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NON-SMOOTHABLE CURVE SINGULARITIES

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Abstract. For curves singularities all smoothing components of the deformation space have the same dimension, but there can be components of different dimensions. We are interested in the question of what the generic singularities are that appear in the fibre over a component. To this end we revisit the known examples of non-smoothable singularities and study their deformations.

There are two general methods available to show that a curve is not smoothable. In the first method one exhibits a family of singularities of a certain type and then uses a dimension count to prove that the family cannot lie in the closure of the space of smooth curves. The other method uses the semicontinuity of a certain invariant, related to the Dedekind different. This invariant vanishes for Gorenstein singularities, so in particular for smooth curves. With these methods and computations with computer algebra systems we study monomial curves and cones over point sets in projective space.

We also give new explicit examples of non-smoothable singularities. In particular, we find non-smoothable Gorenstein curve singularities. The cone over a general self-associated point set in $\mathbb{P}^{g-2}$ is not smoothable if g is at least 11, as then the point set can not be a hyperplane section of a canonical curve of genus g.

Introduction

Deformation spaces of singularities can be very singular [43]. Reduced curve singularities seem to be better behaved, as the dimension of a smoothing component of the versal deformation, that is a component over which the general fibre is smooth, is an invariant of the singularity [9]. But if the singularity deforms into a non-smoothable curve singularity, there are in general also components of other dimensions.

One of the purposes of this paper is to investigate which singularities can occur as general fibre over a (non smoothing) component. Such singularities may be called generic singularities. To this end we revisit the known examples of non-smoothable singularities and study their deformations. We also give new explicit examples of non-smoothable singularities. In particular, we find non-smoothable Gorenstein curve singularities.

The existence of non-smoothable curves was first shown by Mumford [26]. He constructed families of curves with one irreducible singularity, such that the general element of the family is not smoothable, basically because the family is too large to come from a closure of the moduli space of smooth curves. New examples, of lines through the origin, were given by Pinkham [29]. These examples were treated and extended by Greuel [14, 15], see also the survey [16], using a local argument. The dimension of a smoothing component can be expressed in terms of invariants of the curve singularity C, see Proposition 1.1; we call this number the *Deligne number* $e(C)$. The idea is now to exhibit a large family of singularities $\pi\colon C_T \to T$ with singular section, such that for $t \neq 0$ the singularity $(C_t, \sigma(t))$ is not isomorphic to $(C_0, \sigma(0))$. Here large means that the dimension of T is at least the Deligne number $e(C_0)$. If $(T, 0)$ is irreducible, then the general C_t is not smoothable, because the image of T in the versal deformation has dimension $\dim T \geq e(C_0)$, but cannot be a smoothing component, as there are no smooth fibres

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over T. For quasi-homogeneous curves the number e can be expressed in familiar invariants of the curve, making this an effective criterion.

This *large family argument* shows the existence of non-smoothable singularities, but does not give specific examples. The first such examples of monomial curves are due to Buchweitz [4]. He observed that a necessary condition for smoothability, in fact for deforming into a Gorenstein singularity, is that the length of the Dedekind different is at most 2δ, something conjectured, or at least seen as a possibility, to hold for all singularities by Herzog [17]. More generally, Buchweitz defined a k-th normalised Dedekind invariant. Furthermore he showed how to compute this invariant for monomial curves in terms of the semigroup. This has applications to Weierstrass points. A semigroup is the semigroup of a Weierstrass point on a smooth projective curve, if the curve occurs as a deformation of the completion of the corresponding monomial curve in an appropriate weighted projective space, with the Weierstrass point at infinity. The non-smoothability condition is thus also a condition that the semigroup cannot be a Weierstrass semigroup, and this application can be proven directly from Riemann-Roch. In this form Buchweitz' criterion occurs most often in the literature, but the original criterion applies more generally.

These two methods to prove that singularities are not smoothable are the only known general methods. In certain cases one method succeeds, in other cases the other. For not too complicated singularities direct computation of infinitesimal deformations with computer algebra systems, in particular Singular [8] and Macaulay2 [13], can also be used in studying smoothability. Experimentation with such singularities has led to conjectures, which can be found throughout this paper. Not too complicated means in practice quasi-homogeneous. The $\mathbb{G}_m$-actions vastly simplifies computations. Also the occurring invariants of singularities are easier to compute in the quasi-homogeneous case. For monomial curves the generators of the defining ideal are particular simple, as they are binomials. We do give examples of non quasi-homogeneous non-smoothable curve singularities, but they occur as deformations of non-smoothable quasi-homogeneous ones. If a singularity is not smoothable, then by openness of versality every singularity into which it deforms, is also not smoothable.

The Buchweitz criterion does not give any non-trivial condition for Gorenstein curves and therefore it cannot be used to show that symmetric semigroups are not Weierstrass semigroups. A double cover construction yields symmetric non-Weierstrass semigroups [41]. We study deformations of the simplest example and show that it deforms into a Gorenstein curve consisting of lines through the origin, which is not smoothable. This singularity is the cone over a self-associated point set, a concept introduced by Coble [7]. The condition that the general cone is smoothable is that the general self-associated point set is a hyperplane section of a canonical curve. The large family argument shows that this is in general not the case if the genus of the curves is at least 11.

Besides showing that the general singularity of a certain type is not smoothable, we give explicit examples of non-smoothable curve singularities. We mention in particular a non-smoothable Gorenstein curve singularity (Proposition 4.2), an irreducible smoothable but not negatively smoothable quasi-homogeneous curve (Example 2.13) and a singularity where the dimension of the base space is less than the Deligne number e (Example 1.9).

The evidence collected in this paper leads to the conjecture that generic singularities have only smooth branches. The tangents to these branches are not necessarily in general position.

1. Preliminaries

1.1. Let C be a reduced affine curve over an algebraically closed field $\mathbf{k}$ of characteristic 0, lying in $\mathbb{A}^n$, and let $0 \in C$ be a closed point. We denote by $\mathcal{O} = \mathcal{O}_{C,0}$ its local ring with maximal ideal $\mathfrak{m}$. Let $n\colon (\overline{C}, n^{-1}(0)) \to (C,0)$ be the normalisation with semi-local ring $\overline{\mathcal{O}} = n_*\mathcal{O}_{\overline{C},n^{-1}(0)}$. It is the integral closure of $\mathcal{O}$ in its total ring of fractions K. Then the δ-invariant of $(C,0)$, sometimes called the degree of singularity or (mostly for plane curves) the number of virtual double points, is $\delta = \delta(C,0) = \dim_k \overline{\mathcal{O}}/\mathcal{O}$. For reducible curves $C_1 \cup C_2$ the δ-invariant can be computed as

$$\delta(C_1 \cup C_2) = \delta(C_1) + \delta(C_2) + (C_1 \cdot C_2),$$

where the intersection multiplicity $(C_1 \cdot C_2)$ is given by $(C_1 \cdot C_2) = \dim_k \mathcal{O}_n/(I_1 + I_2)$ with I_1 and I_2 the ideals of C_1 and C_2 in the local ring $\mathcal{O}_n$ of $(\mathbb{A}^n, 0)$.

The conductor ideal is $\mathcal{C} = \operatorname{Ann}_{\overline{\mathcal{O}}}(\overline{\mathcal{O}}/\mathcal{O}) = \{f \in \overline{\mathcal{O}} \mid f\overline{\mathcal{O}} \subset \mathcal{O}\}$ and $c = \dim_k \overline{\mathcal{O}}/\mathcal{C}$ is its multiplicity.

Let $\omega = \omega_{C,0}$ be the dualising module of $(C,0)$. It can be described by Rosenlicht differentials: if Ω is the module of Kähler differentials on $(C,0)$ and $\overline{\Omega} = n_*\Omega_{\overline{C},n^{-1}(0)}$ the module of differentials on the normalisation, then $\omega = \{\alpha \in \overline{\Omega} \otimes K \mid \sum_{P \in n^{-1}(0)} \operatorname{res}_P(\alpha) = 0\}$ [35, IV.9]. Composing the exterior derivation $d\colon \mathcal{O} \to \Omega$ with the map $\Omega \to \overline{\Omega} \hookrightarrow \omega$ gives a map $d\colon \mathcal{O} \to \omega$, which allows to define the Milnor number as $\mu = \dim_k \omega/d\mathcal{O}$ [6]. It satisfies Milnor's formula $\mu = 2\delta - r + 1$, where r is the number of branches. We say that the genus of the curve singularity is $g = \delta - r + 1$. Over the complex numbers the genus is the genus of the Milnor fibre in a smoothing and the first Betti number of the Milnor fibre is $\mu = 2\delta - r + 1$.

Let $(C,0)$ be embedded in $\mathbb{A}^n$ and write $\mathcal{O}_n$ for the local ring of $\mathbb{A}^n$ at the origin. Let $I = \langle f_1, \dots, f_k \rangle$ be the ideal of $(C,0)$. Then there is a minimal free resolution

$$0 \longleftarrow \mathcal{O} \longleftarrow \mathcal{O}_n \longleftarrow \mathcal{O}_n^k \longleftarrow \mathcal{O}_n^l \longleftarrow \cdots \longleftarrow \mathcal{O}_n^t \longleftarrow 0$$

of length $n-1$. The rank of the last free module is the Cohen-Macaulay type t of the curve. It is also the number of generators, $\dim_k \omega/\mathfrak{m}\omega$, of the dualising module. In fact, the dual of the free resolution gives a resolution of the dualising module.

1.2. A deformation of $(C,0)$ over $(S,0)$ consists of a flat morphism $\pi\colon (C_S,0) \to (S,0)$ together with an isomorphism of $(C,0)$ with the special fibre $(C_0,0)$ of π. Often we identify $(C,0)$ with $(C_0,0)$. Flatness can be characterised by the property that every free resolution of $\mathcal{O}$ lifts to a free resolution of $\mathcal{O}_{C_S,0}$ over the local ring of $\mathbb{A}^n \times S$. It suffices that every relation $\sum f_i r_i$ between the generators lifts to a relation $\sum F_i R_i$ between the generators $\langle F_1, \dots, F_k \rangle$ of the ideal of $(C_S,0)$.

There exists a formally versal formal deformation C_B with B the spectrum of a complete local $\mathbf{k}$-algebra with $\mathbf{k}$ as residue field, see e.g., [2]. A component E of the deformation space B is called a smoothing component if "the generic fibre is smooth", that is, if the image of the formal scheme giving the singular locus does not contain the generic point of E. For curve singularities Deligne has given a formula for the dimension of smoothing components [9, Thm. 2.27].

Let $\Theta = \operatorname{Hom}_{\mathcal{O}}(\Omega, \mathcal{O})$ be the module of derivations on $(C,0)$ and let $\overline{\Theta} = \operatorname{Hom}_{\overline{\mathcal{O}}}(\overline{\Omega}, \overline{\mathcal{O}})$ be the module of derivations on $(\overline{C}, n^{-1}(0))$. Then define $m_1 = \dim_k \overline{\Theta}/\Theta$.

Proposition 1.1 (Deligne)**.** *Every smoothing component E of $(C,0)$ has dimension equal to the Deligne number $e = 3\delta - m_1$.*

The formula also holds in finite characteristic, but as then not every derivation of $\mathcal{O}$ lifts to a derivation of $\overline{\mathcal{O}}$, the definition of m_1 has to be modified. The derivations of $\mathcal{O}$ and $\overline{\mathcal{O}}$ lift to derivations of the total ring of fractions K and under these embeddings Θ and $\overline{\Theta}$ are commensurable, allowing to define $m_1 = \dim_k \overline{\Theta}/(\Theta \cap \overline{\Theta}) - \dim_k \Theta/(\Theta \cap \overline{\Theta})$.

