rk(H) = k \geq 1$ and consider $G \subset H$ a sub-module of finite co-dimension in H with the rank of H equal to $k+1$. If B_H has no root in $-\beta+\mathbb{N}$ then there exists a normal rank 1 sub-module E_γ in $S_1(H)$ and since the roots of B^1_H are roots of B_H, we have $\gamma \notin \beta-\mathbb{N}$. The sub-module $E_\gamma \cap G$ is isomorphic to $E_{\gamma+m}$ for some integer $m \geq 0$, and it is normal in G. So the Bernstein polynomial of $G\big/E_{\gamma+m}$ has still the root $-\beta$ and since $G\big/E_{\gamma+m}$ is a sub-module with finite co-dimension in $H\big/E_\gamma$ the induction hypothesis gives the existence of a root in $-\beta+\mathbb{N}$ for the Bernstein polynomial of $H\big/E_\gamma$. Since this polynomial divides B_H we obtain a contradiction. □

Corollary 2.3.2. *Let $G \subset H$ be two geometric (a,b)-modules such that $H\big/G$ is a finite-dimensional complex vector space. If $-\beta$ is a root of B^p_G then there exists $q \geq p$ such that B^q_H has a root in $-\beta+\mathbb{N}$.*

Proof. We may assume that G and H has simple poles.

Let $\tilde{G} := G\big/S_{p-1}(G)$ and $\tilde{H} := H\big/S_{p-1}(H)$. Then we have $B^1_{\tilde{G}} = B^p_G$ and $B^{p+h-1}_H = B^h_{\tilde{H}}$. Since $B^p_G = B^1_{\tilde{G}}$ divides $B_{\tilde{G}}$ we may apply Lemma 2.3.1 and we find that $B_{\tilde{H}}$ has a root in $-\beta+\mathbb{N}$ and so there exists $h \geq 1$ such that $B^h_{\tilde{H}} = B^{p+h-1}_H$ has such a root. □

Lemma 2.3.3. *Let $G \subset E$ be two geometric (a,b)-modules. Assume that B_E has no root in $-\beta+\mathbb{N}^*$ and that $-\beta$ is a root of B^p_G. Then $-\beta$ is a root of a polynomial B^q_E for some $q \geq p$.*

Proof. We may assume that G and E has a simple pole. Since G has finite co-dimension in $N_E(G)$, using Lemma 2.3.2 we may assume, replacing G by $N_E(G)$, that G is normal in E with the same hypothesis (but may be replacing $-\beta$ by $-\beta+m$ with m a non-negative integer). Since B^p_G divides $B_{G/S_{p-1}(G)}$ and since we have the exact sequence

$$0 \to G\big/S_{p-1}(G) \to E\big/S_{p-1}(G) \to E\big/G \to 0$$

because we assume that G is normal in E (so $S_{p-1}(G)$ is normal in E), B^p_G divides the Bernstein polynomial of $E\big/S_{p-1}(G)$.

But $B_{E\big/S_{p-1}(E)}$ divides the product of the $B^j_{E\big/S_{p-1}(E)}$ and since we have the equality

$$B^j_{E\big/S_{p-1}(E)} \simeq B^{j+p-1}_E,$$

we find that $-\beta$ must be a root of some B^q_E for at least one integer $q \geq p$. □

The following simple lemma is proved in [6] Lemma 6.3.6.

Lemma 2.3.4. *For any geometric (a,b)-module E and any $\alpha \in \mathbb{Q}\big/\mathbb{Z}$ we have*

$$(E^{[\alpha]})^\sharp = (E^\sharp)^{[\alpha]}.$$

Lemma 2.3.5. *Let E be a geometric (a,b)-module and assume that B_E has a root $-\beta$ of multiplicity p. Then $-\beta$ is also a root of multiplicity p in $B_{E^{[\alpha]}}$, where $[\alpha]$ is the class of β in $\mathbb{Q}\big/\mathbb{Z}$.*

Proof. We may assume that E has a simple pole, thanks to the previous lemma. Now the exact sequence

$$0 \to E_{[\neq\alpha]} \to E \to E^{[\alpha]} \to 0$$

splits and B_E is the product of the Bernstein polynomials of $E_{[\neq\alpha]}$ and of $E^{[\alpha]}$, thanks to the corresponding exact sequence of vector spaces

$$0 \to E_{[\neq\alpha]}\big/bE_{[\neq\alpha]} \to E\big/bE \to E^{[\alpha]}\big/bE^{[\alpha]} \to 0$$

compatible with the respective actions of $-b^{-1}a$ and the fact that there is no eigenvalue of the action of $-b^{-1}a$ on $E_{[\neq\alpha]}\big/bE_{[\neq\alpha]}$ in $-\alpha+\mathbb{Z}$.

Then the existence of the root $-\beta \in -\alpha+\mathbb{Z}$ of multiplicity p in E/bE implies the same property for $E^{[\alpha]}\big/bE^{[\alpha]}$. □

Lemma 2.3.6. *Let E be a simple pole geometric (a,b)-module. Then for each $[\alpha]$ in $\mathbb{Q}/\mathbb{Z}$ and each integer $j \in [1, d(E)]$ we have*

$$B^j_E = B^j_{E_{[\neq\alpha]}} B^j_{E^{[\alpha]}} \quad \text{and also} \quad B_E = B_{E_{[\neq\alpha]}} B_{E^{[\alpha]}}.$$

Proof. This is obvious since we know that $E = E_{[\neq\alpha]} \oplus E^{[\alpha]}$ as a $B[a]$-module. Then for each j we have $S_j(E) = S_j(E_{[\neq\alpha]}) \oplus S_j(E^{[\alpha]})$.

The conclusion follows because the eigenvalues of the action of $-b^{-1}a$ on the finite-dimensional vector spaces $G\big/bG$ where G is respectively $S_j(E_{[\neq\alpha]})\big/S_{j-1}(E_{[\neq\alpha]})$ and $S_j(E^{[\alpha]})\big/S_{j-1}(E^{[\alpha]})$ are mutually disjoint. □

Corollary 2.3.7. *Let E be a geometric (a,b)-module and let $[\alpha]$ be in $\mathbb{Q}/\mathbb{Z}$. Then the polynomial B_E and for each integer $j \in [1, d(E)]$ the polynomial $B^j_{E^{[\alpha]}}$ are respectively obtained by deleting in B_E and in B^j_E all roots which do not belong to $[\alpha]$.*

Proof. Thanks to Lemma 2.3.4 and Lemma 2.3.6 we may assume that E has a simple pole. Then Lemma 2.3.6 gives the result. □

3. The case of a fresco

3.1. **Some known facts.** First let me remind some previous results and fix some notation.

(1) A **fresco** is a geometric (a,b)-module F such that $F\big/(aF+bF)$ is a one-dimensional complex vector space. This is equivalent of the fact that F is a sub-module of some $S\Xi^{(N)}_{\mathscr{A}} \otimes V$ generated by one element over $B[a]$.

(2) It is proved in [7][8] that any fresco F is isomorphic to a quotient $B[a]\big/B[a]P$ where P is an element in $B[a]$ of the type

$$P := (a-\lambda_1 b)S_1^{-1}(a-\lambda_2 b)S_2^{-1}\cdots(a-\lambda_k b)$$

where k is the rank of F as a B-module and where $-\lambda_j + k - j$ are negative rational numbers which are exactly the roots (counting multiplicities) of B_F.

Note that for any fresco F, B_F is the **characteristic polynomial** of the action of $-b^{-1}a$ on the vector space $F^\sharp/bF^\sharp$ which is equal to the minimal polynomial in this case.

(3) If we have an exact sequence of geometric (a,b)-modules

$$0 \to E_1 \to E \to E_2 \to 0$$

and if E is a fresco, then E_1 and E_2 are frescos and we have the equality

$$B_E(x) = B_{E_1}(x+r)B_{E_2}(x),$$

where r is the B-rank of E_2.

[8]The formal results of [7] extend to the convergent frescos. The division by $(a-\lambda b)$ in $B[a]$ is an easy exercise.

3.2. **The B^j_F when F is a fresco.** Our aim is now to prove that the definition given above for the higher Bernstein polynomials of a geometric (a,b)-module is compatible with the definition of the higher Bernstein polynomials of a fresco which is given in a previous version of this paper (see [2]). For instance, we will verify that in the case of a fresco F the product of the B^j_F is equal to B_F.

The definition of the higher Bernstein polynomial of a fresco F given in [2] is the following.

Definition 3.2.1. *Let F be a fresco and let j be an integer in $[1, d(F)]$. Then $B^{(j)}_F$, the j-th Bernstein polynomial of F is defined by the formula*

$$B^{(j)}_F(x) = B_{S_j(F)\big/S_{j-1}(F)}(x + r_j), \qquad (@)$$

where r_j is the rank of $F\big/S_{j+1}(F)$.

Our goal is to prove the following compatibility of this definition with the Definition 2.2.1 given in Section 2.

Theorem 3.2.2. *For each fresco F and each integer j we have $B^j_F = B^{(j)}_F$.*

This result will be the content of Corollary 3.2.6 below.

Recall that for any (a,b)-module E with rank r a **Jordan-Hölder sequence**[9] for E is an increasing sequence of normal sub-modules

$$\{0\} = G_0 \subset G_1 \subset \cdots \subset G_r = E$$

such that $G_{j+1}\big/G_j$ is a rank 1 (a,b)-module for each $j \in [0, k-1]$. Any geometric (a,b)-module[10] admits a Jordan-Hölder sequence.

Proposition 3.2.3. *Let F be a fresco with rank r and let*

$$\{0\} = F_0 \subset F_1 \subset \cdots \subset F_r = F$$

be a Jordan-Holdër sequence for F. Then the normalization of $F^\sharp_j$ in $F^\sharp$ is equal to $b^{j-r}F^\sharp_j$. Moreover we have

$$F^\sharp = \sum_{j=1}^{r} b^{j-r}F_j.$$

Note that this equality shows that $b^{j-r}F_j$ is a sub-module of $F^\sharp$ where $r - j$ is the co-rank of F_j in F.

Proof. Since our assertions are obvious for $r = 1$, assume that they are already proved for $r \geq 1$ and consider the case where F is of rank $r + 1$.

So we assume that $\{0\} = F_0 \subset F_1 \subset \cdots \subset F_r \subset F_{r+1} = F$ is a J-H sequence for F and that $F^\sharp_r = \sum_{j=1}^r b^{j-r}F_j$. Then, assuming that $F/F_r \simeq E_\alpha$, let x be a generator of F (as a $B[a]$-module) whose image in E_α is the standard generator of E_α (see the point 4 in paragraph 2.1 which recalls the classification of rank 1 geometric (a,b)-modules).

Claim 1. Then $y := (a - \alpha b)x$ is a generator of F_r.

[9]Corollary 3.2.5 in [6] implies the existence of a normal sub-module of rank 1 in a regular (a,b)-module. An induction is then enough to obtain the existence of a J-H. sequence for any regular (convergent) (a,b)-module.

[10]The regularity of E is enough in fact, as explained in the previous footnote.

Proof of the claim 1. As x is a generator of F it is enough to show that if $P \in B[a]$ satisfies $Px \in F_r$ then there exists $Q \in B[a]$ such that $Px = Qy$.
Make the division of P by $(a - \alpha b)$ in $B[a]$. This gives $P = Q(a - \alpha b) + R$ with $R \in B$. Since $Px \in F_r$ is equivalent to $Pe_\alpha = 0$ in $E_\alpha = F\big/F_r$, we find that $Re_\alpha = 0$ in E_α and so $R = 0$ and we obtain $Px = Qy$ proving the claim.

This implies that $b^{-1}y$ is in $F^\sharp$ and so that $b^{-1}(F_r^\sharp) = \big(b^{-1}F_r\big)^\sharp$ is contained in $F^\sharp$.

Indeed we have in any simple pole (a, b)-module and for each integer $p \geq 0$ the equality $b(b^{-1}a)^p = (b^{-1}a - 1)^p b$.

Remark now that the surjective $B[a]$-linear map $\pi_0 : F \to E_\alpha$ extends to a surjective $B[a]$-linear map $\pi : F^\sharp \to E_\alpha$ because E_α has a simple pole. Since F_r is the kernel of π_0, the kernel of π is the normalization of $(F_r)^\sharp$ in $F^\sharp$ (the inclusion between the kernel of π and the normalisation of $(F_r)^\sharp$ is clear and both are normal sub-modules with the same rank).

Claim 2 . We have $F^\sharp = Bx + b^{-1}(F_r)^\sharp$ as a B-module and the kernel of π is equal to $b^{-1}(F_r)^\sharp$; thus the normalisation of $(F_r)^\sharp$ is $b^{-1}(F_r)^\sharp$.

Note that $b^{-1}(F_r)^\sharp$ is an (a, b)-sub-module of $F^\sharp$ but that Bx is not stable by a (at least when $r \geq 1$); so $F^\sharp$ is the direct sum of Bx and $b^{-1}(F_r)^\sharp$ as a B-module but not as a $B[a]$-module.
Proof of the claim 2. We know (see Claim 1) that y is a generator of F_r and that $b^{-1}y$ is in $F^\sharp$, so the inclusion $Bx + b^{-1}(F_r)^\sharp \subset F^\sharp$ is clear.

Conversely, if z is in $F^\sharp$ there exists $z_1, \dots, z_p \in F$ such that $z = \sum_{j=1}^p (b^{-1}a)^j z_j$. Now, for each $j \in [1, p]$ write $z_j = S_j(b)x + t_j$ where $S_j \in B$ and $t_j \in F_r$.

Then we obtain the opposite inclusion because $(b^{-1}a)^j Bx \subset Bx + b^{-1}(F_r)^\sharp$ is consequence of the following computation in which $S \in B$:

$$(b^{-1}a)S(b)x = b^{-1}\big(\alpha S(b)bx + S(b)y + b^2 S'(b)x\big) \in Bx + b^{-1}By \subset Bx + b^{-1}(F_r)$$

Thus we have $(b^{-1}a)^j t_j \in (F_r)^\sharp \subset b^{-1}(F_r)^\sharp$, proving our first assertion.

