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Linear components of the tangent cone in the Nash modification of a complex surface singularity

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Abstract

We prove that each linear component of the tangent cone of a complex surface singularity corresponds to at least one singular point in the normalized Nash modification, whenever the minimal resolution factors through the blow-up of the origin of the germ. We give an example of a surface whose tangent cone has no linear component and the normalized Nash modification is singular.

1 Introduction

Let (X, x) be a germ of equidimensional complex analytic singularity. The Nash modification of (X, x) is a modification that consists in replacing each singular point of a representative of the germ by all the possible limits of directions of tangent spaces. More precisely, let $X \subset \mathbb{C}^n$ be a representative of $(X, 0)$, and suppose it has pure dimension d. Call $\mathrm{Sing}(X)$ the singular locus of X. We define the map:

$$\begin{array}{rrcl} \lambda: & X \setminus \mathrm{Sing}(X) & \to & \mathbf{G}(d,n) \\ & y & \mapsto & T_yX \end{array}$$

where $\mathbf{G}(d, n)$ is the Grassmannian of d-dimensional vector space in $\mathbb{C}^n$ and T_yX is the direction of tangent space to X at y. The closure of the graph of λ in $X \times \mathbf{G}(d, n)$ is a complex analytic space of dimension d. Call it $\tilde{X}$ and endow it with the restriction ν of the projection on the first factor. The datum $\nu : \tilde{X} \to X$ is the Nash modification of X.

If y is a singular point of X, the fibre $\nu^{-1}(y)$ consists of the vector spaces T obtained as limits of directions of tangent spaces $T_{y_n}X$ of a sequence of non singular points of X converging to y. We will simply call it a limit of tangent spaces to X at y. The limits of tangent spaces to analytic varieties have been studied in the 70's and 80's by D.T Lê, B. Teissier, J.P.G. Henry, M. Merle, T. Gaffney and others, see [8, 6, 5, 2].

We will focus in this work on the 2-dimension case. Let (S, s) be a complex surface singularity. We will denote by $C_{S,s}$ its tangent cone. We recall that this is the algebraic variety defined by the ideal generated by all the initial forms at s of the holomorphic functions of the ideal defining the surface S. The tangent cone is strongly related to the limits of tangent spaces to the surface. This relation is established and explained in [7].

In particular, any two-dimensional plane, tangent to the tangent cone is a limit of tangent spaces to the surface at s. So whenever $C_{S,s}$ has a linear two-dimensional plane as an irreducible component, this plane will correspond to a limit of tangent spaces to S at s. These linear components will be called the planar components of the tangent cone.

The Nash modification of a surface singularity is not necessarily normal, not even when the original surface is normal. Call $n : \bar{S} \to \tilde{S}$ the normalization of the surface obtained by Nash modification. The composition map $\nu \circ n : \bar{S} \to S$ will be called the normalized Nash modification of the surface singularity (S, s).

The normalization map being finite, each point in the exceptional fiber of the Nash modification $\tilde{S}$ has finitely many pre-images in the normalized Nash modification. In particular, since each planar component of the tangent cone corresponds to a point in the exceptional fiber of ν, then each of these components corresponds to finitely many points in the normalized Nash modified surface $\bar{S}$.

In terms of this last correspondence, our main result states that whenever (S, s) is a surface singularity for which the minimal resolution factors through the blow-up of the point s, then any planar component of the tangent cone corresponds to at least a singular point in the normalized Nash modification of (S, s).

Then we give an example showing that the converse is false. Namely we exhibit a minimal singularity whose tangent cone has no planar component and the normalized Nash modification has two singular points.

This example shows some limit of the analogy between the pairs (Nash modification, polar curves) and (point blow-up, hyperplane sections) for normal surfaces. Indeed, the singular points of the normalized blow-up of the origin of a normal germ of a surface are fixed points of the family of polar curves. Meanwhile a singular point of the normalized Nash modification of a normal germ of surface need not be a fixed point of the family of hyperplane sections ; see [10, 11].

2 The result

Let (S, s) be a germ of complex surface singularity and let $f_1, \dots, f_r$ be holomorphic functions on it. For each $\alpha \in \mathbb{P}^{r-1}$ we define the curve $\mathcal{C}_\alpha$ to be the zero set on (S, s) of a linear combination $\Sigma_{i=1}^r a_i f_i$, where $(a_1 : \dots : a_r)$ are homogeneous co-ordinates of α. The family of curves $(\mathcal{C}_\alpha)_\alpha$ is the linear system of curves generated by $f_1, \dots, f_r$ and parametrised by $\mathbb{P}^{r-1}$.

Consider now a modification $\mu : X \to (S, s)$.