1.3. Many computations simplify considerably if the singularity is quasi-homogeneous. In that case one can work degree by degree. In computer algebra systems computations with weighted homogeneous equations are much faster.

In the quasi-homogeneous case the multiplicative group $\mathbb{G}_m$ of units of $\mathbf{k}$ acts diagonally, for suitably chosen coordinates $(x_1, \dots, x_n)$ on $\mathbb{A}^n$, by $(g, x_i) \mapsto g^{w_i} x_i$; the integers w_i are called weights. The coordinate ring P/I, where $P = \mathbf{k}[x_1, \dots, x_n]$, is a graded ring. We continue to denote it by $\mathcal{O}$. The generators of the ideal $I = \langle f_1, \dots, f_k \rangle$ can be chosen to be equivariant, with f_j of degree q_j. A quasi-homogeneous deformation can be described by power series F_j of the same degree q_j, lifting the f_j, if one assigns appropriate weights to the deformation variables. If all the deformation variables have positive weights, then the F_j are polynomials. In that case the deformation describes a deformation of the closure of the curve in a weighted projective space, which is trivial at infinity. If we also allow weight zero, then the deformation still globalises. A deformation with deformation variables of positive weight is said to be a deformation of negative weight, as the degree of the terms in the space variables x_i decreases. This is in particular true for trivial deformations, those induced by coordinate transformations. We note that Pinkham [29, 30] uses the Bourbaki definition of positive and negative, according to which 0 is both positive and negative. It is therefore customary to call a weighted homogeneous curve negatively smoothable if there exists a smoothing that globalises to a smoothing of the closure of the curve, in the same weighted projective space.

In the quasi-homogeneous case, the formula for the dimension of smoothing components simplifies.

Proposition 1.2 (Greuel [15])**.** *For a quasi-homogeneous curve $(C, 0)$ the Deligne number e is $e = \mu + t - 1 = 2\delta - r + t$.*

Most known examples of non-smoothable curves fall under one of two extremes, either they are irreducible, or have as many branches as possible, each of them smooth.

1.3.1. *Monomial curves.* Let $\mathcal{S} = \langle a_1, \dots, a_n \rangle$ be a numerical semigroup (for the general theory of numerical semigroups, see the book [33]). The semigroup ring $\mathbf{k}[\mathcal{S}]$ can be identified with the ring $\oplus_{a \in S} \mathbf{k} t^a$, a ring generated by monomials. The curve $C_{\mathcal{S}} = \operatorname{Spec} \mathbf{k}[\mathcal{S}]$ is the associated monomial curve. The complement $\mathbb{N} \setminus \mathcal{S} =: \mathrm{L}$ is the set of gaps. The number of gaps is called the genus of the semigroup. It is equal to $g = \delta$ of the associated monomial curve. The largest gap is called the Frobenius number $F(\mathcal{S})$ and $c = F(\mathcal{S}) + 1$ is the conductor: the element t^c generates the conductor ideal of $\mathbf{k}[\mathcal{S}]$.

The dualising module ω is easy to describe. It contains $\overline{\Omega}$ and $\omega/\overline{\Omega}$ is the vector space generated by the differentials $\frac{dt}{t^{l+1}}$, $l \in \mathrm{L}$. The type of the curve is the number of generators of ω, which is the number of gaps l such that $l + n \in \mathcal{S}$ whenever whenever $n \in \mathcal{S} \setminus 0$.

1.3.2. *Lines though the origin.* A special case of the other extreme concerns homogeneous singularities. In this case the curve is the cone over a set Γ of points in $\mathbb{P}^{n-1}$, and consists of a set of lines through the origin in $\mathbb{A}^n$. We use the notation L^n_r for any homogeneous curve consisting of r lines in $\mathbb{A}^n$. We are mostly interested in lines in general position; for the definition we follow the definitions in [15] and [12].

Definition 1.3. A set Γ of r points in $\mathbb{P}^{n-1}$ is said to be in *general position*, if it imposes the maximal number of conditions on forms of degree d, for all $d \geq 1$. The points are in *uniform position* if every subset is in general position. A set of r lines through the origin in $\mathbb{A}^n$ is in general (uniform) position if the corresponding point set is in general (uniform) position.

The conditions can also be formulated with the d-tuple Veronese embedding

$$v_d \colon \mathbb{P}^{n-1} \to \mathbb{P}^{N-1}, \text{ where } N = \binom{n+d-1}{d}.$$

The point set $\{p_1, \dots, p_r\}$ is in general position, if for all d the points $v_d(p_1), \dots, v_d(p_r)$ span a linear subspace of dimension $\min(r, N) - 1$. After choosing homogeneous coordinates for the points and an ordering of the monomials of degree d we can write the matrix A_d of homogeneous coordinates of the image of the points under the d-tuple Veronese embedding. General position just means that the matrix A_d has maximal rank, namely rank $\min(r, N)$.

As the dimension of the space of forms of degree d on $\mathbb{P}^{n-1}$ is $\binom{n+d-1}{d} = \binom{n+d-1}{n-1}$, the Hilbert function of the homogeneous coordinate ring $\mathcal{O}$ of Γ in general position is

$$H(l) = \min\left(r, \binom{n+l-1}{n-1}\right).$$

If $r \geq \binom{n+l-1}{n-1}$, there are no forms of degree l in the homogeneous ideal of the points. We can construct a uniform set Γ_r of r points by adding one point at a time; note that this does not hold for general position. This is a practical way to do experiments with (cones over) "random" point sets.

We compute δ for a curve L^n_r of r lines through the origin in $\mathbb{A}^n$ in uniform position. We add a line L to a L^n_{r-1} in uniform position. Let d be the lowest degree of a form vanishing on L^n_{r-1}. Then L^n_r imposes independent conditions on forms of degree d, if there exist a form in the ideal of L^n_{r-1} that does not vanish on L. This gives that the intersection multiplicity $(L \cdot L^n_{r-1}) = d$. Therefore δ grows with d. Using the formula for $H(l)$ we see that the lowest degree d of a form vanishing on L^n_{r-1} is determined by the conditions $\binom{n+d-2}{d-1} < r \leq \binom{n+d-1}{d}$. We introduce a notation for this number.

Definition 1.4. Given two integers $n < r$, let $d(n, r)$ be the unique integer d such that

$$\binom{n+d-2}{d-1} < r \leq \binom{n+d-1}{d}.$$

With this notation, the discussion above gives the following result.

Lemma 1.5. *Let L^n_r be a curves of lines in uniform position and set $d = d(n, r)$. Then*

$$\delta(L^n_r) = dr - \binom{n+d-1}{d-1}.$$

This result holds under the weaker condition of general position [15, Lemma 3.3]. It can also be proved using the generalisation to higher embedding dimension of Noether's formula for the δ-invariant [39].

We describe the homogeneous parts of the dualising module ω. As the conductor ideal $\mathcal{C}$ is equal to $\mathfrak{m}^d$, where $d = d(n, r)$, the lowest degree part is ω_{-d}. The condition that a rational differential form $\alpha = \sum a_i \frac{dt_i}{t_i^l} \in \overline{\Omega} \otimes K_{-l}$ lies in ω_{-l} is that $\sum \operatorname{res} m\alpha = 0$ for each monomial of degree $l - 1$. This gives a system of homogeneous linear equations on the coefficients of α, whose matrix is the $\binom{n+l-2}{l-1} \times r$ matrix A_l with entries the coordinates of the r points under the Veronese embedding v_{l-1}. This matrix has maximal rank if the points are in general position. Therefore the dimension of ω_{-l} is $r - \binom{n+l-2}{l-1}$. The minimal number of generators for ω as $\mathcal{O}$-module is obtained when multiplication with $\mathcal{O}_1$ generates a subspace of ω_{-d+1} of largest possible dimension. The resulting formula for the type of the singularity was conjectured by Roberts [32] and the conjecture was proved by Trung and Valla [42] and Lauze [24], to hold for at least one curve. Therefore it holds for a generic L^n_r in uniform position; in fact a generic L^n_r is in uniform position. Therefore we have:

Proposition 1.6. *For generic L^r_n write $r = \binom{n+d-2}{d-1} + s$, where $d = d(n, r)$. The type t is $\max\{s, \binom{n+d-3}{d-1} - (n-2)s\}$.*

In fact it had been conjectured that all Betti numbers of the minimal free resolution have the minimal possible value given the Hilbert function. For systematic counterexamples see [10].

1.3.3. *Non-smoothable L^n_r.* By a result of Greuel [15, 3.5] (see also Corollary 1.11), a curve L^n_r of lines in general position has no non-trivial deformations of positive degree if $n < r \leq \binom{n+1}{2}$, so it is smoothable if and only if the points lie in the closure of the space of hyperplane sections of (non-special) curves of genus $g = r - n$ in $\mathbb{P}^n$. One calls an embedded curve non-special if the linear system of hyperplane sections is non-special. For being a hyperplane section there is a criterion in terms of the Gale transform of the points, that is in terms of associated point sets.

Definition 1.7. Let $\Gamma = \{p_1, \dots, p_r\}$ be a set of r ordered points in $\mathbb{P}^{n-1}$, whose homogeneous coordinates are given by a $n \times r$-matrix P. The Gale transform of Γ is a set of r points $\{q_1, \dots, q_r\}$ in $\mathbb{P}^{r-n-1}$, given by a $(r-n) \times r$ matrix Q satisfying $PQ^t = 0$. The two point sets are said to be associated.

This concept was introduced by Coble [7]. For a detailed study, including an extensive historical overview, we refer to [11]. If no $r-1$ points lie in a hyperplane we can normalise the matrix P in the form (I, A) with I the $n \times n$ identity matrix and no row of A is zero. Then a solution of $PQ^t = 0$ is $Q = (-A^t, I)$.

We now have the following criterion [37, Thm. 9], which is a special case of [11, Cor. 3.2].

Theorem 1.8. *Let $\Gamma = \{p_1, \dots, p_r\}$ be a hyperplane section of a non-special curve C of genus g in $\mathbb{P}^{r-g}$. Then the Gale transform of Γ lies on the canonical image of C in $\mathbb{P}^{g-1}$.*

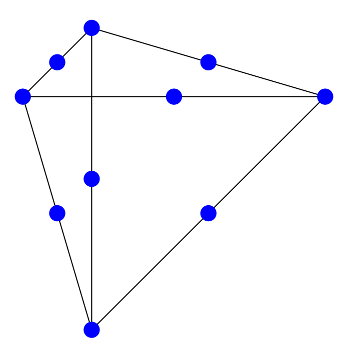
This result enables us to construct simple explicit examples of non-smoothable curves. The lowest possible embedding dimension is obtained for genus $g = 4$. The general canonical curve is the complete intersection of a quadric and a cubic, so for smoothability of the cone over a point set the points in the Gale transform have to lie on a quadric. The minimal number of points r for which this is in general not the case is 10. Then $n = r - g = 6$. Choosing a set of ten points on a quadric leads to a smoothable L^6_{10} that deforms into a non-smoothable one and therefore has a reducible base space.

Example 1.9. As a set of ten points that do not lie on a quadric we take the four vertices of the coordinate tetrahedron and the six midpoints of the edges.