Since the restriction of π to Bx is bijective, we conclude that $Ker(\pi) \subset b^{-1}(F_r)^\sharp$. But there exists an integer $n \geq 1$ such that $b^n(F_r)^\sharp$ is contained in F_r and so $b^{n-1}(F_r)^\sharp$ is in $b^{-1}F_r$. This implies that $b^{-1}(F_r)^\sharp = (b^{-1}F_r)^\sharp \subset Ker(\pi)$ by B-linearity and normality of $Ker(\pi)$. So Claim 2 is proved.

Now our induction hypothesis gives, since $Bx \subset F_{r+1}$ that $F^\sharp = \sum_{j=1}^{r+1} b^{j-r-1}F_j$. □

We have proved in fact the more precise result for a fresco F with rank r and generator x:

Corollary 3.2.4. *For a fresco F with rank r and generator x there is a direct sum decomposition of $F^\sharp$ as a B-module*

$$F^\sharp = \oplus_{j=1}^r B b^{j-r} x_j,$$

where x_j for each $j \in [1, k]$ is the generator of the j-th term F_j of a J-H sequence of F. The $x_j, j \in [1, r]$ are obtained as follow:

Let P be a generator in $B[a]$ of the annihilator of x in F which may be written as (see point 2 above)

$$P := (a - \alpha_1 b)S_1(a - \alpha_2 b)S_2 \dots S_{r-1}(a - \alpha_r b)S_r$$

where $S_1, \dots, S_r$ are invertible elements in B and where the x_j are defined by the formula

$$x_j := (a - \alpha_{j+1}b)S_{j+1} \dots (a - \alpha_r b)S_r x \quad j \in [1, r-1] \quad \text{and} \quad x_r = x.$$

Corollary 3.2.5. *For a fresco F and any normal sub-module G of co-rank g in F the normalisation of $G^\sharp$ in $F^\sharp$ is equal to $b^{-g}G^\sharp$.*

Proof. With the notation of the previous proposition, we have shown that $b^{-1}(F_r)^\sharp$ is normal in $F^\sharp = (F_{r+1})^\sharp$. This implies, since each F_h is normal and is a fresco, that for each h the sub-module $b^{h-r-1}(F_h)^\sharp$ is normal in $F^\sharp = (F_{r+1})^\sharp$. Since it contains $(F_h)^\sharp$ and has the same rank, it is the normalization of $(F_h)^\sharp$ in $F^\sharp$.

Now the conclusion follows because we may assume that $G = F_h$ for some J-H. sequence of F: choose a J-H. sequence for F/G , lift it in F and complete with a J-H. sequence for G. $\square$

As a consequence we obtain that for any fresco F and any $j \in [1, d(F)]$ the equality:

$$S_j(F^\sharp) = b^{-r_j} S_j(F)^\sharp,$$

where r_j is the co-rank of $S_j(F)$ in F. So the Bernstein polynomial of $S_j(F^\sharp)$ is deduced from the Bernstein polynomial of $S_j(F)$ by the formula

$$B_{S_j(F^\sharp)}(x) = B_{S_j(F)}(x + r_j).$$

Corollary 3.2.6. *For each fresco F and each integer $j \in [1, d(F)]$, the equality*

$$B_{S_j(F^\sharp)/S_{j-1}(F^\sharp)}(x) = B_{S_j(F)/S_{j-1}(F)}(x + r_j)$$

holds true.

Proof. We shall prove that there is an isomorphism of (a, b)-modules between $\big(S_j(F)/S_{j-1}(F)\big)^\sharp$ and $b^{r_j}\big(S_j(F^\sharp)/S_{j-1}(F^\sharp)\big)$. We have, thanks to Lemma 2.1.5:

$$\big(S_j(F)/S_{j-1}(F)\big)^\sharp = S_j(F)^\sharp/N_{S_j(F)^\sharp}(S_{j-1}(F)^\sharp).$$

Recall that $N_Y(X)$ denotes the normalisation of X in Y (this notation is introduced in paragraph 2.1 point 2).

For an inclusion $X \subset Y$ of regular (a, b)-modules, the following three properties are easy to prove and left to the reader (see [6]):

(1) For any non-negative integer s, $N_Y(b^s X) = N_Y(X)$.
(2) For any positive integer r, we have $b^{-r}X \subset b^{-r}Y$ and $N_{b^{-r}Y}(b^{-r}X) = b^{-r}\big(N_Y(X)\big)$.
(3) For any positive integer j and any $n \in \mathbb{Z}$ we have $S_j(b^n X) = b^n S_j(X)$.

Denote by r_j the co-rank of $S_j(F)$ in F and by s_j the co-rank of $S_{j-1}(F)$ in $S_j(F)$. Then we have $S_j(F^\sharp) = b^{-r_j}S_j(F)^\sharp$ and $S_{j-1}(F^\sharp) = b^{-r_j-s_j}S_{j-1}(F)^\sharp$ thanks to Corollary 3.2.5. Using the three properties above we see that

$$N_{S_j(F)^\sharp}(S_{j-1}(F)^\sharp) = b^{r_j}\big(N_{S_j(F^\sharp)}(b^{s_j}S_{j-1}(F^\sharp)) = b^{r_j}N_{S_j(F^\sharp)}(S_{j-1}(F^\sharp)) = b^{r_j}S_{j-1}(F^\sharp)$$

as $S_{j-1}(F^\sharp)$ is normal in $S_j(F^\sharp)$. Since $S_j(F)^\sharp = b^{r_j}S_j(F^\sharp)$ the proof is complete using the easy fact that for X a normal sub-module in an (a, b)-module Y we have for each integer $r \in \mathbb{Z}$ a natural isomorphism $b^r(Y/X) \simeq b^rY/b^rX$. $\square$

As a consequence, the definition of the polynomials B_F^j given in [2] for a fresco F coincide with the general definition of the higher Bernstein polynomials of a geometric (a, b)-module given in the previous section.

Corollary 3.2.7. *For a fresco F we have the equality*

$$B_F = \prod_{j=1}^{d(F)} B_F^j.$$

Proof. We know that B_F and the product $\prod_{j=1}^{d(F)} B_F^j$ have the same degree equal to the rank of F. Since B_F divides $\prod_{j=1}^{d(F)} B_F^j$ and they are both monic, the equality follows. $\square$

3.3. **Complements.** Since in Theorem 1.2.1 we consider a fresco $\mathcal{F}$ with $d(\mathcal{F}^{[\alpha]}) = d$ and a root $-\alpha - m$ of $B^d_{\mathcal{F}}$ we want to show that we may reach this situation if we begin with the hypothesis that $B^d_{H_0^{n+1}}$ has a root $-\alpha - m'$. Proposition 3.3.1 below shows that in this situation we may find such a fresco $\mathcal{F}$ inside H_0^{n+1} with $B^d_{\mathcal{F}}$ having a root $-\alpha - m$ with $m \geq m'$. In the case where $d = d(\mathcal{H}^{[\alpha]})$ we may choose $\mathcal{F}$ such that $m = m'$.

Proposition 3.3.1. *Let $\mathcal{E}$ be an $[\alpha]$-primitive geometric (a, b)-module and let j be an integer in $[1, d(\mathcal{E})]$ where $d(\mathcal{E})$ is the nilpotent order of $\mathcal{E}$. Assume that $-\alpha - m$ is a root of the Bernstein polynomial $B^j_{\mathcal{E}}$.*

Then there exist an integer $m' \geq m$ and a fresco $\mathcal{F} \subset \mathcal{E}$ with that $d(\mathcal{F}) \geq j$ such that $-(\alpha+m')$ is a root of some h-th Bernstein polynomial $B^h_{\mathcal{F}}$ of $\mathcal{F}$, for some $h \geq j$.

Moreover, if $j = d(\mathcal{E})$ then we may take $m' = m$ with $h = d(\mathcal{E})$.

Note that, thanks to Corollary 2.2.5, we have also a root $-(\alpha + m')$ for $B^j_{\mathcal{F}}$ but may be with $m' \in \mathbb{Z}$.

Proof. Our hypothesis implies that there exists a $B[a]$-linear surjective map, where $\beta := \alpha + m$:

$$\pi : S_j(\mathcal{E}^\sharp)/S_{j-1}(\mathcal{E}^\sharp) \to E_\beta.$$

Let $n \in \mathbb{N}$ be such that $b^n S_j(\mathcal{E}^\sharp) \subset S_j(\mathcal{E})^\sharp$. Remark that if we assume that $j = d(\mathcal{E})$, we may take $n = 0$, since we have $S_j(\mathcal{E}^\sharp) = \mathcal{E}^\sharp$.

We also denote by π the map $S_j(\mathcal{E}^\sharp) \to \mathcal{E}_\beta$ obtained by composition with the quotient map to $S_j(\mathcal{E}^\sharp)/S_{j-1}(\mathcal{E}^\sharp)$.

Take $x \in S_j(\mathcal{E}^\sharp)$ such that $\pi(x) = e_\beta$ where e_β is the standard generator of E_β (so $E_\beta = B.e_\beta$ and $ae_\beta = \beta be_\beta$). Since there exists some non-negative integer n such that $b^n x$ is in $S_j(\mathcal{E})^\sharp$ we may find $y_0, \dots, y_p$ in $S_j(\mathcal{E})$ such that $b^n x = \sum_{p=0}^N (b^{-1}a)^p y_p$. Then there exists at least one $p_0 \in [0, N]$ such that $\pi(y_p)$ is not contained in $b^{n+1}E_\beta$. Since π vanishes on $S_{j-1}(\mathcal{E}^\sharp)$ we have $y_{p_0} \notin S_{j-1}(\mathcal{E})$. Then applying a suitable invertible element in B to y_{p_0} we obtain an element $z \in S_j(\mathcal{E}) \setminus S_{j-1}(\mathcal{E})$ such that the fresco $\mathcal{F} := B[a]z$ satisfies $d(\mathcal{F}) \geq j$ and $\mathcal{F}\big/S_{j-1}(\mathcal{F})$ has a surjective map to $E_{\alpha+m'}$ with $m' \leq m + n$. So $-(\alpha + m')$ is a root of the Bernstein polynomial of $\mathcal{F}\big/S_{j-1}(\mathcal{F})$ and also of at least one polynomial $B^h_{\mathcal{F}/S_{j-1}(\mathcal{F})} = B^{j+h-1}_{\mathcal{F}}$ for some $h \geq 1$. □

The following example shows that, in general, for $1 \leq j < d(\mathcal{E})$ we cannot avoid the integer shift for the root of the Bernstein polynomial of the fresco constructed in the previous proposition.

Example. Fix $\beta \neq \gamma$ in $]0, 1[\cap\mathbb{Q}$. Let $\mathcal{E}$ be the geometric (a, b)-module generated by $\xi := s^{\beta-1}Log\, s \otimes v_1$ and $\eta := s^{\gamma-1} \otimes v_2$ inside $\Xi^{(1)}_{\beta,\gamma} \otimes V$ where (v_1, v_2) is a basis of V. Then we have $S_2(\mathcal{E}) = \mathcal{E}$, so $d(\mathcal{E}) = 2$,

$$\mathcal{E}^\sharp = \mathcal{E} + \mathbb{C}s^{\beta-1} \otimes v_1, \quad S_1(\mathcal{E}) = S_1(\mathcal{E})^\sharp = (E_{\beta+1} \otimes v_1) \oplus (E_\gamma \otimes v_2)$$

and

$$S_1(\mathcal{E}^\sharp) = (E_\beta \otimes v_1) \oplus (E_\gamma \otimes v_2).$$

So $-\beta$ is a root of $B^1_{\mathcal{E}}$.

The only surjective map $\pi : S_1(\mathcal{E}^\sharp) \to E_\beta$ is given (up to a multiplicative constant) by the projection of $S_1(\mathcal{E}^\sharp)$ onto $E_\beta \otimes v_1$ and the only choice for an element x such that $\pi(x) = e_\beta$ is $x = s^{\beta-1} \otimes v_1$ up to a multiplicative constant and an element in $E_\gamma \otimes v_2$. Since we have $x = -(b^{-1}a - \beta)\xi$ modulo $E_\gamma \otimes v_2$ the map π is not defined on ξ but only on bx which is in $S_1(\mathcal{E})^\sharp = S_1(\mathcal{E})$. So the fresco $B[a]bx$ which is semi-simple has $-(\beta + 1)$ (and may be $-\gamma$) as root of its Bernstein polynomial. There is no fresco in $\mathcal{E}$ which is semi-simple and having $-\beta$ for root of its Bernstein polynomial.

3.4. **Jordan blocks.** Recall that a **theme** is, by definition, a fresco which may be embeded in some $S\Xi_{\mathscr{A}}^{(N)}$ (so with $V = \mathbb{C}$) (see Definition 5.1.3 in [6]).

Lemma 3.4.1. *Let $\alpha \in]0,1[\cap\mathbb{Q}$ and let φ be an element in $S\Xi_{\alpha}^{(N)}$ which is of degree N in $Log\, s$. Then inside the rank $N+1$ theme $T := B[a].\varphi \subset S\Xi_{\alpha}^{(N)}$, there exists an element*

$$\psi_N := s^{\alpha+m-1}\frac{(Log\, s)^N}{N!}$$

where m is an integer.

For $\alpha = 1$, if $\varphi \in S\Xi_1^{(N)}$ is the class of a series which has degree $N+1$ in $Log\, s$, then inside the rank $N+1$ theme $T := B[a]\varphi \subset S\Xi_1^{(N)}$ there exists an element which is the class of $s^m\frac{(Log\, s)^{N+1}}{(N+1)!}$, where m is an integer.