Definition 2.1. *A point $\eta \in \mu^{-1}(s)$ is called a fixed point, or a base point, of the family of curves $(\mathcal{C}_\alpha)_\alpha$ if there exists an open set $U \subset \mathbb{P}^{r-1}$, such that for any $\alpha \in U$ the strict transform of the curve $\mathcal{C}_\alpha$ by the modification μ contains the point η.*

In [11, Thms 3.2 and 4.2], we proved the following:

Proposition 2.2. *Let (S, s) be a reduced equidimensional germ of complex surface singularity. The normalized Nash modification of (S, s) factors through the blow-up of the point s in S if and only if the tangent cone $C_{S,s}$ does not have any planar component.*

Moreover, the planar components of the tangent cone correspond exactly to the fixed points of the linear system of hyperplane sections on S at s in the Nash modification.

The aim of this short note is to show how the planar components of the tangent cone contribute in the singularities of the surface obtained by Nash modification.

Theorem 2.3. *Let (S, s) be a germ of reduced and equidimensional complex surface singularity. Suppose that the minimal resolution of (S, s) factors through the blow-up of the point s. Then to any planar component of the tangent cone $C_{S,s}$ corresponds at least one singular point in the surface obtained by the normalized Nash modification.*

Proof:

Let P be a planar component of the tangent cone $C_{S,s}$. It is a limit of tangent planes to the surface at s. So it corresponds to a point η in the exceptional fibre of the Nash Modification. Call $\eta_1, \ldots, \eta_d$ the inverse image of η by the normalization map.

Let us consider the following commutative diagram:

$$\begin{tikzcd} \bar{X} \arrow[r, "\tau"] \arrow[d, "\pi"'] & X \arrow[d, "\rho"] \\ \bar{S} \arrow[d, "n"'] & S' \arrow[d, "e"] \\ \tilde{S} \arrow[r, "\nu"] & S \end{tikzcd}$$

where $n : \bar{S} \to \tilde{S}$ is the normalization of the Nash modified surface, the map $\pi : \bar{X} \to \bar{S}$ is the minimal resolution of the singularities of $\bar{S}$, $e : S' \to S$ is the blow-up of the point s and $\rho : X \to S'$ is the minimal resolution of the singularities of S'.

Since the composition $\nu \circ n \circ \pi$ is a resolution of S it factors through the minimal resolution of S which coincides by hypothesis with the minimal resolution of S'. Let us call $\tau : \bar{X} \to X$ the factorisation map.

By proposition 2.2, η is a fixed point of the linear system of hyperplane sections of S at s in the surface $\tilde{S}$. Since the normalization map is finite, at least one of the points $\eta_i \in \bar{S}$ is still a fixed point of the hyperplane sections. Suppose η_1 is. And suppose it is not a singular point of the surface $\bar{S}$. Then, the minimal resolution π induces an isomorphism over a neighborhood of η_1. So the inverse image $\pi^{-1}(\eta_1)$ is again a fixed point of the hyperplane sections of S at s in the resolution $\bar{X}$ and hence in the minimal resolution X and in the blow-up of the origin S'.

The universal property of the blowing up asserts that the linear system of hyperplane sections does not have any fixed point in the blown-up surface. So this contradicts the assumption of η_1 being a non singular point of $\bar{S}$.

So at least one of the $\eta_i \in \bar{S}$ is a singular point. □

Remark 2.4. *There exists a class of normal two-dimensional singularities, called rational surface singularities having the property that any resolution factors through the blow-up of the singularity. For more details on rational surface singularities see for example [1] and [9]. So our hypothesis in theorem 2.3 is satisfied by a non trivial class of surface singularities. However it would be interesting to see if the conclusion of theorem 2.3 is valid for a general complex reduced purely two-dimensional singularity.*

Example 2.5. *The A_2 singularity given by the equation $x^2+y^2+z^3$ is a rational double point singularity. Its tangent cone is a union of two planes. The Nash modification of this surface was studied by G. Gonzalez-Sprinberg in [4]. There it is shown that the Nash modification of this surface has exactly two singular points corresponding precisely to the planes of the tangent cone at the origin.*

3 The converse is false

Consider a normal two-dimensional singularity $(S, 0)$ having the diagram of figure 1 as dual graph of its minimal resolution $\pi : X \to S$.