After taking a suitable basis the 11 equations of the Gale transform in $\mathbb{P}^5$ are

$$x_1x_4 = x_2x_5 = x_3x_6 = 0, \quad x_1x_2 = x_1x_3 = x_2x_3, \quad x_3x_4 = x_3x_5 = x_4x_5,$$
$$x_5x_6 = x_5x_1 = x_6x_1 \quad \text{and} \quad x_2x_4 = x_2x_6 = x_4x_6.$$

These points are not in uniform position, as each coordinate hyperplane contains seven points. The advantage is that the equations are very simple. A computation with Macaulay2 [13] or Singular [8] shows that T^1 of the corresponding L^6_{10} is concentrated in degree 0 and has dimension 15, while $T^2 = 0$. This shows that the versal deformation is smooth, and that this particular curve has only equisingular deformations. The Deligne-Greuel formula for the dimension of a smoothing component gives $e = 20$. Therefore we have here an example where the dimension of the base space of the versal deformation is less than the Deligne number e.



A smoothable L^6_{10} is obtained by replacing two opposite edge midpoints of the coordinate tetrahedron in $\mathbb{P}^3$ by the center $(1:1:1:1)$ and a point on the quadric through the centre and the remaining eight points, say the point $(1:2:2:4)$ on the quadric $y_1y_4 = y_2y_3$. The resulting L^6_{10} has $\dim T^1 = 21$, with $\dim T^1_0 = 15$ and $\dim T^1_{-1} = 6$, while $\dim T^2 = \dim T^2_{-1} = 6$. The versal deformation has two smooth components, one smoothing component of dimension 20, and the 15-dimensional component of equisingular deformations, intersecting in a 14-dimensional space.

1.4. To compute deformations the first step is to compute the vector space of infinitesimal deformations and the obstruction space, that is T^1 and T^2. In general these vector spaces are part of the theory of cotangent cohomology, whose main properties relevant for the case at hand are summarised in [31].

An elementary definition of T^1 is

$$T^1(C,0) = \operatorname{Coker} \Theta_n \otimes \mathcal{O} \to \operatorname{Hom}_{\mathcal{O}_n}(I,\mathcal{O}).$$

One can compute $\operatorname{Hom}_{\mathcal{O}_n}(I,\mathcal{O})$ from equations and relations. Let f be the row vector of generators of the ideal of C and r the matrix of relations, so $fr = 0$. An infinitesimal deformation is of the form $f + \varepsilon f'$ and flatness requires that $fr = 0$ can be lifted to

$$(f + \varepsilon f')(r + \varepsilon r') \equiv 0 \pmod{\varepsilon^2}.$$

This says that $r^t(f')^t \cong 0 \pmod f$. Furthermore $\Theta_n \otimes \mathcal{O}$ is the free $\mathcal{O}$-module generated by the derivations $\partial_i = \frac{\partial}{\partial x_i}$. Computing f', that is syzygies over the quotient ring of the transpose of the relation matrix, can be done with a computer algebra system. In particular, to show that a homogeneous singularity has no deformations of (strictly) negative weight, it suffices to compute generators of the normal sheaf $\operatorname{Hom}_{\mathcal{O}_n}(I,\mathcal{O})$ up to degree -1. If there as many generators as the number of variables, then there are no deformations of (strictly) negative weight, besides the trivial ones coming from the ∂_i.

For monomial curves the elementary approach suffices to give a dimension formula for the graded parts of $T^1(\mathbf{k}[\mathcal{S}])$. The ideal of $C_{\mathcal{S}} := \{(t^{a_1},\dots,t^{a_n})\,;\, t \in \mathbf{k}\} \subset \mathbb{A}^n$ can be generated by binomials f_i of the form

$$f_i := x_1^{\alpha_{i1}} \dots x_n^{\alpha_{in}} - x_1^{\beta_{i1}} \dots x_n^{\beta_{in}},$$

with $\alpha_i \cdot \beta_i = 0$. As usual, the weight of f_i is $q_i := \sum_j a_j\alpha_{ij} = \sum_j a_j\beta_{ij}$. For each i, let $v_i := (\alpha_{i1} - \beta_{i1},\dots,\alpha_{in} - \beta_{in})$ be the vector in $\mathbf{k}^n$ induced by f_i.

Theorem 1.10 ([3, Thm. 2.2.1])**.** *Let $\mathbf{k}[\mathcal{S}]$ be the semigroup ring of a numerical semigroup $\mathcal{S} = \langle a_1,\dots,a_n\rangle$. For $l \in \mathbb{Z}$ let $A_l := \{i \in \{1,\dots,n\} \mid a_i + l \notin \mathcal{S}\}$ and let V_l be the vector subspace of $\mathbf{k}^n$ generated by the vectors v_i such that $d_i + l \notin \mathcal{S}$. Then for $l \notin \operatorname{End}(\mathcal{S})$*

$$\dim T^1(\mathbf{k}[\mathcal{S}])_l = \#A_l - \dim V_l - 1,$$

while $\dim T^1(\mathbf{k}[\mathcal{S}])_s = 0$ *for* $s \in \operatorname{End}(\mathcal{S})$ *.*

For reducible quasi-homogeneous singularities we need a more sophisticated description of T^1. We follow [31] and [14].

Let as before K be the total ring of fractions of $\mathcal{O}$. The exact sequence

$$0 \to \mathcal{O} \to K \to K/\mathcal{O} \to 0$$

gives an exact sequence of cotangent modules

$$0 \to T^0(\mathcal{O},\mathcal{O}) \to T^0(\mathcal{O},K) \to T^0(\mathcal{O},K/\mathcal{O}) \to T^1(\mathcal{O},\mathcal{O}) \to 0,$$

where $T^1(\mathcal{O},K) = 0$ because $\mathcal{O}$ is generically smooth over $\mathbf{k}$, and $T^0(\mathcal{O},K) \cong K$. Here $T^1(\mathcal{O},\mathcal{O})$ is the previously defined $T^1(\mathcal{O})$. For any $\mathcal{O}$ module M one has

$$T^0(\mathcal{O},N) = \operatorname{Der}_k(\mathcal{O},N) = \operatorname{Hom}_{\mathcal{O}}(\Omega,N).$$

If $I = \langle f_1,\dots,f_k\rangle$ is the ideal of $(C,0)$, then

$$T^0(\mathcal{O},K/\mathcal{O}) = \operatorname{Hom}_{\mathcal{O}}(\Omega,K/\mathcal{O}) \subset \operatorname{Hom}_{\mathcal{O}}(\Omega^1_n,K/\mathcal{O}) \cong (K/\mathcal{O})^n$$

is isomorphic to the kernel of the map

$$\partial f\colon (K/\mathcal{O})^n \to (K/\mathcal{O})^k$$

induced by the Jacobi matrix $\left(\frac{\partial f_j}{\partial x_i}\right)$.

For quasi-homogeneous $(C,0)$ all the modules are graded and the maps have degree 0. The module $(K/\mathcal{O})^n$ has the vector fields $\frac{\partial}{\partial x_i}$ as basis, so $(K/\mathcal{O})^n_l = \oplus_i (K_{l+w_i}/\mathcal{O}_{l+w_i})$, while $(K/\mathcal{O})^k_l = \oplus_j (K_{l+q_j}/\mathcal{O}_{l+q_j})$. We embed $\overline{\Omega}$ in K via the map $(dt_1,\dots,dt_r) \mapsto (t_1,\dots,t_r)$. Then for quasi-homogeneous curves the image of Ω is $\mathfrak{m}$ [15, Lemma 2.2]. Therefore

$$T^0(\mathcal{O},\mathcal{O}) \cong \operatorname{Hom}_{\mathcal{O}}(\mathfrak{m},\mathcal{O}) = \{a \in K \mid a\mathfrak{m} \subset \mathcal{O}\}.$$

This gives us the exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{O}}(\mathfrak{m},\mathcal{O})_l \to K_l \to (\ker \partial f)_l \to T^1_l \to 0. \tag{1}$$

We apply this in particular to homogeneous curves, so $w_i = 1$ for all i.

Corollary 1.11. *For homogeneous $(C,0)$ we have $T^1_l = 0$ if $K_{l+1} = \mathcal{O}_{l+1}$. In particular, for L^n_r with $\binom{n+d-2}{d-1} < r \le \binom{n+d-1}{d}$ in general position we have $T^1_l = 0$ for $l \ge d-1$.*

2. Buchweitz criterion

2.1. Buchweitz defined an invariant for curve singularities that varies semicontinously in families. Being published only in a preprint and as second part of his Thèse [4], this invariant is not very well known. The only place in the literature we are aware of, that treats Buchweitz' original approach is [23, § 6]. Buchweitz' definition uses Noether normalisation and differents for finite mappings, in particular the Kähler and Dedekind differents (see e.g. [36, Tag 0DWH]), and actually he defines several invariants. Here we use only the Dedekindian one and use an alternative form [4, Remark after Lemma 3.1].

We embed ω in K via the map $(dt_1,\dots,dt_r) \mapsto (1,\dots,1)$. For any two $\mathcal{O}$-ideals $\mathfrak{a},\mathfrak{b}$ in K containing a non-zero divisor one has

$$\operatorname{Hom}_{\mathcal{O}}(\mathfrak{b},\mathfrak{a}) = \mathfrak{a} : \mathfrak{b} = \{x \in K \colon x\mathfrak{b} \subset \mathfrak{a}\},$$

where the homomorphism is given by multiplication with $x \in K$. As $\mathcal{O} \subset \omega$, we get under this identification $\operatorname{Hom}(\omega,\mathcal{O}) \subset \mathcal{O}$.

Definition 2.1. The k-th normalised Dedekind invariant of the curve $(C,0)$ is

$$d_k = d_k(C) = \dim \operatorname{Coker} \operatorname{Hom}(\omega^{\otimes k},\mathcal{O}) \to \mathcal{O}.$$

The following proposition is in [4] a consequence of the definition. Here we give a direct proof.

Proposition 2.2. *For Gorenstein curves $d_k = 2k\delta$.*

Proof. If C is Gorenstein, the module ω is free, with generator α satisfying $v_P(\alpha) = -c_P$ for all $P \in n^{-1}(0)$, and $\operatorname{Hom}(\omega^{\otimes k},\mathcal{O})$ is generated by the map which sends α^k to $1 \in \mathcal{O}$. This corresponds to an element $f^k \in \mathcal{O}$, where f generates the conductor ideal $\mathfrak{c}$ in $\widetilde{\mathcal{O}}$ and the image of $\operatorname{Hom}(\omega^{\otimes k},\mathcal{O})$ is generated by f^k as $\mathcal{O}$-module. As $\dim \widetilde{\mathcal{O}}/\mathfrak{c} = 2\delta$ for Gorenstein curves [35, IV.11], we get $\dim \mathcal{O}/f^k\mathcal{O} = \dim \widetilde{\mathcal{O}}/\mathfrak{c}^k - \dim \widetilde{\mathcal{O}}/\mathcal{O} + \dim f^k\widetilde{\mathcal{O}}/f^k\mathcal{O} = 2k\delta - \delta + \delta = 2k\delta$. ☐

The essential result proven by Buchweitz is the semicontinuity of the invariant $b_k := d_k - 2k\delta$. His proof uses Noether normalisation. It is written in the context of complex analytic curve germs, but it is remarked that the proof also works in the formal category of complete Noetherian rings.