Proof. Note first that $T := B[a]\varphi$ is a rank $(N+1)$ fresco thanks to Lemma 5.2.4 in [6].

We shall prove the lemma for $\alpha \neq 1$ by induction on $N \geq 0$. We leave the case $\alpha = 1$ which is similar as an exercise for the reader.
Since the case $N = 0$ is clear, assume that the lemma is proved for $N-1$ and let $T \subset S\Xi_{\alpha}^{(N)}$ be a rank $N+1$ fresco. Then $S_1(T)$ is equal to $T \cap S\Xi_{\alpha}^{(0)}$ and we may embed the rank N fresco $T/S_1(T)$ in

$$S\Xi_{\alpha}^{(N-1)} \simeq S\Xi_{\alpha}^{(N)}/S\Xi_{\alpha}^{(0)}.$$

Thanks to our inductive hypothesis there exists an integer m' such that

$$s^{\alpha+m'-1}(Log\, s)^N/N! \qquad modulo\ S\Xi_{\alpha}^{(0)}$$

is in $T/S_1(T)$ and, since $S_1(T) \subset S\Xi_{\alpha}^{(0)}$, we may find an invertible element S in B such that $\varphi := s^{\alpha+m'-1}(Log\, s)^N/N! + S(b)s^{\alpha+M-1}$ is in T. Since $S_1(T)$ is isomorphic to $E_{\alpha+q}$ for some positive integer q, for an integer m'' large enough, $s^{\alpha+m'+m''-1}(Log\, s)^N/N!$ will be in T, since $S(b)s^{\alpha+M+m''-1}$ is in $S_1(T)$, concluding the proof. □

Corollary 3.4.2. *Let $\mathcal{F}$ be a fresco and assume that the p-th Bernstein polynomial of $\mathcal{F}$ has a root in $-\alpha-\mathbb{N}$, where α is in $]0,1]\cap\mathbb{Q}$. Then there exists $w_1,\dots,w_p$ in $\mathcal{F}$ (in fact in $\mathcal{F}_{[\alpha]}$) and an integer $m \in \mathbb{N}$ satisfying the relations:*

$$(\star) \qquad aw_j = (\alpha+m)bw_j + bw_{j-1} \quad \forall j \in [1,p] \quad \text{with the convention} \quad w_0 \equiv 0$$

and which are B-linearly independent in $\mathcal{F}$.

Proof. We assume that $\alpha \neq 1$ leaving the case $\alpha = 1$ which is analogous as an exercise for the reader.

Since $S_p(\mathcal{F}_{[\alpha]}) = S_p(\mathcal{F})_{[\alpha]}$, thanks to Lemma 4.2.5 in [6], we may find an $[\alpha]$-primitive theme T of rank p in $\mathcal{F}_{[\alpha]}$, thanks to Proposition 6.3.3 in [6]. As we may assume that T is embedded in $S\Xi_{\alpha}^{(p-1)}$ the previous lemma shows that there exists an integer m_0 such that $s^{\alpha+m_0-1}(Log\, s)^p/p!$ is an element in T.

Define $w_j = s^{\alpha+m_0-1}(Log\, s)^j/j!$ for $j \in [1,p]$. Then the relations $(\star)$ are satisfied and imply that $w_1,\dots,w_p$ are elements in $T^\sharp$.

To show that $w_1,\dots,w_p$ are B-linearly independent, note J the B-sub-module generated by $w_1,\dots,w_p$. Then it has rank at most p. But the relation $(\star)$ shows that J is an (a,b)-sub-module of $T^\sharp$ with a simple pole. Moreover, as $w_1,\dots,w_p$ are clearly linearly independent over $A = \mathbb{C}\{s\}$, we have

$$\dim_{\mathbb{C}} J/aJ = \dim_{\mathbb{C}} J/bJ \geq p$$

and so J has rank p as a B-module.

Since there exists a non-negative integer q such that $b^qT^\sharp \subset T \subset \mathcal{F}_{[\alpha]}$ the conclusion follows considering $b^qw_1,\dots,b^qw_p$ and $m := m_0 + q$. □

Remark. Let $J := \sum_{j=1}^{p} Bw_j$ the sub B-module generated by $w_1, \dots, w_p$. Then J is a sub (a,b)-module of $\mathcal{F}$ which has a simple pole and is $[\alpha]$-primitive; it is equal to $\mathcal{E}(J_{\alpha+m,p})$ where $J_{\alpha+m,p}$ is the matrix of the standard Jordan block with rank p and eigenvalue $\alpha + m$ (see the end of Section 2.3 in [6]). The action of $b^{-1}a$ on J/bJ is given by $J_{\alpha+m,p}$. So the Bernstein polynomial of J is equal to $(x + \alpha + m)^p$.

It is interesting to compare this result with Corollary 3.2.5 in [6]. Here we do not assume that the action of $b^{-1}a$ on $\mathcal{F}^\sharp/b\mathcal{F}^\sharp$ has a Jordan block of size p for some λ in $\alpha + \mathbb{N}$ but, in a way, that this happens for the eigenvalue $\exp(2i\pi\alpha)$ of $\exp(2i\pi b^{-1}a)$ acting of $\mathcal{F}^\sharp/b\mathcal{F}^\sharp$. And this hypothesis is precisely formulated by the existence of a root in $-\alpha - \mathbb{N}$ for the p-th Bernstein polynomial of the fresco $\mathcal{F}$.

Note that contrary to the result in Corollary 3.2.5 in [6] we have no control here on the integral shift between the root of the p-th Bernstein polynomial and the (multiple) root of the Bernstein polynomial of the Jordan block which is obtained.

4. Existence of poles

4.1. The complex of sheaves $(Ker\, df^\bullet, d^\bullet)$.

The standard situation. We consider now the following situation:

Let $f : U \to \mathbb{C}$ be a holomorphic function on an open polydisc U with center 0 in $\mathbb{C}^{n+1}$. We shall assume that U is small enough in order that the inclusion $\{df = 0\} \subset \{f = 0\}$ holds in U.

We denote Y the hypersurface $\{f = 0\}$ in U and we assume that Y is reduced. For each point $y \in Y$ we denote $f_y : X_y \to D_y$ a Milnor representative of the germ of f at y. So X_y is constructed by cutting a small ball, with center y and with radius $\varepsilon > 0$ very small, with $f^{-1}(D_\delta)$ where D_δ is an open disc with center 0 and radius $\delta \ll \varepsilon$. For $y = 0$ we simply write $f : X \to D$ such a Milnor representative of the germ of f at the origin.

Let $\pi : H \to D^*$ be the universal cover of the punctured disc $D^* := D \setminus \{0\}$ and choose a base point $\tilde{s}_0$ in H over a chosen base point s_0 in D^*. Fix a point $y \in Y$ and take for D the disc of a Milnor representative of f_y. Then we identify the Milnor fiber F_y of f at y with $f^{-1}(s_0)$.

For any p-cycle γ in $H_p(F_y, \mathbb{C})$ let $(\gamma_{\tilde{s}})_{\tilde{s}\in H}$ be the horizontal family of p-cycles in the fibers of $f \times_{D^*} \pi$ over H taking the value γ at the point $\tilde{s}_0$. Then the regularity of the Gauss-Manin connection of f at y insures that for any $\omega \in \Omega_y^{p+1}$ which satisfies $d\omega = 0$ and $df \wedge \omega = 0$ the (multi-valued) function $s \mapsto \int_{\gamma_s} \omega/df$ has a convergent asymptotic expansion when s goes to 0, which is in $\Xi_{\mathscr{A}}^{(p-1)}$, where $\exp(2i\pi\mathscr{A})$ contains the eigenvalues of the monodromy of f at the point y.

We define on Y the following sheaves (see [8] or [9]) for each integer $p \in [1, n]$:

First let $Ker\, df^{p+1} \subset \Omega^{p+1}$ be the kernel of the map $\wedge df : \Omega^{p+1} \to \Omega^{p+2}$ of coherent sheaves on U and $Ker\, d^{p+1}$ be the kernel of the ($\mathbb{C}$-linear) de Rham differential

$$d^{p+1} : \Omega^{p+1} \to \Omega^{p+2}.$$

Then for $p \in [1, n]$ define the sheaf $\mathcal{H}^{p+1}$ as the (topological) restriction on Y of the sheaf $Ker\, df^{p+1} \cap Ker\, d^{p+1}/d(Ker\, df^p)$.

By convention we put $\mathcal{H}^{p+1} = 0$ for $p \notin [1, n]$.

Then we have a natural structure of $A := \mathbb{C}\{s\}$-modules on the sheaves $\mathcal{H}^{p+1}$ for each p induced by the natural action of A on $\Omega_{|Y}$ given by $(g, \omega) \mapsto f^*(g)\omega$ where g is in A and ω is in Ω_y^{p+1}, for each $y \in Y$.

We have also an action of $\mathbb{C}[b]$ on $\mathcal{H}^{p+1}$ for each $p \in [1, n]$ which is defined as follows:

- For $\omega_y \in Ker\, d^{p+1} \cap Ker\, df^{p+1}$ write $\omega_y := du_y$ for some $u_y \in \Omega_y^p$ (holomorphic de Rham Lemma) and put $b[\omega_y] := [df \wedge u_y]$.

Then clearly $d(df \wedge u_y) = 0$ and $df \wedge (df \wedge u_y) = 0$.

- Of course, if we change the choice of u_y (for $p \in [1, n]$) in $u_y + dv_y, v_y \in \Omega_y^{p-1}$, the class of $b[\omega_y] \in \mathcal{H}^{p+1}$ is the same since $df \wedge dv_y = -d(df \wedge v_y)$ is in $d(Ker\, df^p)$.

The sheaf $\mathcal{H}^{p+1}$ modulo its a-torsion[11], denoted by H^{p+1}, is the (a, b)-module version of the Gauss-Manin connection in degree p. As we assume that Y is reduced, the 0-th cohomology of the Milnor fiber is $\mathbb{C}$ and the corresponding monodromy is trivial.

Lemma 4.1.1. *The actions of a and b on $\mathcal{H}^{p+1}$ satisfy the commutation relation $ab - ba = b^2$.*

Proof. For $\omega_y = du_y \in Ker\, df^{p+1} \cap Ker\, d^{p+1}$ we have

$$b(a[\omega_y] + b[\omega_y]) = b[f du_y + df \wedge du_y] = b[d(f u_y)] = [df \wedge f u_y] = ab[\omega_y],$$

which gives the relation $b(a + b) = ab$ concluding the proof. □

Note that the action of a is well-defined on $Ker\, df^{p+1}$ but the action of b is only well-defined on the cohomology $\mathcal{H}_y^{p+1}$ for each $p \in [1, n]$ and each $y \in Y$.

Theorem 4.1.2. *We keep the notation introduced above and let p be an integer in $[1, n]$. Let $\omega \in \Omega_y^{p+1}$ be in $Ker\, df^{p+1}$ such that $d\omega = 0$. Then for each $\gamma \in H_p(F_y, \mathbb{C})$ define $\Phi(\gamma, \omega)$ as the element in $S\Xi_{\mathscr{A}}^{(p)}$ given by the singular part of the asymptotic expansion* [12] *of the period-integral $\int_{\gamma_s} \omega/df$, where $\mathscr{A}$ is the image in $]0, 1]$ of the opposite of the roots of the reduced Bernstein polynomial of f at the point $y \in Y$ and where $(\gamma_s)_{s \in H}$ is the horizontal family of p-cycles taking the value γ at the base point $\tilde{s}_0$. So we have:*

$$\Phi(\omega, \gamma) := \int_{\gamma_s} \omega/df \in S\Xi_{\mathscr{A}}^{(p)}.$$

Then using the fact that $H^p(F_y, \mathbb{C})$ is the dual of $H_p(F_y, \mathbb{C})$ and the linarity of Φ in γ, Φ defines a map

$$\Psi : \mathcal{H}_y^{p+1} \to S\Xi_{\mathscr{A}}^{(p)} \otimes_{\mathbb{C}} H^p(F_y, \mathbb{C}), \quad \Psi(\omega) := [\gamma \mapsto \Phi(\omega, \gamma)]$$

which is A-linear and b-linear and whose kernel is equal to the a-torsion of $\mathcal{H}_y^{p+1}$. So Ψ is well-defined and injective on H_y^{p+1}.

Proof. The A-linearity of Ψ is obvious. The b-linearity is an easy consequence of the derivation formula

$$\partial_s(\int_{\gamma_s} u) = \int_{\gamma_s} du/df$$

when u is in Ω_y^p satisfies $df \wedge du = 0$.

Consider now $\omega \in Ker\, df^{p+1}$ such that $d\omega = 0$ and assume that ω is in the kernel of Ψ. Then for each γ the corresponding period-integral vanishes because the asymptotic expansion is convergent (thanks to the regularity of the Gauss-Manin connection). So the class induced by ω/df in $H^p(F_y, \mathbb{C})$ vanishes, and this implies that the class defined by ω in the f-relative de Rham cohomology vanishes. So we may find a meromorphic form $v \in \Omega_y^{p-1}[f^{-1}]$ such that $\omega = df \wedge dv$ (see [4] and [3] for $\alpha = 1$). This implies that $a^N[\omega] = 0$ in $\mathcal{H}_y^{p+1}$. □

[11] An element ξ is of a-torsion if there is an positive integer m such that $a^m \xi = 0$. In $\mathcal{H}_x^{p+1}$ for each $x \in \{f = 0\}$ the a-torsion subspace is stable b and contains the b-torsion sub-space; see [8] paragraph 2.3.

[12] Since for the eigenvalue 1 we consider only the *singular part* of the asymptotic expansion, so we replace $\Xi_1^{(p-1)}$ by $S\Xi_1^{(p-1)} := \Xi_1^{(p)}/\Xi_1^{(0)}$ which is isomorphic to $\Xi_1^{(p-1)}$ (see [3]), the Γ-factor that we introduce below shifts the order of poles at points in $-\mathbb{N}$ in the complex Mellin transform $F_h^{\omega,\omega'}(\lambda)$ (see [11]) of the associated hermitian periods $\int_{f=s} \rho\omega \wedge \bar{\omega}'/df \wedge d\bar{f}$.