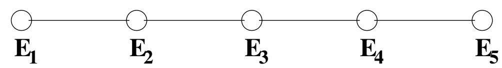

Figure 1: Dual graph of the minimal resolution

Where all the irreducible components E_i, $1 \leq i \leq 5$, of the exceptional divisor are rational smooth curves intersecting transversally. The self intersections are given as follows: $E_1^2 = E_5^2 = -3$ and $E_2^2 = E_3^2 = E_4^2 = -2$. This singularity is rational, and actually even minimal, i.e. the fundamental cycle (or equivalently in this case, the maximal ideal cycle) in the minimal resolution is $Z = \Sigma_{i=1}^5 E_i$. A surface with minimal singularity has also the property that its tangent cone is reduced.

It is well known that the surface S' obtained by the contraction of the irreducible components of the exceptional divisor satisfying the property $Z \cdot E_i = 0$ (known as Tyurina components) is isomorphic to the surface obtained by the blow-up of the origin in S. The image under this contraction of the irreducible components of the exceptional fibre such that $Z \cdot E_i < 0$ (the non-Tyurina components) is precisely the projective tangent cone of the surface $(S, 0)$.

So in this example, the projective tangent cone has two (reduced) and irreducible components obtained by the images of E_1 and E_5 in the contraction of E_2, E_3 and E_4. Moreover the degree of each of these components is given by the (positive) number $-Z \cdot E_i$. So the projective curve associated to the tangent cone is the union of two irreducible and reduced curves of degree two, intersecting in one point. This intersection point is the only singular point of the surface S'; it is an A_3 singularity. In particular, the tangent cone $C_{S,0}$ has no planar component.

A normalized modification of a normal surface singularity factors through the Nash modification if and only if the family of (local and absolute) polar curves of the surface at the singular point have no fixed point in the normalized modification ([12, Thm. III.1.2]). M. Spivakovsky studied extensively Nash modification on normal surfaces, generalizing a previous work by G. González-Sprinberg in [3]. In particular he gave a precise characterization of the fixed points of the polar curves in the minimal resolution of a minimal surface singularity, and determined the components of the exceptional fibre that intersect a general polar curve; see [12, Thm.III 5.4.]. Let us state this result:

Consider $\pi : X \to S$ the minimal resolution of a minimal surface singularity. Call

$E = \bigcup_i E_i$ the decomposition of the exceptional fiber into its irreducible components and Γ the dual graph associated to it. Let V_i be the vertex of Γ corresponding to E_i. The cycle $Z = \Sigma_i E_i$ is in the case the so called fundamental cycle. It satisfies $Z \cdot E_i \leq 0$ for all i. Let V_{i_0} be a vertex such that $Z \cdot E_{i_0} = 0$. The Tyurina component of Γ containing V_{i_0} is the connected component of $\Gamma \setminus \{V_i,\ Z \cdot E_i < 0\}$ that contains V_{i_0}. We define the integer s_i associated to V_i as follows:

if $Z \cdot E_i < 0$ then $s_i = 1$

if $Z \cdot E_i = 0$ call Δ the Tyurina component of Γ containing V_i. In this case the value s_i is the minimal number of edges between V_i and $\Gamma \setminus \Delta$ plus 1.

Let V_i and V_j be two adjacent vertices in Γ. The edge joining both vertices is called a central edge if $s_i = s_j$.

A vertex V_i is called a central vertex if there exist at least two vertices V_j and V_k adjacent to V_i such that $V_j = V_k = V_i - 1$.

The criterion established by M. Spivakovsky in [12, Thm.III 5.4.] says the following:

The fixed points of the polar curves in the minimal resolution are precisely the points of X corresponding to the central edges of the graph Γ. Away from these points, the general polar curve intersects a component E_i if and only if V_i is a central vertex or $Z \cdot E_i \leq -2$.

Applying this criterion to our example, we obtain that:

- The dual graph in figure 1 has no central edge, and hence the minimal resolution has no fixed point of the polar curves. So the minimal resolution $\pi : X \to S$ factors through the Nash modification, and hence it is also the minimal resolution of the normalization $\bar{S}$ of the Nash modification $\tilde{S}$ of S.

- The general polar curve intersects exactly the irreducible components E_1, E_3 and E_5. So the surface obtained by the contraction of the irreducible curves E_2 and E_4 is the normalized Nash modified surface $\bar{S}$.

The surface $\bar{S}$ has then two A_1 singularities.

This examples shows that it is possible the have a rational surface singularity whose tangent cone has no planar component and whose normalized Nash modification has singularities.

Remark 3.1. *In [10], we proved that a singular point of the normalized blow-up of the origin of a germ of a normal surface singularity is always a base point of the linear system of polar curves. We used to think that the similar behavior of the hyperplane sections in the blow-up of the origin with the polar curves in the Nash modification of normal surfaces would suggest that all singular points of the normalized Nash modification of a normal surface are fixed points of the family of hyperplane sections. However this example shows that it is not always the case.*

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