Theorem 2.3 (Buchweitz)**.** *Let $\pi\colon \mathcal{C} \to T$ be a 1-parameter deformation of a reduced curve singularity C, then $b_k(C) \le b_k(\mathcal{C}_t)$, where $\mathcal{C}_t$ is the generic fibre, and the invariant for $\mathcal{C}_t$ is the sum of the invariants of its singular points.*

Corollary 2.4. *If C is deformable into a curve with at most Gorenstein singularities (in particular, if C is smoothable), then $d_k \le 2k\delta$ for all $k \in \mathbb{N}$.*

Example 2.5. The smallest example of a monomial curve, where $d_1 > 2\delta$, is the curve with semigroup $\langle 13, \dots, 18, 20, 22, 23\rangle$. It occurs already in the original paper by Buchweitz [4]. In this case $c = 26$, and $\delta = 16$. A basis for $\omega_X/\Omega^1_{\widetilde{X}}$ is

$$\frac{dt}{t^{26}}, \frac{dt}{t^{25}}, \frac{dt}{t^{22}}, \frac{dt}{t^{20}}, \frac{dt}{t^{13}}, \dots, \frac{dt}{t^{2}},$$

where the first four are generators of ω, so if $\varphi \in \mathrm{Hom}(\omega, \mathcal{O})$ maps $\frac{dt}{t^{26}}$ to t^l, then $\varphi(\frac{dt}{t^{25}}) = t^{l+1}$, $\varphi(\frac{dt}{t^{22}}) = t^{l+4}$ and $\varphi(\frac{dt}{t^{20}}) = t^{l+6}$. This means that with t^l also t^{l+1}, t^{l+4} and t^{l+6} have to be elements of $\mathcal{O}$. The smallest possible value for l is therefore 14, followed by 16, 22 and all values from 26 on. Such a φ corresponds to $t^{l+26} \in \mathcal{O}$ and therefore $d_1 = 52-3-16 = 33 > 2\cdot 16 = 2\delta$.

We describe some deformations of the curve. Its equations can be given in a redundant way by 2×2 minors of the following matrix, where the dots cannot be filled in for reasons of degree. Only minors that do not contain dots lead to equations.

$$\begin{vmatrix} x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & x_{20} & x_{22} & x_{23} \\ x_{14} & x_{15} & x_{16} & x_{17} & x_{18} & \cdot & \cdot & x_{23} & \cdot \\ x_{15} & x_{16} & x_{17} & x_{18} & \cdot & x_{20} & x_{22} & \cdot & \cdot \\ x_{16} & x_{17} & x_{18} & \cdot & x_{20} & \cdot & x_{23} & \cdot & x_{13}^2 \\ x_{17} & x_{18} & \cdot & x_{20} & \cdot & x_{22} & \cdot & x_{13}^2 & x_{13}x_{14} \end{vmatrix}$$

Perturbations of the entries of the matrix lead to deformations of the curve. In this way we can describe the deformation of lowest degree, by replacing x_{13}^2 with $x_{13}^2 + sx_{13}$ and replacing $x_{13}x_{14}$ with $x_{13}x_{14} + sx_{14}$. The tangent cone of the deformed curve for $s \neq 0$ is obtained by replacing x_{13}^2 with sx_{13}, $x_{13}x_{14}$ with sx_{14}. The resulting equations describe a L^9_{13}. Note that this deformation lies on the cone over a projected rational normal curve, with $x_{13} = u^{10}$, $x_{14} = u^9v$, $x_{15} = u^8v^2$, $x_{16} = u^7v^3$, $x_{17} = u^6v^4$, $x_{18} = u^5v^5$, $x_{20} = u^3v^7$, $x_{22} = uv^9$ and $x_{23} = v^{10}$. The deformation is then the image of the curve $u^{23} + su^{13} - v^{13}$. As the tangent cone has no deformations of positive weight, the deformed singularity is in fact isomorphic to this tangent cone. By semicontinuity of Buchweitz' invariant this is a particular L^9_{13}, for which $d_1 > 2\delta$. A direct computation for $s = 1$ shows that T^1 of this L^9_{13} is concentrated in degree 0, of dimension 24 equal to the number of moduli of points. By Theorem 1.8 we know that the general L^9_{13} is not smoothable. We have here however a rather simple to describe specific example.

The computation of d_1 can be done in general for L^r_n with $r \le \binom{n+1}{2}$.

Proposition 2.6. *Let L^{r-g}_r be the cone over r points in $\mathbb{P}^{r-g-1}$ in uniform position, with $r \ge g + \frac{1+\sqrt{8g+1}}{2}$, and $g \ge 4$. If the L^{r-g}_r is smoothable, then the Gale transform of the r points in $\mathbb{P}^{g-1}$ imposes at most $3g-3$ conditions on quadrics.*

Proof. The assumption on r is equivalent to $r \le \binom{r-g+1}{2}$, so by Lemma 1.5 we have $\delta(L^{r-g}_r) = r+g-1$. Let $\mathcal{I}$ be the image of $\mathrm{Hom}(\omega, \mathcal{O})$ in $\mathcal{O}$. As $\mathcal{O}_2 = K_2$ we have $\mathfrak{m}^4 \subset \mathcal{I} \subset \mathfrak{m}^2$. Let $s = \dim \mathfrak{m}^3/\mathcal{I}$. Smoothability implies $d_1 \le 2\delta$, so $1 + r - g + r + s \le 2r + 2g - 1$, that is $s \le 3g-3$.

Let P be the matrix describing the points and Q the matrix describing the Gale transform, so $PQ^t = 0$. Then Q is also the matrix of coefficients of the generators of ω, considered as elements of $\overline{\Omega} \otimes K$. Multiplying a generator α of ω with $f = (f_1, \dots, f_r) \in \mathcal{O}_3$ leads to a vector of coefficients, which should be a linear combination of the rows of P, so its transpose should lie in the kernel of the matrix Q. We obtain therefore the matrix equation $QFQ^t = 0$, where $F = \mathrm{diag}(f_1, \dots, f_r)$. Considered as linear equations for the $f_1, \dots, f_r$ the coefficient matrix becomes, after deleting identical rows, a $\binom{g+1}{2} \times r$ matrix, where the columns are the coordinates of the r points in the Gale transform under the second Veronese embedding v_2. This is the matrix describing the conditions on quadrics, and s is its rank. □

By this proposition the Gale transform of the r points lies on at least $(g-2)(g-3)/2$ linearly independent quadrics. By Max Noether's Theorem a non-hyperelliptic canonical curve of genus g lies on exactly this number of quadrics [1, p. 117]. So the previous result is weaker than Theorem 1.8, but it suffices to show that the L^6_{10} of Example 1.9 is not smoothable. Therefore this non-smoothability also follows from Buchweitz' criterion.

Example 2.7. The following example of a non-smoothable monomial curve with $d_1 = 2\delta$ but $d_2 = 4\delta + 1$ is due to Komeda [21, Example 2.4 (2)]. Consider the semigroup

$$\langle r, r+1, \ldots, 2r-8, 2r-7, 2r-4, 2r-3\rangle$$

of genus $g = r + 3$ with conductor $2r$. The pole orders of differentials are

$$2r, 2r-1,\ 2r-4, 2r-5,\ r, r-1, \ldots, 2.$$

One checks easily that for $r \geq 12$ multiplication with

$$t^{3r}, \ldots, t^{4r-12}, t^{4r-8}, t^{4r-4}, t^{4r}, t^{4r+1}, \ldots$$

maps ω into $\mathcal{O}$, giving that $d_1 = 3r + 9 - (r + 3) = 2\delta$. The pole orders of quadratic differentials, relevant for the computation of d_2 are $4r, \ldots, 4r-10$ with $4r-3$ and $4r-7$ excepted. Therefore only multiplication by $t^{5r}, \ldots, t^{5r-17}$ and from t^{6r} onwards works, giving $d_2 = 4r+13$ for $r \geq 16$.

The monomial curve lies on the cone

$$x_r = u^{r-3},\ x_{r-1} = u^{r-4}v,\ \ldots,\ x_{2r-7} = u^4v^{r-7},\ x_{2r-4} = uv^{r-4},\ \text{and}\ x_{2r-3} = v^{r-3}$$

as the image of $u^{2r-3} = v^r$. It deforms into the L^{r-4}_r one the cone, which is the image of $u^r = v^r$. The point matrix P involves powers of the r-th roots of unity and the matrix Q of the Gale transform also. The Gale transform of the r points lies on a non-degenerate quadric, but not on a curve of type $(3,3)$ if $r \geq 16$. Indeed, as this L^{r-4}_r with $r \geq 16$ is not smoothable, the Gale transform cannot lie on a canonical curve; note that the lower bound 16 is sharp, as 15 points determine a curve of type $(3,3)$.

2.2. The known non-smoothable monomial curves have non-trivial deformations of negative weight. By openness of versality the curves occurring in this way are also not smoothable. Buchweitz' original example is the smallest, with $g = 16$, but there is an example with smaller embedding dimension having $g = 17$ [21]. For computations with explicit equations this is easier.

Example 2.8. Consider the semigroup of embedding dimension 8 with generators

$$\langle 13, \ldots, 18, 21, 23\rangle$$

[21, Example 2.1 (2)]. We do not give equations for the curve here, as their structure is similar to that in Example 2.5. The equations can be given in *rolling factors format*, see e.g., [40, Ch. 12]. There are quadratic equations expressing that the curve lies on the cone over a projected rational normal curve, with $x_{13} = u^{10}$, $x_{14} = u^9v$, $x_{15} = u^8v^2$, $x_{16} = u^7v^3$, $x_{17} = u^6v^4$, $x_{18} = u^5v^5$, $x_{21} = u^2v^8$ and $x_{23} = v^{10}$. Furthermore there are rolling factors equations obtained from the equation $u^{23} - v^{13} = 0$ of the curve on the cone.

A rolling factor deformation is induced by $u^{23} + su^{13} - v^{13}$, to a L^8_{13} lying on the cone. More rolling factor deformations can be obtained from the deformation $u^{23} + su^{13-k}v^k - v^{13}$. Writing this last expression as

$$\frac{(u^{10+k} + sv^k)(su^{13-k} - v^{13-k})}{s} - \frac{u^{10+k}v^{13-k}}{s},$$

we see that the curve consists of $13-k$ smooth branches and a singular branch of multiplicity k.

In particular, for $k=2$ a direct computation with Singular [8] shows that it is a singularity with $\dim T^1 = 28$, that is the same dimension as for the general L^8_{13}. This is maybe unexpected, but one has to keep in mind that the singularity is not quasi-homogeneous, and there is no $\mathbb{G}_m$-action on the base space.

If we take $s=-1$ and consider the 11 lines $u^{11}+v^{11}$ together with the curve, which is the image of $u^{12}-v^2$, so the curve parametrised by $(x_{13},\dots,x_{23}) = (t^2,t^3,t^4,t^5,t^6,t^7,t^{10},t^{12})$, then it is a trivial deformation of the previous curve, but the computation of T^1 does not finish in reasonable time. However it is possible to compute the generators of the normal sheaf $\operatorname{Hom}_{\mathcal{O}_n}(I,\mathcal{O})$, and conclude that there are no non-trivial infinitesimal deformations with linear part, and that the number of generators apart from those coming from coordinate transformations is equal to 28. The same holds for a curve consisting of an ordinary cusp and 11 lines in general position. This shows:

Proposition 2.9. *The general curve consisting of an ordinary cusp and* 11 *lines in general position through the origin in* $\mathbb{A}^8$ *is not smoothable. It deforms only into curves* L^8_{13}.

2.3. For monomial curves the Buchweitz criterion can be expressed in terms of the semigroup. Recall that L is the set of gaps. Denote by kL the k-fold sumset $\mathrm{L}+\cdots+\mathrm{L}$ and by $|k\mathrm{L}|$ its cardinality.