Remark 4.1.3. *The map* Φ *satisfies also the relation* $\Phi(\omega, T\gamma) = \mathscr{T}(\Phi(\omega,\gamma))$ *where* T *is the monodromy acting on* $H_p(F_y,\mathbb{C})$ *and where* $\mathscr{T}$ *is the monodromy acting on* $S\Xi^{(p)}_{\mathscr{A}}$ *via* $Log\, s \mapsto Log\, s + 2i\pi$.

So the image of Ψ *is contained in the sub* (a,b)*-module of* $S\Xi^{(p)}_{\mathscr{A}} \otimes_{\mathbb{C}} H^p(F_y,\mathbb{C})$ *which is invariant by* $\mathscr{T}\otimes T^*$ *where* T^* *is the action of the monodromy on* $H^p(F_y,\mathbb{C})$.

Corollary 4.1.4. *For* $y \in Y$ *let* H^{p+1}_y *be the quotient of* $\mathcal{H}^{p+1}_y$ *by its a-torsion. Then* H^{p+1}_y *is a geometric (convergent)* (a,b)*-module.*

Proof. The point is that H^{p+1}_y is a finite type A-module since A is noetherian and $S\Xi^{(p)}_{\mathscr{A}}$ is a finite type (free) A-module. Then H^{p+1}_y is closed for the natural dual Fréchet topology induced by $S\Xi^{(p)}_{\mathscr{A}} \otimes H^p(F_y,\mathbb{C})$. As it is also a free finite type B-module, it is also stable by the action of $B[a]$. So it is a geometric (a,b)-module. □

Note that it is not obvious to show directly that B acts on H^{p+1}_y contrary to the formal case (corresponding to the algebra $\hat{B} := \mathbb{C}[[b]]$. .

Definition 4.1.5. *In the situation above, let* ω *be a germ at* $y \in Y$ *of the sheaf* $Ker\, df^{p+1}$ *which is d-closed. Then we define the* **fresco** $\mathcal{F}_{f,\omega,y}$ *associated to these data as the fresco* $B[a][\omega] \subset H^{p+1}_y$. *So it is generated in the geometric* (a,b)*-module* H^{p+1}_y *by the class of* ω.

Note that for $p = n$ each germ ω at a point y of Ω^{n+1}_y satisfies $df \wedge \omega = 0$ and $d\omega = 0$.

In the sequel we shall mainly use the case $p = n$ with $y = 0$. So we simplify the notation to $\mathcal{F}_\omega$ when we consider the fresco $\mathcal{F}_{f,\omega,0}$ in H^{n+1}_0.

4.2. **The use of frescos.** Now comes the main hypothesis we make in the sequel on the holomorphic germ f.

Definition 4.2.1. *In the standard situation, fix a rational number* $\alpha \in]0,1]$. *We say that the germ* f *has* **an isolated singularity for the eigenvalue** $\exp(2i\pi\alpha)$ **of its monodromy** *when, at each point* $y \neq 0$ *in the reduced hypersurface* $Y = f^{-1}(0)$*, the local monodromy of* f*, acting on the* reduced *cohomology of the Milnor fiber at the point* y*, does not admit this eigenvalue. This hypothesis is denoted* $H(\alpha,1)$ *in the sequel.*

Let us recall some known facts.

1. The hypothesis $H(\alpha,1)$ is equivalent to the fact that, in open neighborhood of the origin, the local *reduced* b-function of f at any point $x \neq 0$ has no root in $-\alpha - \mathbb{N}$.
2. The hypothesis $H(\alpha,1)$ is equivalent to the fact that, in an open neighborhood of the origin, the polar parts of the meromorphic extension of the distributions
$$\frac{1}{\Gamma(\lambda)}|f|^{2\lambda}\bar{f}^{-h}, \quad \forall h \in \mathbb{Z},$$
at points in $-\alpha - \mathbb{N}$ have their support contained in $\{0\}$.
3. The hypothesis $H(\alpha,1)$ is equivalent to the fact that, for any test form φ in $\mathscr{C}^\infty_c(\mathbb{C}^{n+1})^{n+1,n+1}$ with compact support in $X \setminus \{0\}$, the meromorphic extension of the functions
$$\frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar{f}^{-h}\varphi$$
has no pole in $-\alpha - \mathbb{N}$ for each $h \in \mathbb{Z}$.
4. Since the monodromy of f is defined on $H^p(F_y,\mathbb{Z})$, for $\alpha \in]0,1[$ the hypothesis $H(\alpha,1)$ is equivalent to the hypothesis $H(1-\alpha,1)$.
Thus, for a holomorphic germ f, the hypotheses of isolated singularity at the origin for the eigenvalues $\exp(\pm 2i\pi\alpha)$ of the monodromy are equivalent.

Assume that, in the standard situation, the germ f satisfies the hypothesis $H(\alpha, 1)$. Let ω, ω be in Ω_0^{n+1}, $\rho \in \mathscr{C}_c^\infty(\mathbb{C}^{n+1})$ be such that $\rho \equiv 1$ near 0, with a support small enough in order that $\rho\omega \wedge \bar{\omega}'$ is a well-defined $\mathscr{C}_c^\infty$ differential form of type $(n+1, n+1)$ on $\mathbb{C}^{n+1}$. Then, for any $h \in \mathbb{Z}$, the holomorphic function, defined for $2\Re(\lambda) > \sup\{0, h\}$ by the formula

$$\text{(F)} \qquad F_h^{\omega,\omega'}(\lambda) := \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-h} \rho\omega \wedge \bar{\omega}',$$

has a meromorphic continuation to the all complex plane, with poles contained in $-\mathscr{A} - \mathbb{N}$, where $\mathscr{A}$ is the finite subset of $\mathbb{Q}^{*+}$ of the roots of the reduced Bernstein polynomial $\tilde{b}_{f,0}$ of f at the origin.

Moreover, thanks to our hypothesis $H(\alpha, 1)$, we have the following properties (see [1] for a proof):

(1) The polar parts of $F_h^{\omega,\omega'}(\lambda)$ at the points in $-\alpha - \mathbb{N}$ do not depend on the choice (with the conditions specified above) of the function ρ.

(2) The polar parts of $F_h^{\omega,\omega'}(\lambda)$ at points in $-\alpha - \mathbb{N}$ depend, for given ω' and h, only on the image of ω in the formal (a, b)-module $\widehat{H}_0^{n+1}$, which is the formal b-completion of the geometric (a, b)-module H_0^{n+1} defined in section 4.1.

The following result is proved in [1] Proposition 3.1.1.

Proposition 4.2.2. *In the standard situation assume that hypothesis $H(\alpha, 1)$ is satisfied. Let ω and ω' be holomorphic $(n+1)$-differential forms on X and let ρ be a $\mathscr{C}^\infty$ function with compact support in X which satisfies $\rho \equiv 1$ near 0. We have the following properties:*

i) If there exists $v \in \Omega^n(X)$ satisfying $df \wedge v \equiv 0$ and $dv = \omega$ on X, then $F_h^{\omega,\omega'}(\lambda)$ has no pole in $-\alpha - \mathbb{N}$ for any $h \in \mathbb{Z}$ and any $\omega' \in \Omega_0^{n+1}$.

ii) For any $h \in \mathbb{Z}$ and any $\omega' \in \Omega_0^{n+1}$, the function $F_h^{a\omega,\omega'}(\lambda) - (\lambda+1)F_{h-1}^{\omega,\omega}(\lambda+1)$ has no pole at points in $-\alpha - \mathbb{N}$.

iii) For any $h \in \mathbb{Z}$ and any $\omega' \in \Omega_0^{n+1}$ the function $F_h^{b\omega,\omega'}(\lambda) + F_{h-1}^{\omega,\omega'}(\lambda+1)$ has no pole at points in $-\alpha - \mathbb{N}$

iv) For any complex number μ, for any $h \in \mathbb{Z}$ and for any $\omega' \in \Omega_0^{n+1}$ the function

$$F_h^{(a-\mu b)\omega,\omega'}(\lambda) - (\lambda + \mu + 1)F_{h-1}^{\omega,\omega'}(\lambda + 1)$$

has no pole at points in $-\alpha - \mathbb{N}$.

An easy consequence of the proposition above is the following:

Corollary 4.2.3. *Under the hypothesis $H(\alpha, 1)$ assume that the meromorphic extension of the holomorphic function $F_h^{\omega,\omega'}(\lambda)$ has never a pole of order $\geq p$ at a point in $-\alpha - \mathbb{N}$ for some given $\omega' \in \Omega_0^{n+1}$ but for each $h \in \mathbb{Z}$. Then the same is true replacing ω by any $w \in \Omega_0^{n+1}$ such that $[w]$ is in the fresco $\mathcal{F}_\omega = B[a][\omega] \subset H_0^{n+1}$.*

Proof. Points *ii*) and *iii*) of Proposition 4.2.2 show that for any integers q and r the function $F_h^{a^q b^r \omega,\omega'}(\lambda)$ has no pole of order $\geq p$ at the point $-\alpha - m$. The conclusion follows from Property 2 above. □

The following important tool for the sequel is also a consequence of Proposition 4.2.2, using the Structure Theorem for frescos of [7] extended to the convergent case (see Point 2 in the beginning of Section 3.1). It needs the following terminology:

THE PROPERTY $P(\omega,\omega',p)$. In the standard situation, assuming the hypothesis $H(\alpha,1)$, fix two holomorphic germs ω,ω' in Ω_0^{n+1}. Let $p\geq 1$ be an integer and assume that there exists $h\in\mathbb{Z}$ such that $F_h^{\omega,\omega'}(\lambda)$ has a pole of order at least equal to p at a point in $-\alpha-\mathbb{N}$. Then we say that **the integer m has the property** $P(\omega,\omega',p)$ when m is the smallest integer such that there exists $h\in\mathbb{Z}$ and a pole of order $\geq p$ at the point $\lambda=-\alpha-m$ for the function $F_h^{\omega,\omega'}(\lambda)$.

Proposition 4.2.4. *In the situation described above, assume that, for some $h\in\mathbb{Z}$, there exists a pole of order $\geq p$ at the point $-\alpha-m$ for the function $F_h^{\omega,\omega'}(\lambda)$.*
Then the following properties hold true, :

1. *Assuming that the integer m satisfies the property $P(\omega,\omega',p)$, for each $S\in\widehat{B}$ such that $S(0)\neq 0$ there exists a pole of order at least equal to p for $F_{h+1}^{S(b)\omega,\omega'}(\lambda)$ at the point $-\alpha-m$. Moreover, the integer m satisfies also the property $P(S\omega,\omega',p)$.*
2. *If $\mu\neq\alpha+m$ there exists a pole of order at least equal to p for $F_{h+1}^{(a-\mu b)\omega,\omega'}(\lambda)$ at the point $-\alpha-m-1$. Moreover, if the integer m satisfies the property $P(\omega,\omega',p)$, the integer $m+1$ satisfies the property $P((a-\mu b)\omega,\omega',p)$.*
3. *For $\mu=\alpha+m$, there exists a pole of order at least equal to $p-1$ for $F_{h+1}^{(a-\mu b)\omega,\omega'}(\lambda)$ at the point $-\alpha-m-1$.*

Proof. Assume that X is a sufficiently small open neighborhood of 0 in $\mathbb{C}^{n+1}$ such that the germs ω and ω' are holomorphic on X and that there exists $u\in\Omega^n(X)$ satisfying $du=\omega$ on X. Thanks to Stokes' Formula and Hypothesis $H(\alpha,1)$ (see Proposition 3.1.1 in [1] or Proposition 4.2.2 above) the meromorphic function

$$F_h^{b\omega,\omega'}(\lambda)+(\lambda+1)F_{h-1}^{\omega,\omega'}(\lambda+1)=-\frac{1}{\Gamma(\lambda)}\int_X|f|^{2\lambda}\bar{f}^{-h}d\rho\wedge u\wedge\bar{\omega}'$$

has no poles at points in $-\alpha-\mathbb{N}$ for any choice of ω',h and $\rho\in\mathscr{C}_c^\infty(X)$ which is identically 1 near the origin (since 0 is not in the support of $d\rho$).

Since m satisfies Property $P(\omega,\omega',p)$, it is clear that for any positive integer q, $F_{h'}^{b^q\omega,\omega'}(\lambda)$ has no pole of order $\geq p$ at $-\alpha-m'$ for each $m'\leq m-q$. Since we have never a pole for $F_h^{\omega,\omega'}(\lambda)$ at points where $\Re(\lambda)\geq 0$, we conclude that for any $S\in\widehat{B}$ with $S(0)\neq 0$ we have a pole of order p for $F_{h+1}^{S(b)\omega,\omega'}(\lambda)$ at the point $-\alpha-m$. Moreover m satisfies the property $P(S\omega,\omega',p)$.

With the same arguments (and the same Proposition 3.1.1 in [1] or Proposition 4.2.2) the meromorphic function

$$F_h^{(a-\mu b)\omega,\omega'}(\lambda)-(\lambda+\mu+1)F_{h-1}^{\omega,\omega'}(\lambda+1)$$

has no pole at points in $-\alpha-\mathbb{N}$ for any choice of ω',h and $\rho\in\mathscr{C}_c^\infty(X)$ which is identically 0 near the origin.

Now the same line of proof gives points 2 and 3 of the proposition using point iv) in Proposition 4.2.2. ☐

Now appears the main strategy of proof to locate the bigger order p pole in $-\alpha-\mathbb{N}$ for a given pair ω,ω'.

Corollary 4.2.5. *Assume that there exists a pole of order at least equal to p at the point $-\alpha-m$ for $F_h^{\omega,\omega'}(\lambda)$ for some integer $h\in\mathbb{Z}$ and assume that the integer m satisfies Property $P(\omega,\omega',p)$. Put $\Pi:=(a-\mu_1 b)S_1(a-\mu_2 b)S_2\dots(a-\mu_k b)S_k$ where $S_1,\dots,S_k$ are invertible elements in B and $\mu_1,\dots,\mu_k$ are positive rational numbers.*

1. *Assume that $\mu_j+j-k\neq\alpha+m$ for each $j\in[1,k]$. Then $F_{h+k}^{\Pi\omega,\omega'}(\lambda)$ has a pole of order at least equal to p at the point $-\alpha-m-k$. Moreover the integer $m+k$ satisfies the property $P(\Pi\omega,\omega',p)$.*

(2) If μ_1 is the only value of $j \in [1,k]$ such that $\mu_1 + j - k = \alpha + m$ then $F_{h+k}^{\Pi\omega,\omega'}(\lambda)$ has a pole of order $p-1$ at the point $-\alpha - m - k$.

Proof. The result is easily obtained using inductively the assertions $1,2,3$ of the previous proposition. □

Corollary 4.2.6. *Assume $H(\alpha,1)$ and that the nilpotent order of $(B[a]\omega)^{[\alpha]}$ (the $[\alpha]$-primitive quotient*[13] *of the fresco $B[a]\omega$) is at most $p-1$. Then for any choice of ω' and h, the meromorphic extension of $F_h^{\omega,\omega'}(\lambda)$ has no pole of order $\geq p$ at any point in $-\alpha - \mathbb{N}$.*

Proof. We shall prove the result by induction on $p \geq 1$. For $p = 1$ our hypothesis means that $(B[a]\omega)^{[\alpha]} = \{0\}$ so if $\Pi := (a-\mu_1 b)S_1(a-\mu_2 b)S_2 \dots (a-\mu_k b)S_k$ where $S_1, \dots, S_k$ are invertible elements in B, generates the annihilator of $[\omega]$ in the geometric (a,b)-module H_0^{n+1}, we may assume that $\mu_1, \dots, \mu_k$ are not in $-\alpha - \mathbb{N}$. Then, since $F_h^{\Pi\omega,\omega'}(\lambda)$ has no poles in $-\alpha - \mathbb{N}$ (see Proposition 4.2.2 (i)), we obtain immediately a contradiction with the assertion of Corollary 4.2.5 if we assume that for some choice of ω' and h the meromorphic function $F_h^{\omega,\omega'}(\lambda)$ has a pole at some point $-\alpha - m$.

Thanks to the case proved above, we may replace ω by a generator of the fresco $(B[a]\omega)^{[\alpha]}$, which means that we may assume that $B[a]\omega$ is an $[\alpha]$-primitive fresco with nilpotent order at most $p-1$ with $p \geq 2$ (see again the previous footnote 11).

Define $\mathcal{F} := S_{p-1}(B[a]\omega)/S_{p-2}(B[a]\omega)$. This fresco is $[\alpha]$-primitive, semi-simple and generated by $[\omega]$. So the generator $\Pi := (a-\mu_1 b)S_1(a-\mu_2 b)S_2 \dots (a-\mu_k b)S_k$ where $S_1, \dots, S_k$ are invertible elements in B, of the annihilator of the class $[\omega]$ in this semi-simple fresco[14] may be chosen such that we have any order for the sequence $\mu_j + j$. Since these numbers are pairwise distinct there exists at most one $j \in [1,k]$ such that $\mu_j + j - k = \alpha + m$. So we have have only two possibilities:

either there is no such $j \in [1,k]$ or there exists a unique $j \in [1,k]$ such that $\mu_{j_0} + j_0 - k = \alpha + m$ and in this case we may choose $j_0 = 1$.

So using inductively Corollary 4.2.5 we see that if we assume that $F_h^{\omega,\omega'}(\lambda)$ has a pole of order $\geq p$ at the point $-\alpha - m$, we shall find a pole of order $\geq p-1$ for $F_{h+k}^{\Pi\omega,\omega'}(\lambda)$ at the point $-\alpha - m - k$. Since the fresco $\mathcal{G}$ generated by the class $\Pi[\omega]$ satisfies $\mathcal{G} = S_{p-2}(B[a][\omega])$, its nilpotent order is at most equal to $p-2$. This contradicts our induction hypothesis.

The case where there is no $j \in [1,k]$ such that $\mu_j + j - k = \alpha + m$ leads to a pole of order $\geq p$ at the point $-\alpha - m - k$, so gives also a contradiction. □

4.3. **The final key.** Note that in Section 4.2 we always assume the existence of poles at some point in $-\alpha - \mathbb{N}$ for $F_h^{\omega,\omega'}(\lambda)$ (under our hypothesis $H(\alpha,1)$) and obtain consequences on the Bernstein polynomial of the fresco $\mathcal{F}_\omega$. These results improve the results in [1]. To go in the other direction, that is to say to prove that the existence of such poles is consequence of informations on the Bernstein polynomial of $\mathcal{F}_\omega$, we shall use now the main idea of [4] (and also [3] in the case $\alpha = 1$). This is the point where the use of convergent (a,b)-modules is essential. It allows to show that the non-vanishing of the class induced by ω in the $[\alpha]$-primitive quotient of H_0^{n+1} implies that the cohomology class induces by ω/df in the spectral part for the eigenvalue $\exp(2i\pi\alpha)$ of the monodromy of f acting on $H^n(F_0, \mathbb{C})$ does not vanish, where F_0 is the Milnor's fiber of F at 0.

[13]The nilpotent order of $E^{[\alpha]}$ and of $E_{[\alpha]}$ are the same since the natural injection $E_{[\alpha]} \to E^{[\alpha]}$ has a finite-dimensional co-kernel, because they have same rank; see Lemma 6.3.6 in [6]).

[14]Recall that the saturation of this semi-simple fresco is a direct sum of rank 1 geometric (a,b)-modules corresponding to pairwise distinct numbers since the Bernstein polynomial of a fresco $\mathcal{F}$ is the characteristic polynomial of $-b^{-1}a$ acting of $\mathcal{F}^\sharp / b\mathcal{F}^\sharp$.

Theorem 4.3.1. *Let $f : X \to D$ be a Milnor representative of the holomorphic germ of f near the origin in $\mathbb{C}^{n+1}$ and assume that $H(\alpha, 1)$ is satisfied. Let $u \in \Omega^n(X)$ such that there exists $m \in \mathbb{N}$ with $fdu = (\alpha + m)df \wedge u$ on X and assume that the class induced by u in $H^n(F_0, \mathbb{C})$ is not 0. Then there exists a germ $\omega' \in \Omega_0^{n+1}$ and an integer $h \in \mathbb{N}$ such that for any $\rho \in \mathscr{C}_c^\infty(X)$ which is identically 1 near 0 and with support small enough in order that $\rho\omega'$ is in $\mathscr{C}_c^\infty(X)^{n+1,0}$, the meromorphic extension of*

$$\frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-h} \rho \frac{df}{f} \wedge u \wedge \bar{\omega}' \tag{2}$$

has a pole at $-\alpha - m$.

Proof. Define, for $j \in \mathbb{N}$, the $(n, 0)-$current on X by the formula[15]

$$\langle T_j, \psi\rangle := Pf\big(\lambda = -\alpha - m,\ \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-j} u \wedge \psi\big)$$

where ψ is a test form of type $(1, n + 1)$ which is in $\mathscr{C}_c^\infty(X)$.

Claim. Then we have the following properties for each $j \in \mathbb{N}$

1. $\bar{f} T_{j+1} = T_j$ on X
2. The support of the current $d'T_j$ is contained in $\{0\}$.
3. The support of the current $d''T_j + (\alpha + m + j)d\bar{f} \wedge T_{j+1}$ is contained in $\{0\}$.

proof of the claim. The first assertion is clear.

Let us compute $d'T_j$. Let φ be a $\mathscr{C}_c^\infty(X)$ test form of type $(0, n + 1)$. We have

$$\langle d'T_j, \varphi\rangle := (-1)^n \langle T_j, d'\varphi\rangle = (-1)^n \langle T_j, d\varphi\rangle$$

But for $\Re(\lambda) \gg 1$ the form $|f|^{2\lambda} \bar{f}^{-j} \cdot u \wedge \varphi$ is in $\mathscr{C}_c^1(X)$ and applying Stokes' Formula and meromorphic continuation give

$$0 = \frac{1}{\Gamma(\lambda)} \int_X d\big(|f|^{2\lambda} \bar{f}^{-j} u \wedge \varphi\big) = \frac{(\lambda + \alpha + m)}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-j} \frac{df}{f} \wedge u \wedge \varphi + (-1)^n \langle T_j, d'\varphi\rangle$$

because $du = (\alpha + m)\frac{df}{f} \wedge u$ and $d\bar{f} \wedge \varphi \equiv 0$. Then we obtain[16]

$$\langle d'T_j, \varphi\rangle = Res\Big(\lambda = -\alpha - m,\ \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-j} \frac{df}{f} \wedge u \wedge \varphi\Big). \tag{3}$$

This gives our assertion 2 because we know that the poles of the meromorphic extension of $\frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-j} \square$ at points in $-\alpha + \mathbb{Z}$ are supported by the origin, thanks to our hypothesis $H(\alpha, 1)$.

In an analogous way let us compute $d''T_j$; let ψ be a $\mathscr{C}_c^\infty(X)$ test form of type $(1, n)$. We have:

$$\langle d''T_j, \psi\rangle := (-1)^n \langle T_j, d''\psi\rangle = (-1)^n \langle T_j, d\psi\rangle$$

But for $\Re(\lambda) \gg 1$ the form $|f|^{2\lambda} \bar{f}^{-j} u \wedge \psi$ is in $\mathscr{C}_c^1(X)$ and so:

$$d\big(|f|^{2\lambda} \bar{f}^{-j} u \wedge \psi\big) = (\lambda - j)|f|^{2\lambda} \bar{f}^{-j-1} d\bar{f} \wedge u \wedge \psi + (-1)^n |f|^{2\lambda} \bar{f}^{-j} u \wedge d\psi$$

because the type of du as well as the type of $df \wedge u$ is $(0, n + 1)$. Then Stokes' Formula and the meromorphic continuation give

$$\langle d''T_j + (\alpha + m + j)d\bar{f} \wedge T_{j+1}, \psi\rangle = Res\Big(\lambda = -\alpha - m,\ \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-j-1} d\bar{f} \wedge u \wedge \psi\Big).$$

This proves the assertion 3, again thanks to our hypothesis $H(\alpha, 1)$.

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[15]Here $Pf(\lambda = \lambda_0, F(\lambda))$ denotes the constant term in the Laurent expansion at $\lambda = \lambda_0$ of the meromorphic function $F(\lambda)$.

[16]Here $Res(\lambda = \lambda_0, F(\lambda))$ denotes the residue at $\lambda = \lambda_0$ of the meromorphic function $F(\lambda)$.

Now we argue by contradiction and we assume that for each $j_0 \in \mathbb{N}$, the current $d'T_{j_0}$ induces the zero class in the conjugate of the space $H^{n+1}_{[0]}(X, \mathcal{O}_X)$. This means that there exists a $(n,0)-$current Θ_{j_0} with support $\{0\}$ satisfying $d'\Theta_{j_0} = d'T_{j_0}$ on X. Then, as we have $\bar{f}^k T_{j_0} = T_{j_0-k}$ for any $k \in \mathbb{N}$ thanks to point 1 proved above, we obtain that

$$d'\bar{f}^k\Theta_{j_0} = \bar{f}^k d'\Theta_{j_0} = \bar{f}^k T_{j_0} = T_{j_0-k}.$$

Now we fix $j_0 \gg 1$ and define, for each $j \leq j_0$, $\Theta_j := \bar{f}^{j_0-j}\Theta_{j_0}$. So for any such $j \leq j_0$ this gives $d'\Theta_j = d'T_j$.

Now we use Lemma C_1, C_2 and Lemma D in [4] and Lemma C'_1, C'_2 in [3] in the case $\alpha = 1$, for the family of currents $\tilde{T}_j := T_j - \Theta_j$ for $j \leq j_0$ to obtain a contradiction.

These currents satisfy:

(1) $d'\tilde{T}_j = 0$ on X.
(2) $d''\tilde{T}_j + (\alpha + m + j)d\bar{f} \wedge \tilde{T}_{j+1}$ has its support in $\{0\}$.
(3) The current $\tilde{T}_j$ coincides with $|f|^{-2(\alpha+m)}\bar{f}^{-j}u$ on the Milnor fiber $F_0 = f^{-1}(s_0)$ (these currents are smooth outside Y).

Note that we have $H^p(X \setminus \{0\}, \mathcal{O}_X) = 0$ for $1 \leq p \leq n-1$ which is used for checking the hypothesis of Lemmas C'_1, C'_2 in the case $\alpha = 1$.

Then the cited lemma contradicts our assumption that the class induced by u in $H^n(F_0, \mathbb{C})$ does not vanish.