Lemma 2.10 (Buchweitz)**.** *For a monomial curve of multiplicity at least three*

$$\dim \operatorname{Coker}\left\{\operatorname{Hom}(\omega^{\otimes k},\mathcal{O})\to\mathcal{O}\right\} = |(k+1)\mathrm{L}| + (2k+1) - \delta.$$

Proof. Multiplication by $t^l \in \mathcal{O}$ with $l \ge kc$ is the homomorphism $\varphi_l\colon \omega^{\otimes k} \to \widetilde{\mathcal{O}}$ given by $\varphi_l\left(\frac{(dt)^k}{t^a}\right) = t^{l-a}$. The image of this map is not contained in $\mathcal{O}$ if and only if $l-a\notin\Gamma$ for some a. By the description of ω we can write $a-k=(a_1-1)+\cdots+(a_k-1)$ with $a_i-1\notin\Gamma$. If for some i one has $a_i-1\notin\mathrm{L}$, that is $a_i\le 0$, then $a\le(k-1)c$ and $\varphi_l\left(\frac{(dt)^k}{t^a}\right) = t^{l-a}\in\mathcal{O}$ as $l-a\ge c$. Therefore the image of φ_l is not contained in $\mathcal{O}$ if and only if $l-k$ can be written as the sum of $k+1$ elements of L. For $l=2k+1,\dots,kc-1$ we have that $l-k=k+1,\dots,kc-(k+1)$ and all these numbers belong to $(k+1)$L: as $c-1$ is the maximal element of L, this statement follows by induction from the case $k=1$, which is true by [4, 4.2], [27, Thm. (1.3)].

So $d_k+\delta$ is equal to the number of $t^l\in\widetilde{\mathcal{O}}$ with $l<kc$ plus the number of elements in $(kc-k+\mathbb{N})\cap(k+1)\mathrm{L}$; this sum is equal to $|(k+1)\mathrm{L}|+(2k+1)$. □

Corollary 2.11. *For a monomial curve of multiplicity at least 3 one has* $d_k > 2k\delta$ *if and only if* $|(k+1)\mathrm{L}| > (2k+1)(\delta-1)$.

This is the form in which the Buchweitz criterion is mostly cited, but the original form $d_k>2k\delta$ is wider applicable.

2.4. **Weierstrass semigroups.** For a smooth projective pointed curve (C,P) of genus $g>1$ defined over **k** the set of nonnegative integers n, such that there is a rational function on C whose pole divisor is nP, form a semigroup $\mathcal{S}$, the Weierstrass semigroup at P. By Riemann-Roch the set L of positive integers that are not in $\mathcal{S}$ (the set of gaps) has size exactly g. The point $P\in C$ is a Weierstrass point if its semigroup is different from the ordinary one $\{0,g+1,g+2,\dots\}$.

Let $\mathcal{M}^{\mathcal{S}}_{g,1}$ be the moduli space of smooth pointed curves of genus g whose Weierstrass semigroup at the marked point is $\mathcal{S}$. Pinkham [29, Section 13] observed that it is related to the negative part $\mathcal{C}^-\to B^-$ of the versal deformation of the monomial curve $C_{\mathcal{S}}$. The $\mathbb{G}_m$-action on the curve induces a $\mathbb{G}_m$-action on the base space B and on B^-.

Theorem 2.12 (Pinkham)**.** *Let* B^- *be the base space in negative degrees of the monomial curve* $C_{\mathcal{S}}$ *and denote by* B^-_{sm} *the open subset of* B^- *given by the points with smooth fibres. Then* $\mathcal{M}^{\mathcal{S}}_{g,1}$ *is isomorphic to* $B^-_{sm}/\mathbb{G}_m$.

This means that $C_{\mathcal{S}}$ is negatively smoothable if and only if $\mathcal{S}$ is a Weierstrass semigroup. Buchweitz' condition $|(k+1)\mathrm{L}| > (2k+1)(\delta-1)$ implies that $C_{\mathcal{S}}$ is not smoothable and therefore not negatively smoothable, so $\mathcal{S}$ is not a Weierstrass semigroup. This can also be seen directly by Riemann-Roch. Suppose that P is a Weierstrass point on a smooth curve C. If n is a gap, then there exists a regular differential α with a zero of order $n-1$. If $\alpha_1,\dots,\alpha_k$ are differentials with zeros of order n_i-1, $i=1,\dots,k$, then their product defines a k-fold differential with a zero of order $\sum n_i - k$. By Riemann-Roch $h^0(C,\Omega^k)=(2k+1)(g-1)$, so the number of zero orders can at most be $(2k+1)(g-1)$. Therefore $|(k+1)\mathrm{L}| \le (2k+1)(g-1)$ has to hold. In the literature this argument is most often given to prove Buchweitz' criterion, with the notable exception of [23].

2.5. The known non-smoothable monomial curves have multiplicity at least 13. Komeda has found examples of non-Weierstrass semigroups with multiplicity 8 and 12 [22]. These monomial curves are therefore not negatively smoothable, but they might be smoothable. We show that this is indeed so in the simplest case. This gives an example of a smoothable irreducible quasi-homogeneous curve singularity which is not negatively smoothable. Earlier Pinkham gave an example of a reducible curve [30, p. 70].

Example 2.13. The semigroup $\mathcal{S}=\langle 8,12,18,22,51,55\rangle$ is not a Weierstrass semigroup [22, Example 5.1]. The ideal of the monomial curve $C_{\mathcal{S}}$ is generated by 13 polynomials, and the deformation in negative weight unobstructed, with dimension 57. Fortunately it suffices to write down the deformation for generators of T^1 as $\mathcal{O}$-module, the other deformations can be obtained by substitution. We write the deformation in rolling factors format, with 9 equations given (non-minimally) by the following determinantal and two pairs of remaining equations:

$$\begin{bmatrix} x_8 & x_{12} & x_{18} & x_{22} & x_{51} & x_{55} \\ x_{12} & x_8(x_8+s_8) & x_{22} & x_{18}(x_8+s_8) & x_{55} & x_{51}(x_8+s_8) \end{bmatrix}$$

$$\begin{aligned} &x_{12}^3 - x_{18}^2 + x_{22}s_{14} + x_{18}s_{18} + x_{12}s_{24} + x_8 s_{28},\\ &x_{12}^2 x_8(x_8+s_8) - x_{22}x_{18} + x_{18}(x_8+s_8)s_{14}\\ &\qquad + x_{22}s_{18} + x_8(x_8+s_8)s_{24} + x_{12}s_{28},\\ &x_{22}^3x_{18}^2 - x_{51}^2 + x_{55}s_{47} + x_{51}s_{51} + x_{22}s_{80} + x_{18}s_{84} + x_{12}s_{90} + x_8 s_{94},\\ &x_{22}^4 x_{18} - x_{55}x_{51} + x_{51}(x_8+s_8)s_{47} + x_{55}s_{51} + x_{18}(x_8+s_8)s_{80}\\ &\qquad + x_{22}s_{84} + x_8(x_8+s_8)s_{90} + x_{12}s_{94}. \end{aligned}$$

From these equations one sees immediately that the curve is not negatively smoothable: at the origin the Jacobian matrix has rank at most 4, as only in four equations the x-variables occur linearly. But the singularity is smoothable, because the general fibre of this deformation has only a hypersurface singularity at the origin. To see this, take the 1-parameter deformation $s_8=s^4$, $s_{18}=s^9$, $s_{94}=s^{47}$, while the other deformation variables are zero. A computation shows that for $s\neq 0$ there is indeed only one singularity, at the origin; the Jacobian matrix has rank 4. The first two additional equations become $x_{18}(s^9-x_{18}) = -x_{12}^3$, $x_{22}(s^9-x_{18}) = x_{12}^2x_8(x_8+s^4)$ with (s^9-x_{18}) a unit in the local ring. This allows to eliminate x_{18} and x_{22}. The last two equations then allow elimination of x_8 and x_{12}. What remains is $x_{55}^2 = ux_{51}^2$ with u a unit, so the curve has an ordinary double point.

3. Large families

3.1. The second general method for showing non-smoothability is based on exhibiting large families of singularities, too large to be in the closure of the locus of smooth ones. Iarrobino [18] used it to study zero-dimensional schemes. The first examples of non-smoothable curve singularities were given by Mumford [26], using this method. He constructed a family of singular complete curves which is too large to lie in the closure of the moduli space of smooth curves of the genus in question. Greuel [15] used his version of Deligne's formula for the dimension of smoothing components for quasi-homogeneous singularities to analyse Mumford's examples, and those of Pinkham [29]. In fact, he gave the following general criterion.

Proposition 3.1. *Let $\pi\colon \mathcal{C} \to T$ be a deformation with singular section $\sigma\colon T \to \mathcal{C}$ of the curve singularity $C = \mathcal{C}_0$. If $\mathcal{C}_t$ is not isomorphic to C for $t \neq 0$ and T is irreducible of dimension $\dim T \geq e(C)$, then there is a dense open subset $T' \subset T$ such that $(\mathcal{C}_t, \sigma(t))$ is not smoothable for $t \in T'$.*

Indeed, the image of T in the versal deformation has dimension $\dim T \geq e(C)$ and the image cannot be a smoothing component, as there are no smooth fibres over T.

3.2. **Lines through the origin in general position.** We first analyse in detail the examples of Pinkham and Greuel [29, 15]. Consider the singularity L^n_r of r lines in $\mathbb{A}^n$ through the origin in general position (see Section 1.3.2); it is the cone over r points in $\mathbb{P}^{n-1}$. The number of moduli of r points in $\mathbb{P}^{n-1}$ is $(r-n-1)(n-1)$.

For fixed n the curves L^n_n, L^n_{n+1} and L^n_{n+2} are always smoothable. We now consider $r \geq n+3$ and determine the Deligne number for generic curves.

Let $d = d(n,r)$. By Lemma 1.5

$$\delta(L^n_r) = dr - \binom{n+d-1}{d-1}.$$

For generic curves the type t is by Proposition 1.6 given by

$$t = \max\{r - \tbinom{n+d-2}{d-1}, \tbinom{n+d-3}{d-1} - (n-2)\left(r - \tbinom{n+d-2}{d-1}\right)\}.$$

Therefore the Deligne number $e = \mu + t - 1 = 2\delta - r + t$ is

$$e = \max\begin{cases} 2dr - 2\tbinom{n+d-1}{d-1} - \tbinom{n+d-2}{d-1} \\ (2d+1-n)r - 2\tbinom{n+d-1}{d-1} + \tbinom{n+d-3}{d-1} + (n-2)\tbinom{n+d-2}{d-1}. \end{cases}$$

We first consider the condition $(r-n-1)(n-1) \geq e$ for r such that $t = r - \binom{n+d-2}{d-1}$, which holds for most r. Then also e is given by the first alternative. The condition that the number of moduli is at least e translates into $r(n-1) - (n^2-1) \geq 2dr - 2\binom{n+d-1}{d-1} - \binom{n+d-2}{d-1}$, which we rewrite as

$$(n-2d-1)r \geq (n^2-1) - 2\binom{n+d-1}{d-1} - \binom{n+d-2}{d-1}. \tag{2}$$

We determine a lower bound for r, in the interval $n < r \leq \binom{n+1}{2}$, so $d = 2$. We find $(n-5)r \geq n^2 - 3n - 3 = (n+2)(n-5) + 7$.

For an upper bound we distinguish between even and odd n. For odd $n = 2m+1$ the condition (2) is obviously satisfied if $d = \frac{n-1}{2} = m$ and $m \geq 3$. We claim that it is no longer satisfied for $r = \binom{n+m}{m+1} = \binom{3m+1}{m+1}$. We determine where on the interval $\binom{n+m-1}{m} < r \leq \binom{n+m}{m+1}$ formula (2) ceases to hold:

$$-2r \geq 4m(m+1) - 2\binom{3m+1}{m} - \binom{3m}{m}$$

gives

$$r \leq \binom{3m+1}{m} + \frac{1}{2}\binom{3m}{m} - 2m(m+1) =: M(2m+1), \tag{3}$$

where the right-hand side is indeed less than $\binom{3m+1}{m+1}$. If we use instead the second formula for e, which holds for $r - \binom{3m}{m}$ small, then we obtain a condition of the form

$$(2n-2d-2)r = (2m-2)r \geq F(m)$$

with $F(m)$ an explicit expression independent of r, which gives the correct condition for $r = \binom{3m}{m}$, and we have already noticed that it is satisfied in that case, if $m \geq 3$. The maximal r lies therefore in the range where we need the first expression.