Thus we obtain that there exists $j_0 \in \mathbb{N}$ such that the class induced by $d'T_{j_0}$ does not vanish in the dual of the space $\overline{\Omega^{n+1}_0}$ of the germs at the origin of anti-holomorphic volume forms on $\mathbb{C}^{n+1}$ (and then for any $j_0 + k$ for $k \in \mathbb{N}$ also). So there exists $\omega' \in \Omega^{n+1}_0$ and $\rho \in \mathscr{C}^\infty_c(X)$ which is identically 1 near 0 and with support small enough in order that $\rho\omega'$ is in $\mathscr{C}^\infty_c(X)^{n+1,0}$ such that

$$\langle d'T_{j_0}, \rho\overline{\omega'}\rangle = Res\Big(\lambda = -\alpha - m,\ \frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar{f}^{-j_0}\frac{df}{f} \wedge u \wedge \rho\overline{\omega'}\Big) \neq 0$$

concluding the proof of the theorem. □

Corollary 4.3.2. *Assume that we have holomorphic forms $u_j \in \Omega^n(X)$ for each integer j in $[-N, p]$ such that*

$$f du_j = (\alpha + m)df \wedge u_j + df \wedge u_{j-1}$$

with the hypothesis that $[du_j] = 0$ in H^{n+1}_0 for each $j \in [-N, 0]$ and such that u_1 induces in $H^n(F_0, \mathbb{C})$ a class which is not 0.
Then there exists $h \in \mathbb{N}$ and $\omega' \in \Omega^{n+1}_0$ such that the meromorphic extension of

$$\frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar{f}^{-h}\rho\frac{df}{f} \wedge u_p \wedge \bar{\omega}' \tag{4}$$

has a pole of order at least equal to p at the point $\lambda = -\alpha - m$.

Proof. For $\Re(\lambda) \gg 1$ the differential form $|f|^{2\lambda}\bar{f}^{-h}\rho u_j \wedge \bar{\omega}'$ is of class $\mathscr{C}^1$ and satisfies

$$d(|f|^{2\lambda}\bar{f}^{-h}\rho u_j \wedge \bar{\omega}') = (\lambda + \alpha + m)|f|^{2\lambda}\bar{f}^{-h}\rho(df/f) \wedge u_j \wedge \bar{\omega}' +$$
$$|f|^{2\lambda}\bar{f}^{-h}\rho(df/f) \wedge u_{j-1} \wedge \bar{\omega}' + |f|^{2\lambda}\bar{f}^{-h}d\rho \wedge u_j \wedge \bar{\omega}'.$$

Then Stokes' Formula and the meromorphic continuation give that for each $q \geq 1$ and each j we have:

$$P_{q+1}\Big((\lambda = -\alpha - m, \frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar{f}^{-h}\rho(df/f) \wedge u_j \wedge \bar{\omega}')\Big) =$$
$$-P_q\Big(\lambda = -\alpha - m, \frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar{f}^{-h}\rho(df/f) \wedge u_{j-1} \wedge \bar{\omega}'\Big),$$

where $P_q(\lambda = \lambda_0, F(\lambda))$ means the coefficient of $(\lambda - \lambda_0)^{-q}$ in the Laurent expansion of the meromorphic function F at the point $\lambda = \lambda_0$.
Then the fact that there exists $h \in \mathbb{N}$ and $\omega' \in \Omega_0^{n+1}$ with (here $P_1 = Res$!)

$$P_1\Big(\lambda = -\alpha - m, \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-h} \rho(df/f) \wedge u_1 \wedge \bar{\omega}'\Big) \neq 0$$

completes the proof of the theorem using the formulas above with $q \in [1, p-1]$ with $j = q+1$. □

To be able to use the previous corollary, the following lemma, combined with Corollary 3.4.2 will be useful.

Lemma 4.3.3. *Let $w_1, \dots, w_p$ be in Ω_0^{n+1} such that the induced class in H_0^{n+1} satisfy the relations:*

$(\star)$ $\qquad a[w_j] = (\alpha + m)b[w_j] + b[w_{j-1}] \quad \forall j \in [1,p] \quad$ with the convention $\quad [w_0] = 0.$

Then there exists an integer N and $u_1 \dots, u_p$ in Ω_0^n such that

$(\star\star)$ $\qquad f du_j = (\alpha + m + N) df \wedge u_j + df \wedge u_{j-1} \quad$ with the convention $\quad u_0 = 0$

and such that we have $[du_j] = (a+b)^N [w_j]$ in H_0^{n+1}.

Proof. Choose, for each $j \in [1,p]$, a form $v_j \in \Omega_0^n$ such that $dv_j = w_j$. Then for each $j \in [1,p]$ the class induced in $\mathcal{H}_0^{n+1}$ by the form

$$f dv_j - (\alpha + m) df \wedge v_j - df \wedge v_{j-1} \quad \text{with the convention } v_0 = 0$$

is of a-torsion in $\mathcal{H}_0^{n+1}$. So there exists an integer N and a germ $t_j \in (Ker\, df)_0^n$ such that, for $j \in [1,p]$, we have

$$f^{N+1} dv_j - (\alpha + m) df \wedge f^N v_j - df \wedge f^N v_{j-1} = f dt_j.$$

This equality may be written

$$f d(f^N v_j + t_j) - (\alpha + m + N) df \wedge (f^N v_j + t_j) - df \wedge (f^N v_{j-1} + t_{j-1}) = 0$$

with the convention $t_0 = 0$, using the fact that $df \wedge t_j = 0$ for each j. Then defining $u_j := f^N v_j + t_j$ for $j \in [1,p]$ concludes the proof since the class induced in H_0^{n+1} by du_j is equal to $a^N[w_j] + Na^{N-1}b[w_j]$, thanks to the equality $a^N + Na^{N-1}b = (a+b)^N$ (see the exercise below). □

Exercise. Show that the commutation relation $ab - ba = b^2$ implies the relation

$$(a+b)^q = a^{q-1}(a + qb) \quad \forall q \in \mathbb{N}^*.$$

Note that $b(a+b)^N = a^N b$ implies that in a geometric (a,b)-module an element x which satisfies $(a+b)^N x = 0$ is null, since a and b are injective.

4.4. **Statements and proofs.** Our first result gives an improvement of the result in [1] but is also a precise converse of this statement. It shows the interest in considering the higher Bernstein polynomials introduced in Section 2.

The content of the following result is the direct part of Theorem 1.2.1 in the introduction.

Theorem 4.4.1. *In the standard situation described in Paragraph 1.2, assume that the hypothesis $H(\alpha, 1)$ is satisfied. Consider a germ $\omega \in \Omega_0^{n+1}$ such that the p-th Bernstein polynomial of the fresco $\mathcal{F}_\omega := B[a]\omega$ in H_0^{n+1} has a root in $-\alpha - \mathbb{N}$. Then there exists $\omega' \in \Omega_0^{n+1}$ and an integer $h \in \mathbb{Z}$ such that the meromorphic extension of the integral*

(A) $$F^{\omega'}_{\omega,h}(\lambda) := \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-h} \rho\omega \wedge \bar{\omega}'$$

has a pole of order at least equal to p at $\lambda = -\alpha - m$ for m a large enough integer, where $\rho \in \mathscr{C}_c^\infty(X)$ is identically 1 near zero.

Remark 4.4.2. *The converse of this result which is the second part of Theorem 1.2.1 in the introduction, proves that, for a germ $\omega \in \Omega_0^{n+1}$, the existence of such ω', h, m giving a pole of order p at the point in $-\alpha - m$ for (A) implies that the p-th Bernstein polynomial of the fresco $\mathcal{F}_\omega = B[a]\omega$ has a root in $-\alpha - \mathbb{N}$.*

This result is a consequence of the Theorem 4.5.4, using the following consequence of Corollary 2.2.5:

- *If the q-th Bernstein polynomial of the geometric (a, b)-module $\mathcal{E}$ has a root in $-\alpha - \mathbb{N}$ then for each $p \in [1, q]$ the p-th Bernstein polynomial of $\mathcal{E}$ has also a root in $-\alpha - \mathbb{N}$.*

For the proof of Theorem 4.4.1 we need the following result.

Proposition 4.4.3. *Assume the hypothesis $H(\alpha, 1)$. Suppose that $u_1 \in \Omega_0^n$ satisfies the relation $f du_1 = (\alpha + m) df \wedge u_1$ for some integer m. If $[du_1]$ is not zero in H_0^{n+1}, then the cohomology class induced by u_1 in $H^n(F_0, \mathbb{C})$ is not zero. So $u_{|F_0}$ induces a class which is an eigenvector of the monodromy for the eigenvalue $\exp(-2i\pi\alpha)$.*

Proof. Thanks to Grothendieck (see [13]), the meromorphic relative de Rham complex of f computes the cohomology of $X \setminus f^{-1}(0)$ and under the hypothesis $H(\alpha, 1)$ the spectral subspace $H^n(F_0, \mathbb{C})_{\exp(-2i\pi\alpha)}$ of the monodromy is isomorphic to the n-th cohomology group of the complex

$$\left(\Omega_0^\bullet[f^{-1}]/df \wedge \Omega_0^{\bullet-1}[f^{-1}], (d - \alpha\frac{df}{f}\wedge)^\bullet\right).$$

If we assume that u_1 induces 0 in $H^n(F_0, \mathbb{C})$, since we have

$$d(f^{-m}u_1) - \alpha\frac{df}{f} \wedge f^{-m}u_1 = 0,$$

there exists $v, w \in \Omega_0^{n-1}[f^{-1}]$ such that

$$dv - \alpha\frac{df}{f} \wedge v = f^{-m}u_1 + df \wedge w.$$

This gives

$$d(f^{-m}u_1) = -df \wedge dw + \alpha df \wedge v/f \quad \text{and then}$$

$$f^{-m}du_1 - m\frac{df}{f} \wedge f^{-m}u_1 = (1 - m/(m+\alpha))f^{-m}du_1 = df \wedge d(-w + \alpha v/f).$$

This implies, since α is in $]0, 1]$, that $[du_1]$ is of a-torsion in $\mathcal{H}_0^{n+1}$ and then 0 in H_0^{n+1}. Contradiction. □

Proof of Theorem 4.4.1. Using Corollary 3.4.2 there exist $[w_1], \dots, [w_p]$ in $\mathcal{F}_\omega$ and an integer $m \in \mathbb{N}$ satisfying the relations:

$$(*) \qquad a[w_j] = (\alpha + m)b[w_j] + b[w_{j-1}] \quad \forall j \in [1, p] \quad \text{with the convention} \quad [w_0] = 0$$

and which are B-linearly independent in $\mathcal{F}_\omega$. Assuming that the Theorem does not hold would imply, thanks to Corollary 4.2.3 and to Corollary 4.3.2, that writing $w_1 = du_1$ with $u_1 \in \Omega_0^n$, the class induced by u_1 in $H^n(F_0, \mathbb{C})$ vanishes.

But this contradicts the hypothesis that $[w_1]$ is not zero in $\mathcal{F}_\omega \subset H_0^{n+1}$, thanks to Proposition 4.4.3. □

The following corollary of Theorem 4.4.1 is clear since we may use a Bernstein identity at the origin to describe the poles of the meromorphic extension of the distribution $\frac{1}{\Gamma(\lambda)}|f|^{2\lambda}\bar{f}^{-h}$ for any $h \in \mathbb{Z}$ (see [5] or [12]).

Corollary 4.4.4. *In the situation of the previous theorem, the existence of a germ $\omega \in \Omega_0^{n+1}$ such that the p-th Bernstein polynomial of the fresco $B[a]\omega \subset H_0^{n+1}$ has a root in $-\alpha - \mathbb{N}$ implies the existence of at least p roots of the reduced b-function $b_{f,0}$ of f at the origin in $-\alpha-\mathbb{N}$ counting multiplicities.*

Sketch of proof. We know, thanks to Theorem 4.4.1, that there exists $\omega' \in \Omega_0^{n+1}$ and $h \in \mathbb{Z}$ such that the meromorphic extension in (A) has a pole of order at least equal to p at $-\alpha - m$ for some integer m. But the use of the Bernstein identity for f at the origin implies that such a pole may occur only when the reduced Bernstein polynomial of f has at least p roots (counting multiplicities) in the set $\{-\alpha - \mathbb{N}\} \cap [-\alpha - m, -\alpha]$. The conclusion follows. $\square$

Remark 4.4.5. *The interest of the higher Bernstein polynomials lies in the fact that the existence of p roots in $-\alpha - \mathbb{N}$ for the reduced Bernstein polynomial $b_{f,0}$ does not implies, in general under our hypothesis, the existence of a pole of order p at some point $\lambda = -\alpha - m$ with $m \in \mathbb{N}$, for the meromorphic extension of $\frac{1}{\Gamma(\lambda)}\int_X |f|^{2\lambda}\bar{f}^{-h}\varphi$ for some test $(n+1, n+1)$-form φ. Theorem 4.4.1 gives a sufficient condition to obtain a pole of order at least p.*

Our next result is an improvement of Theorem 4.4.1.

Theorem 4.4.6. *In the standard situation defined in paragraph 1.2, we assume that the hypothesis $H(\alpha, 1)$ is satisfied. Assume that there exists $\omega \in \Omega_0^{n+1}$ such that $B^p_{\mathcal{F}^{[\alpha]}}$ has a root in $-\alpha-\mathbb{N}$ where $\mathcal{F} := B[a]\omega \subset H_0^{n+1}$ and where $p = d(\mathcal{F}^{[\alpha]})$ is the nilpotent order of the fresco $\mathcal{F}^{[\alpha]}$. Let $-\alpha - m$ be the biggest root of $B^p_{\mathcal{F}^{[\alpha]}}$ in $-\alpha - \mathbb{N}$. Then there exists $\omega' \in \Omega_0^{n+1}$ and $h \in \mathbb{Z}$ such that $F_h^{\omega,\omega'}(\lambda)$ has a pole of order p at the point $-\alpha - m$.*

Recall that, of course, in the previous statement $B^p_{\mathcal{F}}$ denotes the p-th Bernstein polynomial of the fresco $\mathcal{F}$.

Proof. First recall that, thanks to Lemma 6.3.6 in [6], we have, for any geometric (a, b)-module $\mathcal{E}$, the equality $d(\mathcal{E}_{[\alpha]}) = d(\mathcal{E}/\mathcal{E}_{[\neq\alpha]}) = d(\mathcal{E}^{[\alpha]})$.

Let $-\alpha - m$ be the biggest root of $B^p_{\mathcal{F}^{[\alpha]}}$. Then we may choose a J-H. sequence of $\mathcal{F}^{[\alpha]}/S_{p-1}(\mathcal{F}^{[\alpha]})$ such that its last quotient is isomorphic to $E_{\alpha+m}$. This is possible because the fresco $\mathcal{F}^{[\alpha]}/S_{p-1}(\mathcal{F}^{[\alpha]})$ is semi-simple and has $-\alpha - m$ as a root of its Bernstein polynomial (recall that the Bernstein polynomial fo $\mathcal{F}^{[\alpha]}/S_{p-1}(\mathcal{F}^{[\alpha]})$ divides $B^p_{\mathcal{F}^{[\alpha]}}$ which also divides $B^p_{\mathcal{F}}$). Then if Π_0 is the generator of the annihilator of $[\omega]$ in $\mathcal{F}^{[\alpha]}/S_{p-1}(\mathcal{F}^{[\alpha]})$, it may be written $\Pi_0 = (a - (\alpha + m + 1 - k)b)\Pi_0'$ where k is the rank of $\mathcal{F}^{[\alpha]}/S_{p-1}(\mathcal{F}^{[\alpha]})$. Then, choosing a J-H. sequence of $\mathcal{F}$ which begins by a J-H. sequence of $\mathcal{F}_{[\neq\alpha]}$ and ending by the J-H. sequence of $\mathcal{F}^{[\alpha]}$ chosen above, we see that the annihilator of ω in $\mathcal{F}$ may be written as $\Pi = \Pi_2\Pi_1(a-(\alpha+m+1-k)b)\Pi_0'$ with $B[a]/B[a]\Pi_0'$ semi-simple $[\alpha]$-primitive with a Bernstein polynomial having roots strictly less than $-\alpha-m$, with $d(B[a]/B[a]\Pi_1) \leq p-1$, since this fresco is isomorphic to $S_{p-1}(\mathcal{F}^{[\alpha]})$ and with $(B[a]/B[a]\Pi_2)_{[\alpha]} = \{0\}$ since $B[a]/B[a]\Pi_2$ is isomorphic to $\mathcal{F}_{[\neq\alpha]}$.

Now, applying Theorem 4.4.1 we find $\omega' \in \Omega_0^{n+1}$, $h \in \mathbb{Z}$ and $m_1 \in \mathbb{N}$ such that $F_h^{\omega,\omega'}(\lambda)$ has an order p pole at the point $-\alpha - m_1$ and such that the integer m_1 satisfies the property $P(\omega, \omega', p)$.

Using then Proposition 4.2.2 we see that if $m \neq m_1$ we obtain a contradiction with Corollary 4.2.3 because we find a pole of order p at a point $-\alpha - m_1 - k$ for the meromorphic extension of $F_{h-q-1}^{\Pi_0\omega,\omega'}(\lambda)$ where k is the degree in a of Π_0.

So we obtain that $m = m_1$, concluding the proof. $\square$

The following corollaries are obvious consequences of the previous result.

Corollary 4.4.7. *In the standard situation defined in paragraph 1.2 and under the assumption $H(\alpha, 1)$, consider a germ $\omega \in \Omega_0$ and assume that $-\alpha-m$ is the biggest possible pole in $-\alpha-\mathbb{N}$ for any choices of $\omega' \in \Omega_0^{n+1}$ and any $h \in \mathbb{Z}$ for the meromorphic functions $F_h^{\omega,\omega'}(\lambda)$. Then $-\alpha-m$ is the biggest root in $-\alpha-\mathbb{N}$ of the Bernstein polynomial of the fresco $\mathcal{F}_\omega := (B[a]\omega) \subset H_0^{n+1}$.*

Corollary 4.4.8. *In the standard situation defined in paragraph 1.2 and under the assumption $H(\alpha, 1)$, assume that $-\alpha - m$ is the biggest root of the Bernstein polynomial in $-\alpha - \mathbb{N}$ of the geometric (a, b)-module H_0^{n+1}. Then there exists $h \in \mathbb{Z}$ such the meromorphic extension of the distribution $\frac{1}{\Gamma(\lambda)}|f|^{2\lambda}\bar{f}^{-h}$ has a pole at $-\alpha - m$.*

The following consequence of the previous corollary is obvious, since in the case of an isolated singularity at 0 for f it is known (see [15]) that the Brieskorn module coincides with H_0^{n+1} and that its Bernstein polynomial coincides with the reduced Bernstein polynomial $\tilde{b}_f$ of f.

Corollary 4.4.9. *Assume that the germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ of holomorphic function has an isolated singularity at the origin. For $\alpha \in]0, 1] \cap \mathbb{Q}$ let $-\alpha - m$ be the biggest root of the reduced Bernstein polynomial of f in $-\alpha - \mathbb{N}$. Then there exists $h \in \mathbb{N}$ such the meromorphic extension of the distribution $|f|^{2\lambda}\bar{f}^{-h}/\Gamma(\lambda)$ has a pole at $-\alpha - m$.*

Remark 4.4.10. *This result is more precise than the information coming from the general theorem of [4]. The singular part of the asymptotic expansion at $s = 0$ of a fiber integral*

$$s \mapsto \int_{f=s} \rho \frac{\omega}{df} \wedge \frac{\bar{\omega}'}{\bar{d}\bar{f}} \quad \text{with } \rho \in \mathscr{C}_c^\infty(X), \rho \equiv 1 \quad \text{near } 0,$$

is described by the polar structure of the meromorphic extension of its complex Mellin transform[17] *given, for $\lambda \in \mathbb{C}$ and $h \in \mathbb{Z}$ by:*

$$F(\lambda, h) := \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda}\bar{f}^{-h}\rho\omega \wedge \bar{\omega}'.$$

The general result of [4] says only in this situation that for some $h \in [-\alpha - n - 1, -\alpha]$ and for some integer $q \in [-\alpha - n - 1, -\alpha]$ we have a pole at $\lambda = -\alpha + q$ for this Mellin transform. The corollary above makes precise that there is a pole for some $h \in [-\alpha - n - 1, -\alpha]$ at the point $-\alpha - m$ where $-\alpha - m$ is the biggest root of the Bernstein polynomial of f in the set $\{-\alpha - \mathbb{N}\}$.

Question. In the case of an isolated singularity for the eigenvalue $\exp(2i\pi\alpha)$ of the monodromy (so with our hypothesis $H(\alpha, 1)$), is the Bernstein polynomial of the geometric (a, b)-module $(H_0^{n+1})^{[\alpha]} := H_0^{n+1}/(H_0^{n+1})_{\neq[\alpha]}$ (which is the biggest polynomial having its root in $-\alpha - \mathbb{N}$ and dividing the Bernstein polynomial of the (a, b)-module H_0^{n+1}) coincides with the biggest polynomial having its root in $-\alpha - \mathbb{N}$ and dividing the reduced Bernstein polynomial of f at the origin?

4.5. Some improvements of Theorem 3.1.2 in [1].

The goal of this paragraph is to show that, using the higher Bernstein polynomials of the fresco $\mathcal{F}_{f,\omega}$ generated by the class of ω in H_0^{n+1} and the tools introduced above, we improve the main result in [1] Theorem 3.1.2.

We begin by some remarks to make clear the correspondence between our present notation with that used in [1].

[17] This combined a radial Mellin transform of $\mathbb{R}^+$ and a Fourier series on the "argument variable" of complex numbers; see [11].

Remark 4.5.1.

(1) *We use here the notation $H(\alpha, 1)$ with $\alpha \in]0, 1] \cap \mathbb{Q}$ instead of the notation $H(\xi, 1)$ with $\xi \in \mathbb{Q}$.*

(2) *To consider a form $\psi \in \mathscr{C}_c^\infty(\mathbb{C}^{n+1})^{0,n+1}$ with small enough support and such that $d\psi = 0$ in a neighborhood of 0 is equivalent to consider $\rho.\bar{\omega}'$ where ω' is in Ω_0^{n+1} and ρ is a function in $\mathscr{C}_c^\infty(\mathbb{C}^{n+1})$ with small enough support which is identically 1 near the origin.*

Indeed any such ψ may be written as $\psi = \bar{\omega}'$ for some $\omega' \in \Omega^{n+1}$ near the origin thanks to Dolbeault' Lemma, and then $\psi - \rho\bar{\omega}'$ is identically 0 near the origin. So replacing ψ by $\rho\bar{\omega}'$ do not change the poles which may appear in $-\alpha - \mathbb{N}$ for the functions we are looking at, what ever is the choice of $h \in \mathbb{Z}$, thanks to our hypothesis $H(\alpha, 1)$).

For ω, ω' in Ω_0^{n+1} we use the notation $F_h^{\omega,\omega'}(\lambda)$ where the function $\rho \in \mathscr{C}^\infty{}_c(\mathbb{C}^{n+1})$ which is identically 1 near the origin and has a sufficiently small support in order that $\rho\omega \wedge \bar{\omega}'$ is smooth, does not appear in this notation because the poles at points in $-\alpha - \mathbb{N}$ do not depend on the choice of this ρ.

This corresponds to the notation $F_h^\psi(\lambda)$ where ψ is in $\mathscr{C}_c^\infty(X)^{0,n+1}$ is d-closed near the origin (where ω is given in $\Omega^{n+1}(X)$) and with $\psi = \rho\bar{\omega}'$.

(3) *Note also that we change the sign of the integer $h \in \mathbb{Z}$ between these two articles.*

Theorem 4.5.2. *Let $\alpha \in]0, 1]$ be rational number and assume that the holomorphic function f satisfies the hypothesis $H(\alpha, 1)$. Assume that ω in Ω_0^{n+1} is such that there exists an integer $h \in \mathbb{Z}$ and a form $\omega' \in \Omega_0^{n+1}$ for which the function $F_h^{\omega,\omega'}(\lambda)$ has a pole of order $p \geq 1$ at some point ξ in $\{-\alpha - \mathbb{N}\}$. Denote by $\xi_p = -\alpha - m$ the biggest such number ξ in $\{-\alpha - \mathbb{N}\}$ for any choice of ω' and $h \in \mathbb{Z}$. Then the p-th Bernstein polynomial of the fresco $\mathcal{F}_{f,\omega} := B[a]\omega \subset H_0^{n+1}$ has a root in $[-\alpha - m, -\alpha] \cap \{-\alpha - \mathbb{N}\}$.*

Proof. Note $P := P_1P_2$ the annihilator of the class of $[\omega]$ in the $[\alpha]$-primitive quotient

$$\mathcal{F}^{[\alpha]} := \mathcal{F}_{f,\omega}\big/\big(\mathcal{F}_{f,\omega}\big)_{\neq\alpha}$$

of the fresco $\mathcal{F}_{f,\omega} := B[a][\omega]$ inside the (a, b)-module H_0^{n+1} associated to f, where P_2 is the annihilator of $[\omega]$ in $\mathcal{F}^{[\alpha]}\big/S_{p-1}\big(\mathcal{F}^\alpha\big)$. If $F_h^{\omega,\omega'}(\lambda)$ has a pole of order at least equal to p at the point $-\alpha - m$ and if $-\alpha - m$ is not a root of the p-th Bernstein polynomial of $\mathcal{F}^{[\alpha]}$, then $-\alpha - m$ is not a root of the (usual) Bernstein polynomial of the fresco $B[a]\big/B[a]P_2$ which is isomorphic to $S_{p-1}(\mathcal{F}^{[\alpha]})$. In this situation, using Corollary 4.2.5 we see $F_{h+k}^{P_2\omega,\omega'}(\lambda)$ has a pole of order at least equal to p at $-\alpha - m - k$, where k is the rank of the fresco $\mathcal{F}^{[\alpha]}\big/S_{p-1}\big(\mathcal{F}^{[\alpha]}\big)$. But this is impossible, according to Corollary 4.2.6 since the nilpotent order of $S_{p-1}\big(\mathcal{F}^{[\alpha]}\big)$ is $p - 1$. So $-\alpha - m$ is a root of some $(p + j)$-th Bernstein polynomial of $\mathcal{F}^{[\alpha]}$ for some integer j ≥ 0.

The conclusion follows using Corollary 2.2.5. □

The end of Theorem 3.1.2 in [1] is also improved as follows:

Corollary 4.5.3. *In the situation of the previous theorem, let ξ_s, for each integer s in $[1, p]$, be the biggest element in $-\alpha - \mathbb{N}$ for which there exists $h \in \mathbb{Z}$ and $\omega' \in \Omega_0^{n+1}$ such that $F_h^{\omega,\omega'}(\lambda)$ has a pole of order at least equal to s at ξ_s. Then ξ_s is a root of some $(s + j)$-th Bernstein polynomial of the fresco $\mathcal{F}_{f,\omega}^{[\alpha]}$ for some $j \in \mathbb{N}$.*

Moreover, if $\xi_s = \xi_{s+1} = \cdots = \xi_{s+p}$ then there exists at least p distinct values of $j \in \mathbb{N}$ such that ξ_s is root of the $(s + j)$-th Bernstein polynomial of the fresco $\mathcal{F}_{f,\omega}^{[\alpha]}$.

Proof. The proof of the first assertion is analogous to the proof of the theorem above.
The second assertion is an immediate consequence of the fact that the roots of the Bernstein

polynomial of a semi-simple fresco are simple, applied to the successive semi-simple quotients

$$S_d(\mathcal{F}^{[\alpha]})\big/S_{d-1}(\mathcal{F}^{[\alpha]})$$

for $d = s+1, s+2, \dots, s+p$. □

Now we conclude by a result which combines the previous results in order to precise, for a given pair (ω, ω'), the link between the first pole of order at least p in $-\alpha - \mathbb{N}$ with the roots of the q-the Bernstein polynomials for $q \geq p$ of the fresco $\mathcal{F}_{f,\omega}$.