For even n this method gives no non-smoothability result for $n = 4$. Let $n = 2m$. For $d = m$ we have that the left-hand side of formula (2) is negative, but for $n \geq 8$ the formula holds on the whole interval. For $n \geq 8$ we take $d = m + 1$ and find in a similar way as above

$$r \leq \frac{2}{3}\binom{3m}{m} + \frac{1}{3}\binom{3m-1}{m} - \frac{4m^2-1}{3} =: M(2m). \tag{4}$$

For $n = 6$ we need $d = m$ and we get $r \leq 2 \cdot \binom{8}{2} + \binom{7}{2} - 35 = 42$.

From Proposition 3.1 we get:

Proposition 3.2. *Let $M(n)$ for $n \geq 7$ be given by the expression in (3) for odd n and in (4) for even n and set $M(6) = 42$. For $n \geq 6$ the generic L_n^r in uniform position is not smoothable if $n + 2 + \frac{6}{n-5} < r \leq M(n)$.*

For $d = 2$ Theorem 1.8 gives an exact criterion for smoothability: the generic L_r^n is smoothable if through the Gale transform of the r corresponding points passes a canonical curve of genus $g = r - n$. We formulate the condition $r > n + 2 + \frac{6}{n-5}$ in terms of g as $r > g + 5 + \frac{6}{g-2}$. According to [37] r points determine a canonical curve when $r \leq g + 5 + \frac{6}{g-2}$, except for $g = 4$ and $g = 6$, when $r \leq g + 5$. For low values of n we have, with this correction, non-smoothability for the following values of r:

n	6	7	8	9	10
r	$\{10, 12\} \cup [15, 42]$	$\{11\} \cup [13, 138]$	$[12, 419]$	$[13, 922]$	$[14, 2636]$

For r larger than $M(n)$ the curve L_r^n has deformations of positive weight, so the homogeneous curves are not the most general on the equisingularity stratum. We compute the dimension of the tangent space.

Proposition 3.3. *For $l > 0$*

$$\dim T_l^1 \geq \max\left\{0, (n-1)\left(r - \binom{n+l}{l+1}\right) - \binom{n+l-1}{l+1}\right\}.$$

Proof. Let $\binom{n+d-2}{d-1} < r \leq \binom{n+d-1}{d}$. Then there are $k_1 = \binom{n+d-1}{d} - r$ equations of degree d and possibly k_2 equations of degree $d + 1$. Furthermore $K_\nu = \mathcal{O}_\nu$ for $\nu \geq d$. Therefore $(K/\mathcal{O})_l^k = (K_{d+l}/\mathcal{O}_{d+l})^{k_1} \oplus (K_{d+1+l}/\mathcal{O}_{d+1+l})^{k_2} = 0$ for $l > 0$. The exact sequence (1) reduces to

$$0 \to \operatorname{Hom}_{\mathcal{O}}(\mathfrak{m}, \mathcal{O})_l \to K_l \to \left(K_{l+1}/\mathcal{O}_{l+1}\right)^n \to T_l^1 \to 0.$$

Here $\operatorname{Hom}_{\mathcal{O}}(\mathfrak{m}, \mathcal{O})_l = \{a \in K_l \mid a\mathfrak{m}_1 \subset \mathcal{O}_{l+1}\}$. The map $K_l \to \left(K_{l+1}/\mathcal{O}_{l+1}\right)^n$ is induced by the Euler vector field, so it is given by $a \mapsto ([ax_1], \dots, [ax_n])$, where $[ax_i]$ denotes the class of ax_i in $K_{l+1}/\mathcal{O}_{l+1}$ and the x_i are the generators of $\mathfrak{m}$. This induces a map $K_l/\mathcal{O}_l \to \left(K_{l+1}/\mathcal{O}_{l+1}\right)^n$ whose kernel consists of the elements of $\operatorname{Hom}_{\mathcal{O}}(\mathfrak{m}, \mathcal{O})_l$ not lying in $\mathfrak{m}_l = \mathcal{O}_l$. Therefore

$$T_l^1 = \operatorname{Coker} K_l/\mathcal{O}_l \to \left(K_{l+1}/\mathcal{O}_{l+1}\right)^n.$$

The dimension of the image is at most $r - \binom{n+l-1}{l}$, so

$$\dim T_l^1 \geq n(r - \binom{n+l}{l+1}) - r + \binom{n+l-1}{l} = (n-1)\left(r - \binom{n+l}{l+1}\right) - \binom{n+l-1}{l+1}.$$

∎

Proposition 3.4. *The general curve on the equisingularity stratum of L_r^n is not smoothable if $r > n + 2 + \frac{6}{n-5}$ for $n \geq 6$, if $r > 18$ for $n = 5$ and $r > 30$ for $n = 4$.*

Proof. The deformations of positive weight are obtained by deformation of the parametrisation and are not obstructed. Therefore the equisingularity stratum has dimension $\dim T^1_{\geq 0}$. We compare the growth of the dimension with the growth of the Deligne number e. For

$$\binom{n+d-2}{d-1} < r \leq \binom{n+d-1}{d},$$

the growth of $\dim T^1_{\geq 0}$, if r increases by 1, is $(n-1)(d-2)$ or $(n-1)(d-1)$ with the second alternative holding if $(n-1)\left(r-\binom{n+d-2}{d-1}\right) \geq \binom{n+l-1}{l+1}$. The same condition determines the growth of the Deligne number, which is $2d+1-n$ or $2d$. So if $(n-3)d \geq n-1$, then $\dim T^1_{\geq 0}$ stays with growing r larger or equal than e once it is at least equal to e.

For $n \geq 6$ the bound follows from Proposition 3.2; it can be improved by 1 for $n=8$. For $n=4,5$ we need also $\dim T^1_+$. We give the computation for $n=4$. With Proposition 3.3 $\dim T^1_0 = 3r-15$, $\dim T^1_1 \geq 3r-36$ and $\dim T^1_2 \geq 3r-70$. With $d=4$ we have for $24 \leq r \leq 35$ that $e = 8r-90$. We have equality $8r-90 = 9r-121$ if $r=31$. □

From this result nothing can be said about smoothability of the homogeneous curve. But the explicit computations for low values of n show that for some values of r there are no infinitesimal deformations of negative weight, which proves that for these values the general L^n_r is not smoothable. For $n=4$ this is the case in the intervals $[96,105]$ and $[132,150]$. We compute T^1_{-1} in general.

Proposition 3.5. *For* $\binom{n+d-2}{d-1} < r \leq \binom{n+d-1}{d}$ *one has*

$$\dim T^1_{-1} \geq \max\left\{0, (n-1)r - n - \left(r - \binom{n+d-2}{d-1}\right)\left(\binom{n+d-1}{d} - r\right)\right\}.$$

For $r = \binom{n+d-1}{d}$ *equality holds, that is* $\dim T^1_{-1} = (n-1)\binom{n+d-1}{d} - n$.

Proof. We use the same notation as in the proof of Proposition 3.3. We have $\operatorname{Hom}_{\mathcal{O}}(\mathfrak{m}, \mathcal{O})_{-1} = 0$ so the exact sequence (1) becomes

$$0 \to K_{-1} \to \ker\left\{\partial f \colon (K_0/\mathcal{O}_0)^n \to (K_{d-1}/\mathcal{O}_{d-1})^{k_1}\right\} \to T^1_{-1} \to 0.$$

The rank of the map ∂f is at most

$$\min\{\dim(K_0/\mathcal{O}_0)^n - \dim K_{-1}, \dim(K_{d-1}/\mathcal{O}_{d-1})^{k_1}\}.$$

Writing $r = \binom{n+d-2}{d-1} + s$ we have $\dim K_{d-1}/\mathcal{O}_{d-1} = s$ and $k_1 = \binom{n+d-1}{d} - r = \binom{n+d-2}{d} - s$. Therefore $\dim T^1_{-1} \geq \max\{0, (n-1)r - n - s(\binom{n+d-2}{d} - s)\}$.

For $s=0$ the map ∂f is zero, so $\dim T^1_{-1} = (n-1)r - n$. □

Conjecture 3.6. For generic L^n_r equality holds in Proposition 3.3 for $\dim T^1_l$, $l > 0$ and in Proposition 3.5 for $\dim T^1_{-1}$, with some exceptions for low values of r, where the existence of smoothings according to Theorem 1.8 forces T^1_{-1} to be non-zero. The generic L^n_r is not smoothable in the range of Proposition 3.4.

The conjecture is supported by direct computations. For $n=4$ the dimension of T^1 is as predicted for r up to 70, and that of T^1_{-1} for $r \leq 151$. For $n=5$ we have computed the dimension of T^1 for $r \leq 50$ and that of T^1_{-1} for $r \leq 62$. As $\dim T^1_{-1} = 0$ for $41 \leq r \leq 60$, the generic L^5_r is not smoothable for those r.

Example 3.7. The case $n=6$. The general L^6_{10} and L^6_{12} are not smoothable, and there are no deformations of negative weight. In both cases $\dim T^1 < e$. The general L^6_{11} is smoothable, with $\dim T^1 = e = 24$. The base space is smooth, but $\dim T^2 = 5$. The general L^6_{13} is also smoothable, but with $\dim T^1 = 33 = e+1$, so $\dim T^1_{-1} = 3$ which is the minimal value according to Proposition 3.5. Furthermore $\dim T^2 = \dim T^2_{-2} = 1$ and a computation shows that the base space in negative degrees is singular, given by one quadratic equation. The case L^6_{14} is described in [39]. Here $\dim T^1_{-1} = 8$ and the dimension of a smoothing component is one more than the number of moduli. The general curve has 16 smoothing components of dimension 36.

The ideal of L^6_r is generated by quadrics for $6 \le r \le 14$, but from $r = 15$ on one needs also cubics and for $r = 21$ the ideal is generated by 35 cubics. The general L^6_r with $15 \le r \le 21$ is not smoothable, but $\dim T^1_{-1}$ is not zero, increasing to 99 for $r = 21$, as predicted by Conjecture 3.6. For $r = 15$ a computation shows that all deformations of negative weight are obstructed. Therefore the base space of the versal deformation is non-reduced. We expect the same to be true for other $r \le 21$.

On the interval $21 < r \le 56$ the ideal is, up to $r = 42$, generated by cubics, from $r = 43$ also quartics are needed, and for $r = 56$ the ideal is generated by 70 quartics. From $r = 25$ there are deformations of positive degree, with $\dim T^1_1 = 5(r-24)$, computed up to $r = 36$. The dimension of T^1_{-1} decreases, in accordance with the conjecture: it is 70 for $r = 22$, 43 for $r = 23$, 18 for $r = 24$ and zero for $25 \le r \le 47$, to increase afterwards. Only for $r = 48$ we have been able to compute the dimension, which indeed turns out to be 18. In the next interval $56 < r \le 126$ the conjecture predicts that there are no deformations of negative weight for $61 \le r \le 116$. We computed only the case $r = 84$, where the size of the syzygy matrix is minimal: the general L^6_{84} has no deformations of negative weight, and is therefore not smoothable.

In [16] Problem 2.57 reads

> Do there exist for fixed $n \ge 4$ non-smoothable curves L^n_r if r goes to infinity? It seems unlikely that this is not the case.

The evidence above makes it very unlikely that this is not the case. The problem was already formulated in [15] without the second sentence, but with the question:

> Do there exist smoothable ones?

More precisely, we ask whether there exist for fixed n smoothable L^n_r with Hilbert function $H(l) = \min\left(r, \binom{n+l-1}{n-1}\right)$ and large $d(n,r)$. No examples are known to us, in fact not even for $d = 3$.

3.3. The curve L^n_r with $r = \binom{n+1}{2}+1$ is the first where $\delta(L^n_r) = \delta(L^n_{r-1})+3$, because L^n_{r-1} is the first where the ideal is generated by cubics only and therefore the intersection multiplicity with a new line in general position is equal to three. It is also possible to construct a curve $S^{n,n+1}_r$ with $\delta(S^{n,n+1}_r) = \delta(L^n_{r-1}) + 2$ having smooth branches, where each branch has as tangent line a line of the L^n_r. To achieve this we replace the last line by a parabola in $\mathbb{A}^{n+1}$, tangent to $\mathbb{A}^n$. If the line in L^n_r is parametrised by $(x_1, \dots, x_n) = (a_1 t, \dots, a_n t)$, then the smooth branch of $S^{n,n+1}_r$ is given by $(x_1, \dots, x_n, x_{n+1}) = (a_1 t, \dots, a_n t, t^2)$. The curve is quasi-homogeneous. Its tangent cone is not the L^n_r, but has an embedded point at the origin.

The invariants of this singularity extend the pattern for $d = 2$. One computes that $e = 4r - 3n - 2$, $\dim T^1_l = 0$ for $l > 0$, $\dim T^1_0 = (n-1)(r-n-1)$ and $\dim T^1_{-1} = (r-2)n$, with $r = \binom{n+1}{2} + 1$. We conclude that the general such curve is not smoothable for $n \ge 6$. For these curves $\dim T^1_{-1}$ is large, but all these deformations should be obstructed.

If we want to add one more line, we have two choices. If we want to increase δ with 2, we need to increase the embedding dimension. To have a singularity better suited to direct computation (for small n), we keep the embedding dimension constant, and increase δ by three each time we add (in a certain range) a smooth branch in $\mathbb{A}^{n+1}$ of the same form $(a_1 t, \dots, a_n t, t^2)$. Also these curves $S^{n,n+1}_r$ are not smoothable.

The curves $S^{n,n+1}_r$ are quasi-homogeneous, with variables x_i, $i = 1, \dots, n$ of weight 1 and x_{n+1} of weight 2. We expect the ideal of S^n_r for $\binom{n+1}{2} < r \le \frac{3}{4}\binom{n+2}{3}$ to be generated by $\binom{n+2}{3} + n - r$ cubic equations and one equation of degree 4 containing the monomial x^2_{n+1}. The $\binom{n+2}{3} + n - r$ cubic equations alone generate the ideal of the union of $S^{n,n+1}_r$ and the x_{n+1}-axis. The number of moduli for both curves is the same. Explicit computations show that this is indeed true for $4 \le n \le 7$.

Definition 3.8. The curve $AS_r^{n,n+1} \subset \mathbb{A}^{n+1}$ is the curve with $r+1$ branches consisting of the union of $S_r^{n,n+1}$, the curve with r smooth branches tangent to $\mathbb{A}^n \subset \mathbb{A}^{n+1}$, with the x_{n+1}-axis.

Explicit computations for $n = 6$ show that the curves $AS_r^{6,7}$ and $S_r^{6,7}$ have no deformations of negative degree in the range $26 \le r \le 47$. These curves do not deform into curves of a different type.

3.4. **Irreducible curves.** We consider Mumford's construction of non-smoothable curve singularities [26]. Let $(t) \subset \mathbf{k}[t]$ be the ideal defining the point $0 \in \mathbb{A}^1$. Choose integers $1 < n < d$ and let V be a $\mathbf{k}$-vector space with $(t^{2d}) \subset V \subset (t^d)$ and $\dim V/(t^{2d}) = n$. Then $\mathbf{k} + V$ is the affine coordinate ring of a curve C_V having a singular point with $\delta = 2d - n - 1$. These curves are parametrised by a Grassmannian $G(n, d)$, but this family is too large too apply Proposition 3.1, as it in general contains isomorphic curves. We observe that the general C_V is a deformation of positive weight of the monomial curve $C_{d,n}$ where $V/(t^{2d})$ has basis $(t^d, t^{d+1}, \dots, t^{d+n-1})$, and that every deformation of positive weight is of the form C_V.

Moreover, the deformations of positive weight are unobstructed, as we can perturb the parametrisation arbitrarily with terms of higher weight.

Theorem 3.9. *The general equisingular deformation of the curve $C_{d,n}$ is not smoothable, if $(n-6)(d-n-3) \ge 14$.*

Proof. As noted, the equisingularity stratum is smooth with as tangent space $T^1_+ = \oplus_{l>0} T^1_l$. We compute T^1_l, using Buchweitz' formula in Theorem 1.10. The semigroup of the curve $C_{d,n}$ is $\langle d, d+1, \dots, d+n-1\rangle$ if $2n > d$ and $\langle d, d+1, \dots, d+n-1, 2d+2n-1, \dots, 3d-1\rangle$ if $2n < d+1$. Generators larger than the conductor do not contribute to A_l, so

$$A_l = \{i \in \{1, \dots, n\} \mid d + i - 1 + l \notin \mathcal{S}\}.$$

As the degree of the equations is at least $2d$, there are also no positive dimensional V_l. Therefore $\dim T^1_l = |A_l| - 1$ and $T^1_l = 0$ for $l \ge d$. The condition defining A_l is $d+n-1 < d+i-1+l < 2d$ so $n < i + l < d + 1$. For fixed i there are $d - n$ values of l where these conditions are satisfied, and those satisfy $1 \le l \le d - 1$. Therefore $\dim T^1_+ = n(d - n) - (d - 1) = (n - 1)(d - n - 1)$.

To compute t we observe that the generators of ω are $\frac{dt}{t^{2d}}, \dots, \frac{dt}{t^{d+n+1}}$, so $t = d - n$ and $e = 2\delta + t - 1 = 5d - 3n - 3$. Therefore the general equisingular deformation is not smoothable if $(n-1)(d-n-1) \ge 5d - 3n - 3$, so $(n-6)(d-n-1) \ge 2n + 2 = 2(n-6) + 14$. □

Example 3.10. The example of smallest embedding dimension is the curve $C_{17,9}$ of genus $g = 24$ with semigroup

$$\langle 17, 18, 19, 20, 21, 22, 23, 24, 25\rangle.$$

We cannot use Buchweitz' criterion to conclude that an irreducible curve with $g = \delta = 24$ and conductor $c = 34$ is not smoothable, because $(t^{2c}) \subset \mathrm{Hom}(\omega, \mathcal{O})$ and

$$\dim \mathcal{O}/(t^{68}) = 44 < 48 = 2\delta.$$

On the other hand, Buchweitz' monomial curve (Example 2.5) with semigroup

$$\langle 13, \dots, 18, 20, 22, 23\rangle$$

deforms to other irreducible curves with the same δ. In particular, the deformation of the parametrisation where only $x_{22} = t^{22} + st^{19}$ and $x_{23} = t^{23} + st^{21}$ are deformed, gives a curve with semigroup $\langle 13, 14, \dots, 21\rangle$. The general fibre of the deformation, say the one with $s = 1$, can also be seen as deformation of positive weight of the smoothable monomial curve $C_{13,9}$ with semigroup $\langle 13, 14, \dots, 21\rangle$. Here we have that $(n-6)(d-n-3) = 3 < 14$, but the general equisingular deformation of the curve $C_{13,9}$ is not smoothable.

Remark 3.11. The curve $C_{d,n}$ is always smoothable. For $2n \leq d+1$ the singularity is determinantal, with equations coming from the matrix

$$\begin{vmatrix} x_d & \dots & x_{d+n-2} & x_{d+n-1}^2 & x_{2d+2n-1} & \dots & x_{3d-2} & x_{3d-1} \\ x_{d+1} & \dots & x_{d+n-1} & x_{2d+2n-1} & x_{2d+2n} & \dots & x_{3d-1} & x_d^3 \end{vmatrix}.$$

For $2n > d+1$ the curve deforms into an L_d^n which lies on the cone over a rational normal curve and is therefore smoothable.

Proposition 3.12. *Every curve on the equisingularity stratum of $C_{n,d}$ deforms into curve singularity with only smooth branches; if $d \leq 2n-1$ it deforms into L_d^n and if $d \geq 2n$ it deforms into a curve $S_d^{n,N}$ with $N = \max\{n, n+d-\binom{n+1}{2}\}$.*

Proof. In the case $d \geq 2n$ a curve equisingular with $C_{n,d}$ can be parametrised as

$$\begin{cases} x_i = t^i(1+\varphi_i(t)), & d \leq i \leq d+n-1 \\ x_j = t^j, & 2d+2n-1 \leq j \leq 3d-1 \end{cases},$$

where the polynomials $\varphi_i(t)$ contain only powers of t in the range $[d+n-i, 2d-1-i]$. The deformation of the parametrisation

$$\begin{cases} x_i = (t^d-s)t^{i-d}(1+\varphi_i(t)), & d \leq i \leq d+n-1 \\ x_j = (t^d-s)^2 t^{j-2d}, & 2d+2n-1 \leq j \leq 3d-1 \end{cases}$$

is δ-constant and therefore flat. For $s \neq 0$ the curve has smooth branches, each of which has as tangent line a line through the origin in $\mathbb{A}^n \subset \mathbb{A}^{d-n+1}$. The embedding dimension is $\max\{n, n+d-\binom{n+1}{2}\}$.

For $d < 2n$ the deformation $x_i = (t^d-s)t^{i-d}(1+\varphi_i(t))$ for $d \leq i \leq d+n-1$ is a δ-constant deformation into L_d^n. □

4. Gorenstein curves

4.1. Buchweitz' criterion concerns deformations to curves with at most Gorenstein singularities (see Corollary 2.4), so it does not apply to symmetric semigroups. Based on a observation of Stöhr, Torres gave a construction [41, Scholium 3.5] of Gorenstein non-Weierstrass semigroups. Let $\widetilde{\Gamma}$ be a non-Weierstrass semigroup of genus γ and let $g \geq 6\gamma + 4$. Then the semigroup

$$\Gamma = \{2n \mid n \in \widetilde{\Gamma}\} \cup \{2g-1-2t \mid t \in \mathbb{Z} \setminus \widetilde{\Gamma}\}$$

is a symmetric non-Weierstrass semigroup. We refer to these as semigroups of Stöhr-Torres type.