Theorem 4.5.4. *In the standart situation, assume that the hypothesis $H(\alpha, 1)$ is satisfied. Let ω be in Ω_0^{n+1} and define the fresco $\mathcal{F}_\omega := B[a]\omega$. Assume that $p := d(\mathcal{F}_\omega^{[\alpha]})$ is at least equal to 1 and choose*[18] *$\omega' \in \Omega_0^{n+1}$ and $h \in \mathbb{Z}$ such that $F_h^{\omega,\omega'}(\lambda)$ has a pole of order p at some point in $-\alpha - \mathbb{N}$. For each $j \in [1, p]$, let m_j be the integer which has the property $P(\omega, \omega', j)$. Then $-\alpha - m_j$ is a root of at least one of the polynomials $B^{j+q}(\mathcal{F}_\omega^{[\alpha]})$, for some integer q in $\mathbb{N}$.*

Proof. If, for some $j \geq 1$, no root of the polynomials $B^{j+q}_{\mathcal{F}^{[\alpha]}}$ for $q \geq 0$ is equal to $-\alpha - m_j$, we can find a J-H. sequence of $\mathcal{F}_\omega$ such that the corresponding generator of the annihilator of ω is of the form $\Pi := \Pi_2\Pi_1$ where Π_1 has no factor $(a - \lambda_h b)$ with $\lambda_h + h - k$ equal to $\alpha + m_j$, where the nilpotent order of the fresco $(B[a]/B[a]\Pi_2)^{[\alpha]})$ is at most $j-1$ and where k is the rank of $\mathcal{F}_\omega$. Then we conclude as in the previous Theorem using Corollary 4.2.3. □

5. Examples

It is, in general, rather difficult to compute the Bernstein of the fresco associated to a given pair (f, ω), even in the case where f has an isolated singularity.

Nevertheless, in the case where f is a polynomial in $\mathbb{C}[x_0, \dots, x_n]$ having $(n+2)$ monomials, we describe in the article [1], a rather elementary method to obtain an estimation for the Bernstein polynomial of the fresco $\mathcal{F}_{f,\omega}$ associated to a monomial $(n+1)$-form ω.

Of course, when the full Bernstein polynomial has a root of multiplicity $k \geq 2$ then this root is also a root of the j-th Bernstein polynomial for at least k values of j (see Proposition 2.2.3 and Corollary 2.2.5).

But when the Bernstein polynomial has only simple roots, the computation of the higher Bernstein polynomials, even in the special situation of [1], is not easy. We present below some examples where the second Bernstein polynomial is not trivial but where the full Bernstein polynomial has no multiple root.

Proposition 5.0.1. *Let $f(x, y, z) := xy^3 + yz^3 + zx^3 + \lambda xyz$ where $\lambda \neq 0$ is any complex number which is a parameter, and consider the holomorphic forms*

$$\omega_1 := dx \wedge dy \wedge dz,\ \omega_2 := y^3z^2\omega_1,\ \omega_3 = y^7\omega_1,\ \text{and}\ \omega_4 := xy^3\omega_1.$$

Then, in each of these cases, the fresco $\mathcal{F}_{f,\omega_i}$ is a rank 2 theme and the second Bernstein polynomial is equal respectively to $x+1$, $x+4$, $x+5$ and $x+3$.

Moreover, for $i = 3, 4$ the corresponding (full) Bernstein polynomial of the corresponding frescos has only simple roots.

Note that this proposition allows to apply Theorem 4.5.2 to conclude that for each $i \in \{1, 2, 3, 4\}$, there exists some integer h and some germ $\omega'_i \in \Omega_0^3$ such that the meromorphic extension of

$$F_h^{\omega_i,\omega'_i}(\lambda) = \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-h} \rho \omega_i \wedge \bar{\omega}'_i$$

has a double pole at the point λ_i equal to the root of the second Bernstein polynomial of the fresco $\mathcal{F}_{f,\omega_i}$.

[18] such a ω' exists thanks to Theorem 4.4.1.

The proof of this proposition uses several lemmas and the technique of computation described in [1] (see paragraph 4.3.2 in in this article).

Lemma 5.0.2. *Let e be a generator of the rank 2 theme $T := B[a]/B[a](a-2b)(a-b)$ (which is the unique fresco with Bernstein polynomial $(x+1)^2$). Assume that we have three homogeneous polynomials P, Q and R in $B[a]$ of respective degrees $3, 4$ and k with the following conditions*

(1) P, Q and R are monic in a.
(2) Then exists a non-zero constant c such that $P + cQ$ kills e in T.
(3) The Bernstein polynomial of Q[19] is not a multiple of $(x+1)$ or of $(x+2)$.
(4) The Bernstein polynomial of R is not a multiple of $(x+3)(x+2)(x+1)$

Then Re generates a rank two sub-theme in T.

Proof. First, remark that our hypothesis implies that $P = (a-\nu b)(a-2b)(a-b)$ for some $\nu \in \mathbb{C}$ since T is isomorphic to $B[a]\big/B[a](a-2b)(a-b)$. We may realize T in the simple pole asymptotic expansion module with rank 2 which is isomorphic to $T^\sharp$

$$S\Xi_1^{(1)} = \Xi_1^{(2)}\big/\Xi_1^{(0)} \simeq \mathbb{C}[[s]](Log\, s)^2 \oplus \mathbb{C}[[s]Log\, s$$

where a is the multiplication by s and b is defined by $ab - ba = b^2$ and

$$b(Log\, s) = sLog\, s \quad \text{and} \quad b((Log\, s)^2) = s(Log\, s)^2 - 2sLog\, s.$$

Then let us prove that image of e in $\Xi_1^{(2)}\big/\Xi_1^{(0)}$ may be written

$$(@) \qquad e = u(Log\, s)^2 + vs(Log\, s)^2 + ws^3(Log\, s)^2 + s^4\mathbb{C}[[s]](Log\, s)^2 + \mathbb{C}[[s]](Log\, s)$$

where φ is in $\Xi_1^{(2)}\big/\Xi_1^{(0)}$ and where $uvw \neq 0$ are complex numbers.

Remark that the only restrictive condition for writing e as in (@) is the condition $uvw \neq 0$. The condition $u \neq 0$ is easy because we assume that e is a generator of T with Bernstein polynomial $(x+1)^2$; so, writing e as a $\mathbb{C}[[b]]$-linear combination of the $\mathbb{C}[[b]]$-basis $e_1 = (Log\, s)^2$ and $e_2 = Log\, s$ of T, we see that the coefficient of e_1 must be invertible in $\mathbb{C}[[b]]$.

But the condition $(P + cQ)(e) = 0$ implies, since the Bernstein element of T is $(a-2b)(a-b)$, that we may write[20] $P = (a-\nu b)(a-2b)(a-b)$.

The annihilator of $(Log\, s)^2$ in $\Xi_1^{(2)}\big/\Xi_1^{(0)}$ is the ideal $B[a](a-2b)(a-b)$ and so we have $P((Log\, s)^2) = 0$ in T. Since $Q((Log\, s)^2)$ has a non-zero term in $s^4(Log\, s)^2$, because -1 is not a root of B_Q, only the term coming from

$$P(s(Log\, s)^2) = \frac{4-\nu}{24}s^4(Log\, s)^2 \quad \text{modulo } \mathbb{C}[[s]]Log\, s$$

can compensate for this term, in order to obtain the equality $(P + cQ)(e) = 0$. Then $u \neq 0$ implies $v \neq 0$.

But now, the only term which can kill the non-zero term in $s^5(Log\, s)^2$ coming from $Q(vs(Log\, s)^2)$ (using that B_Q is not a multiple of $(x+2)$) can only come from $P(ws^2(Log\, s)^2)$ and this proves that $w \neq 0$. So the assertion (@) holds true.

Now if R is homogeneous of degree k in (a, b) a necessary condition on R such that $R(e)$ has no term in $s^{k+i}(Log\, s)^2$, for $i = 0, 1, 2$, is that B_R divides $(x+1)(x+2)(x+3)$. So, when it is not the case, Lemma 5.2.4 in [6] implies that $R(e)$ is a rank 2 theme and that its second Bernstein polynomial has a (unique) root equal to $-(k+j)$ where $-j$ is the smallest integer among $\{-1, -2, -3\}$ which is not a root of B_R. $\square$

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[19] By definition B_P is defined by the formula

$$(-b)^p B_P(-b^{-1}a) = P$$

where P is in $B[a]$, is homogeneous in (a,b) of degree p and monic in a. This is the Bernstein polynomial of the fresco $B[a]/B[a]P$.

[20] With our choices of f and ω_1, we have $\nu = 3$.

Note that the Lemma above may be easily generalized to many $[\alpha]$-primitive frescos provided that the nilpotent order is known and that it has a generator which admits a enough simple element in $B[a]$ belonging to its annihilator.

Lemma 5.0.3. *In the situation of Proposition 5.0.1, the frescos generated by the forms*

$$\omega_1 := dx \wedge dy \wedge dz,\ \omega_2 := y^3z^2\omega_1,\ \omega_3 := y^7\omega_1,\ \text{and } \omega_4 := xy^3\omega_1$$

generate rank 2 $[1]$*-primitive themes. Their Bernstein polynomials are respectively equal to*

$$(x+1)^2, \quad (x+3)^2 \text{ or } (x+2)(x+3), \quad (x+3)(x+5) \quad \text{and} \quad (x+2)(x+3)$$

and their respective 2*-Bernstein polynomials are* $(x+1)$*,* $(x+3)$*,* $(x+5)$ *and* $(x+3)$*. In the cases* $i = 3, 4$ *there is no double root for the Bernstein polynomial of* $\mathcal{F}_{f,\omega_i}$*.*

Proof. The first point is to show that $\mathcal{F}_{f,\omega_1}$ has rank 2. Since f has an isolated singularity at the origin, we have $Kerdf^n = df \wedge \Omega^{n-1}$ and then $H_0^{n+1}/bH_0^{n+1} \simeq \mathcal{O}_0/J(f)$ and H_0^{n+1} has no b-torsion and no a-torsion. Since f is not[21] in $J(f)$, the image of ω_1 and $a\omega_1 = f\omega_1$ in H_0^{n+1} are linearly independent (over $\mathbb{C}$) and then the rank of $B[a]\omega_1$ is at least equal to 2. Now the computation in [1] (see 4.3.2) shows that the Bernstein polynomial of this fresco divides $(x+1)^3$ (see also the detailed computation below). So it is a theme of rank 2 or 3. But using our main result, the rank 3 would imply that there exists a pole of order 3 for some $F_h^{\omega_1,\omega'}(\lambda)$ which is impossible[22] in $\mathbb{C}^3$. So $\mathcal{F}_{f,\omega_1}$ is a rank 2 theme with Bernstein polynomial $(x+1)^2$. The computation in [1] gives that $P_3 + c\lambda^{-4}P_4$ kills ω_1 in H_0^{n+1}, where

$$P_3 := (a-3b)(a-2b)(a-b), P_4 = (a-(13/4)b)(a-(5/2)b)(a-(7/4)b)a, \text{and } c = 4^4.$$

This is easily obtained by using the technique of the computation of the cited article (see the detailed computation in the Appendix of [2]). Then we may apply Lemma 5.0.2 to see that $\lambda m_1 m_2 \omega_1 = \lambda(a-2b)(a-b)\omega_1$ generates rank 2 themes in H_0^{n+1}. But the identity $\lambda m_1 m_2 = m_4 y^3 z^2$ shows that ω_2 generates also rank 2 fresco in H_0^{n+1}, since

$$m_4\omega_2 = \lambda m_1 m_2 \omega_1 = \lambda(a-2b)(a-b)\omega_1$$

thanks to Lemma 5.0.2 with $R = (a-2b)(a-b)$. Moreover we see that $R(e)$ has a non-zero term in $s^3(Log\, s)^2$.

Since $m_4\omega_2$ generates a rank 2 theme, then ω_2 generates a rank 2 theme also (the rank 3 is again excluded because it would imply that $f^2 \notin J(f)$ which is impossible as explained above).

The technique of computation in [1] applied to ω_2 gives now that the Bernstein polynomial of the rank 2 theme $B[a]\omega_2$ has to divide[23] the polynomial $(x+2)(x+3)^2$.

But the fact that $m_4\omega_2$ has a non-zero term in $s^3(Log\, s)^2$ (and no term in $(Log\, s)^2$ or in $s(Log\, s)^2$) implies, since we have

$$m_4\omega_2 = 4(a-2b)\omega_2,$$

that ω_2 has a non-zero term in $s^2(Log\, s)^2$ and then -3 is a root of the second Bernstein polynomial of the fresco $\mathcal{F}_{f,\omega_2}$. So the Bernstein polynomial is either $(x+2)(x+3)$ or $(x+3)^2$.

We know that the Bernstein polynomial of $\mathcal{F}_{f,\omega_3}$ divides $(x+5)(x+3)(x+2)$ by using the technique of [1].

We know also that $m_1^2 m_4\omega_1 = \lambda m_3\omega_3$ has a non-zero term in $s^5(Log\, s)^2$ (as a consequence of Lemma 5.0.2) and, since $-m_3\omega_3 = (a-2b)\omega_3$ implies that ω_3 has a non-zero term in $s^4(Log\, s)^2$, the second Bernstein polynomial of $\mathcal{F}_{f,\omega_3}$ is $x+5$.

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[21] This point is not so easy to check directly. But the rank is not 1 since this would implies that this fresco has a simple pole and the argument used in Lemma 5.0.2 gives then a contradiction.

[22] This would give an order 4 pole for the meromorphic continuation of $|f|^{2\lambda}$!

[23] This computation gives that $Q_3 + d\lambda^{-4}Q_4$ kills ω_2 in H_0^{n+1}, with $Q_3 := (a-4b)(a-4b)(a-3b)$.

Note that the Bernstein polynomial of the fresco $\mathcal{F}_{f,\omega_3}$ has two simple roots.

The last case is similar, since we know that $m_1\omega_1$ has a non-zero term in $s^2(Log\, s)^2$. So our assertion is consequence of the estimation of the Bernstein polynomial. $\square$

The reader may find more details on the previous computations in [2].

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