Example 4.1. The smallest example is the semigroup

$$\mathcal{S} = 2\langle 13, \dots, 18, 20, 22, 23\rangle + \langle 149, 151, 157, 161\rangle$$

with $g = 100$. The embedding dimension is 13. There are 66 equations. Many of them can be given by an incomplete determinantal, the remaining ones are rolling factors. It is possible to compute the generators of T^1 as $\mathcal{O}$-module. The deformation of lowest weight is of rolling factors type and changes only the equations of highest weight. We also can determine a deformation which covers the deformation of the Buchweitz curve to L_{13}^9. Combining these one finds a deformation to a singularity with reduced tangent cone. Explicitly it is given by the following partial determinantal, which we write transposed compared with the determinantal for the Buchweitz curve in Example 2.5, and rolling factors equations expressing the products of the variables $x_{149}, \dots, x_{161}$.

$$\begin{vmatrix}
x_{26} & x_{28} & x_{30} & x_{32} & x_{34} & \cdot \\
x_{28} & x_{30} & x_{32} & x_{34} & x_{36} & x_{40} \\
x_{30} & x_{32} & x_{34} & x_{36} & \cdot & \cdot \\
x_{32} & x_{34} & x_{36} & \cdot & x_{40} & x_{44} \\
x_{34} & x_{36} & \cdot & x_{40} & \cdot & x_{46} \\
x_{36} & \cdot & x_{40} & \cdot & x_{44} & \cdot \\
x_{40} & \cdot & x_{44} & x_{46} & \cdot & x_{26}(x_{26}+s_{26}) \\
x_{44} & x_{46} & \cdot & \cdot & x_{26}(x_{26}+s_{26}) & x_{30}(x_{26}+s_{26}) \\
x_{46} & \cdot & \cdot & x_{26}(x_{26}+s_{26}) & x_{28}(x_{26}+s_{26}) & x_{32}(x_{26}+s_{26}) \\
x_{149} & x_{151} & \cdot & \cdot & x_{157} & x_{161} \\
x_{151} & \cdot & \cdot & x_{157} & \cdot & \cdot \\
x_{157} & \cdot & x_{161} & \cdot & \cdot & \cdot
\end{vmatrix}$$

$$\begin{gathered}
x_{149}^2 - (x_{36}x_{28}x_{26}^7 + s_{246})x_{26}^2, \\
x_{151}x_{149} - (x_{36}x_{28}x_{26}^7 + s_{246})x_{28}x_{26}, \\
x_{151}^2 - (x_{36}x_{28}x_{26}^7 + s_{246})x_{30}x_{26}, \\
x_{157}x_{149} - (x_{36}x_{28}x_{26}^7 + s_{246})x_{30}^2, \\
x_{157}x_{151} - (x_{36}x_{28}x_{26}^7 + s_{246})x_{32}x_{30}, \\
x_{161}x_{149} - (x_{36}x_{28}x_{26}^7 + s_{246})x_{34}x_{30}, \\
x_{161}x_{151} - (x_{36}x_{28}x_{26}^7 + s_{246})x_{36}x_{30}, \\
x_{157}^2 - (x_{36}x_{28}x_{26}^7 + s_{246})x_{40}x_{28}, \\
x_{161}x_{157} - (x_{36}x_{28}x_{26}^7 + s_{246})x_{46}x_{26}, \\
x_{161}^2 - (x_{36}x_{28}x_{26}^7 + s_{246})x_{46}x_{30}.
\end{gathered}$$

Retaining only the quadratic part of all these equations with $s_{26} = s_{246} = 1$ gives a homogeneous singularity, which describes a L^{13}_{26}. We rename the variables: x_{26+2k} becomes x_k and x_{149+2k} becomes y_k. The singularity is a double cover of the non-smoothable L^9_{13} into which Buchweitz' curve deforms, see Example 2.5. This curve lies on the projection of the cone over the rational normal curve of degree 10 onto $\mathbb{A}^9$: writing $x_k = u^{10-k}v^k$ describes the lines as $u^{13} = v^{13}$. The double cover is then given by $y_6 = \pm u^4v^6$, $y_k = \pm x_k$, with the same signs. This singularity L^{13}_{26} has no non-trivial deformations of positive weight, so the fibre of the deformation over $s_{26} = s_{246} = 1$ is isomorphic to it.

Proposition 4.2. *The L^{13}_{26} described above is a non-smoothable Gorenstein curve singularity.*

Proof. A computer computation shows that T^1 is concentrated in degree 0, of dimension 78, the dimension of the moduli space. □

The fact that this deformation of lowest degree leads to a non-smoothable singularity, suggests that the singularity is not smoothable at all. More evidence is given by a computation of the generators of T^1 as $\mathcal{O}$-module, which shows that all the perturbations of the equations lie in the square of the maximal ideal. We expect this to hold in general.

Conjecture 4.3. The Gorenstein monomial curves of Stöhr-Torres type are not smoothable.

4.2. **Self-associated point sets.** The explicit Gorenstein curve L^{13}_{26} of Proposition 4.2 is a cone over a well-known type of point set: self-associated point sets [7].

Definition 4.4. A set Γ of $2n$ ordered points in $\mathbb{P}^{n-1}$ is self-associated if its Gale transform is projectively equivalent to Γ.

We consider in general cones L^n_{2n} over self-associated point sets. We assume that every subset of $2n-1$ points is in uniform position (this is not the case for the L^{13}_{26} above). In particular the $2n-1$ points pose independent conditions on quadrics. If the $2n$ points are self-associated, then they fail by one to impose independent conditions on quadrics and the L^n_{2n} is Gorenstein, see [11, Sect. 7]. The configuration is characterised by the fact that quadrics passing through any $2n-1$ of the points pass through remaining point. The number of moduli is $n(n-1)/2$; this was classically known, see [11, Cor. 8.4]. Ordered sets of $2n$ self-associated points can be parametrised as (I, P) with $P \in \mathrm{SO}(n, \mathbf{k})$. For computations it is more convenient to parametrise $\mathrm{SO}(n, \mathbf{k})$ using the Cayley transform $A \mapsto (I+A)^{-1}(I-A)$ on the set of matrices with $I+A$ invertible; it induces a birational map between skew-symmetric and special orthogonal matrices. A general self-dual configuration has a skew normal form $(I+S, I-S)$ with S skew-symmetric [5, Thm. 2.9].

Theorem 4.5. *The general Gorenstein L^n_{2n} is not smoothable if $n > 9$.*

Proof. Let C be a Gorenstein L^n_{2n}. Then the points fail to impose independent conditions on quadrics but every maximal proper subset imposes independent conditions on quadrics [11, Sect. 7]. We may also assume that every subset of n points span $\mathbb{P}^{n-1}$. Therefore a subcurve $C_{2n-1} = L^n_{2n-1}$ has $\delta = 3n-3$ and as all quadrics vanishing on C_{2n-1} also vanish on the remaining line L we have that $L \cdot C_{2n-1} = 3$. So $\delta(C) = 3n$ and the Deligne number is $e = \mu = \delta + (\delta - r + 1) = 3n + (n+1) = 4n+1$. As the family of Gorenstein L^n_{2n} has dimension $n(n-1)/2$, the general such curve is not smoothable if $n(n-1)/2 \geq 4n+1$. □

We can also find this conclusion in another way. Petrakiev observed that for large n the general self-associated point set is not a hyperplane section of a canonical curve [28, Sect. 4]. The number of moduli of hyperplane sections of canonical curves of genus g in P^{g-1} is $(3g-3)+(g-1)$. Therefore the general L^{g-1}_{2g-2} is not negatively smoothable if $(g-1)(g-2)/2 > 4g-4$, which is the same bound as the Theorem above. In fact, for such curves negatively smoothable and smoothable are the same.

Proposition 4.6. *A Gorenstein L^n_{2n} is always negatively graded.*

Proof. We compute in the same way as in Proposition 3.3. We have $K_l = \mathcal{O}_l$ for $l \geq 3$. Therefore $(\ker \partial f)_l = (K/\mathcal{O})^n_l$ for $l \geq 1$. This shows that $T^1_l = 0$ for $l > 1$. For $l = 1$ we have $\dim(K/\mathcal{O})_1 = \dim K_2/\mathcal{O}_2 = 1$. Because $\dim K_1 = 2n$, in order to show that $T^1_1 = 0$ we have to show that $\dim\{a \in K_1 \mid a\mathfrak{m}_1 \subset \mathcal{O}_2\} = n$. An element $a \in K_1$ has the form $(a_1t_1, \dots, a_{2n}t_{2n})$. By subtracting elements in $\mathfrak{m}_1 \subset \{a \in K_1 \mid a\mathfrak{m}_1 \subset \mathcal{O}_2\}$, we can achieve that $a_1 = \dots = a_n = 0$. There exist a linear form $l \in \mathfrak{m}_1$ such that l vanishes in the points

$$P_{n+1},\ \dots,\ P_{n+j-1},\ P_{n+j+1},\ \dots,\ \mathbb{P}_{2n},$$

but not in P_{n+j}, for $1 \leq j \leq n$. Then l has as element of K_2 the form

$$(0, \dots, 0, a_{n+j}l(P_{n+j})t^2_{n+j}, 0 \dots, 0)$$

and this is not an element of $\mathcal{O}_2$ if $a_{n+j} \neq 0$. Therefore $\{a \in K_1 \mid a\mathfrak{m}_1 \subset \mathcal{O}_2\} = \mathfrak{m}_1$ and has dimension n. □

Theorem 4.7. *The general Gorenstein L^{g-1}_{2g-2} is smoothable if $g \leq 8$ or $g = 10$ and not smoothable otherwise.*

Proof. A Gorenstein L^{g-1}_{2g-2} is negatively smoothable if the corresponding point set Γ is a hyperplane section of a canonically embedded curve. This is classically known for $g \leq 6$. The cases $g = 7$ and $g = 8$ are the main results of [28]. The general canonical curve of genus 7 and 8 is a linear section of a Mukai Grassmannian, and Petrakiev shows that the same holds for the general Γ. For $g = 9$ the general canonical curve is a codimension 5 linear section of the Lagrangian

Grassmannian $LG(3,6) \subset \mathbb{P}^{14}$. The number of moduli of codimension 6 linear sections is 27 [28, Sect. 4], whereas the moduli space of Γ has dimension 28. The dimension of a smoothing component of L^8_{16} is 33, and the cone over $LG(3,6)$ is the total space of the versal deformation in negative degrees, so the smoothing component intersects the equisingularity stratum in a $33-6=27$-dimensional space. Therefore the general L^8_{16} is not smoothable.

For $g=10$ the general canonical curve does not lie on a $K3$ surface [25]. A direct computation for a random L^9_{18} (constructed from the skew normal form) shows that $\dim T^1_{-1}=1$ and that this infinitesimal deformation can be extended to a deformation, whose total space is the cone over a canonically embedded curve.

For $g \geq 11$ the result follows from Theorem 4.5. □

5. Generic curves

The first occurrence of the term generic singularity seems to be in a famous paper by Schlessinger [34]. He says that a singularity is "generic" if it is not the specialisation of any other singularity X', where a specialisation is defined as 1-parameter deformation with X the special fibre and the other fibres all isomorphic to X'. Under such a definition the curve consisting of an ordinary cusp and 11 lines in general position through the origin in $\mathbb{A}^8$ of Proposition 2.9 is "generic".

Iarrobino [19, 20] defines the term for zero-dimensional singularities using the Hilbert scheme. A generic singularity is one parametrised by a generic point of a component of the Hilbert scheme parametrising only irreducible schemes. To give explicit examples Iarrobino and Emsalem [20] look at almost-generic thick points, meaning that such a point deforms only to other thick points of the same type (a notion they deliberately leave vague) and the parametrising point lies on a single component of the Hilbert scheme. They use the term "generic" for such singularities.

This point of view suggests to define a generic (curve) singularity as one parametrised by a generic point of a component of a base space of a versal deformation, excluding smooth points. A singularity is "generic" if its base space has only one component, and it has only equisingular deformations (for space curves not a precise concept either, see [6]). A general non-smoothable homogeneous L^r_n having deformations of positive weight is "generic", while the generic singularity with the same tangent cone is not quasi-homogeneous. The curve of Proposition 2.9 is not "generic" in this sense.

All our examples of non-smoothable curves are basically based on curves with smooth branches or on monomial curves. This is not a severe restriction, as monomial curves are in a certain sense the most singular ones. The ones we encountered deform into curves with smooth branches, see e.g. Proposition 3.12. We did not find irreducible curves deforming into L^n_r with large $d(n,r)$. This shows how limited our knowledge is. Nevertheless, based on our examples we offer:

Conjecture 5.1. All branches of generic curve singularities are smooth.

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